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Infinitesimal adjunction and polar curves

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Abstract. The polar curves of foliations $\mathcal{F}$ having a curve C of separatrices generalize the classical polar curves associated to hamiltonian foliations of C. As in the classical theory, the equisingularity type $\wp(\mathcal{F})$ of a generic polar curve depends on the analytical type of $\mathcal{F}$, and hence of C. In this paper we find the equisingularity types $\epsilon(C)$ of C, that we call kind singularities, such that $\wp(\mathcal{F})$ is completely determined by $\epsilon(C)$ for Zariski-general foliations $\mathcal{F}$. Our proofs are mainly based on the adjunction properties of the polar curves. The foliation-like framework is necessary, otherwise we do not get the right concept of general foliation in Zariski sense and, as we show by examples, the hamiltonian case can be out of the set of general foliations.

Keywords: singular foliation, polar curve, Newton polygon, equisingularity type, adjoint curves.

Mathematical subject classification: 32S65.

1 Introduction

Let $\mathcal{F}$ be a germ of holomorphic foliation of $(\mathbb{C}^2, 0)$ having a curve of separatrices C. The *polar curve* Γ of $\mathcal{F}$ with respect to a direction $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$ is given by $\omega \wedge (ady - bdx) = 0$, where ω is a 1-form defining $\mathcal{F}$. There is a Zariski-open set of directions such that the equisingularity type $\epsilon(\Gamma \cup C)$ of $\Gamma \cup C$ is the same one, independent of ω and of the coordinates. We denote $\wp(\mathcal{F})$ this generic type of equisingularity. This paper is devoted to provide an accurate description of the types $\wp(\mathcal{F})$ in terms of the equisingularity type $\epsilon(C)$ of C.

We work with foliations in the class $\mathbb{G}^*_C$ of the generalized curves without "bad resonances" defined as follows. A foliation $\mathcal{F}$ belongs to $\mathbb{G}^*_C$ if

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(1) It is a generalized curve in the sense of Camacho-Lins Neto-Sad ([3]) having C as curve of separatrices. Note that, in this case, the minimal morphism of reduction of singularities π_C of C is also the reduction of singularities of $\mathcal{F}$.

(2) For any C-*ramification* $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ (that is, ρ is transversal to C and $\rho^{-1}C$ has only non-singular branches), there is no corner in the reduction of singularities of $\rho^*\mathcal{F}$ with Camacho-Sad index equal to -1.

If $C = (f = 0)$, the hamiltonian foliation $df = 0$ belongs to $\mathbb{G}_C^*$. But the class $\mathbb{G}_C^*$ is wider than that. Let us write $f = \prod_{i=1}^r f_i$, then the logarithmic foliations

$$\mathcal{L}_\lambda = \left(\sum_{i=1}^r \lambda_i \frac{df_i}{f_i} = 0 \right)$$

belong to this class if $\lambda = (\lambda_1, \cdots, \lambda_r)$ avoid certain rational resonances. More generally, each generalized curve foliation $\mathcal{F}$ has a well defined *logarithmic model* $\mathcal{L}_\lambda$, $\lambda = \lambda(\mathcal{F})$, of the above type such that the Camacho-Sad indices of $\mathcal{F}$ and $\mathcal{L}_\lambda$ coincide along the reduction of singularities [5].

There is a first relationship between $\epsilon(C)$ and $\wp(\mathcal{F})$ described in the *decomposition theorem of the polar curve* [5], proved by several authors in different contexts [13, 11, 10, 15]. It can be stated as follows:

Theorem (Decomposition [5]). *Let ρ be a C-ramification. If Γ is a generic polar curve of $\mathcal{F} \in \mathbb{G}_C^*$, then $\rho^{-1}\Gamma$ is a strict adjoint of $\rho^{-1}C$.*

If $Y \subset (\mathbb{C}^2, 0)$ is a curve with only non-singular branches, we say that a curve $Z \subset (\mathbb{C}^2, 0)$ is a *strict adjoint* of Y if the multiplicities satisfy $m_p(Z) = m_p(Y) - 1$ at the infinitely near points p of Y and Z does not go through the corners of the desingularization of Y. (Compare with the definition in [4], p. 152).

There are infinitely many possible equisingularity types $\epsilon(Y \cup Z)$ for a fixed Y and Z being strict adjoint of Y. In section 3 we prove the following result of finiteness by using a control of the Newton polygon of a generic polar curve Γ (a similar result for the case of hamiltonian foliations can be deduced from the virtual behaviour of the polar curves described in [4]).

Theorem. *There exists a finite number of equisingularity types $\wp(\mathcal{F})$, where $\mathcal{F} \in \mathbb{G}_{C'}^*$ and C' is such that $\epsilon(C') = \epsilon(C)$.*

Take as above $Y \subset (\mathbb{C}^2, 0)$ with only non-singular branches. A strict adjoint curve Z of Y is a *perfect adjoint* curve of Y if π_Y desingularizes Z. In this case the equisingularity type $\chi_Y = \epsilon(Y \cup Z)$ does not depend on Z. Section 4 is devoted to prove the following result of genericity.

Theorem (of genericity)**.** *Assume that C has only non-singular branches. There is a non-empty Zariski-open set $U_C \subset \mathbb{P}^{r-1}_{\mathbb{C}}$ defined by*

$$\text{“}\lambda \in U_C \text{ if there exists } \mathcal{F} \in \mathbb{G}^*_C \text{ with } \wp(\mathcal{F}) = \chi_C \text{ and } \lambda = \lambda(\mathcal{F})\text{”.}$$

Moreover, for each $\mathcal{F} \in \mathbb{G}^*_C$ with $\lambda(\mathcal{F}) \in U_C$ we have that $\wp(\mathcal{F}) = \chi_C$.

In general, it is not possible to define χ_C in a way compatible with C-ramifications. This is the characteristic property of the *kind equisingularity types* that we introduce below.

Let $G(C)$ be the dual graph of C oriented by its first divisor. Associate to each divisor E the multiplicity $m(E)$ given by any E-“curvette” and the number b_E of edges and arrows which leave from E. Thus E is a *bifurcation divisor* if $b_E \geq 2$ and a *terminal divisor* if $b_E = 0$. A *dead arc* joins a bifurcation divisor with a terminal divisor, with no other bifurcations. We say that $\epsilon(C)$ is *kind* if $m(E_b) = 2m(E_t)$, for each dead arc of $G(C)$ starting at E_b and ending at E_t. The next proposition, proved in section 5, gives a characterization of kind equisingularity types in terms of adjunction.

Proposition. *The equisingularity type $\epsilon(C)$ is kind if and only if there is a germ of curve $Z \subset (\mathbb{C}^2, 0)$ such that $\rho^{-1}Z$ is a perfect adjoint of $\rho^{-1}C$ for any C-ramification ρ. Moreover $\epsilon(C \cup Z)$ does not depend on the choice of Z.*

For kind equisingularity types we define $\chi_C = \epsilon(C \cup Z)$ and we say that such Z are perfect adjoint curves of C. The next proposition, proved in section 5, gives a precise description of χ_C for kind equisingularity types. (For classical polar curves, our description is slightly more precise than the one in [12]).

Proposition. *Let C be a curve with kind equisingularity type and Z a perfect adjoint curve of C. Then π_C gives a reduction of singularities of $Z \cup C$. Moreover, the branches of Z intersect an irreducible component E of the exceptional divisor of π_C as follows:*

- *If E is a bifurcation divisor of $G(C)$, the number of branches of Z cutting E equals to $b_E - 2$ if E is in a dead arc and to $b_E - 1$ otherwise.*
- *If E is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of Z through E.*
- *Otherwise, no branches of Z intersect E.*

Finally, in section 6, we relate the polar curves to the adjoint curves in the case of kind equisingularity types. As a consequence we obtain a precise description of $\wp(\mathcal{F})$ if $\epsilon(C)$ is kind. Let us define the Zariski open set $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ by

> "$\lambda \in U_C$ if there exists $\mathcal{F} \in \mathbb{G}_C^*$ with $\lambda = \lambda(\mathcal{F})$ having a generic polar curve Γ such that $\rho^{-1}\Gamma$ is a perfect adjoint of $\rho^{-1}C$, for any C-ramification ρ"

Then we prove the following theorem.

Theorem. *The curve C has a kind equisingularity type if and only if $U_C \neq \emptyset$. In this case $\wp(\mathcal{F}) = \chi_C$ for any $\mathcal{F} \in \mathbb{G}_C^*$ such that $\lambda(\mathcal{F}) \in U_C$.*

The hamiltonian foliations $df = 0$ have vector of exponents $\lambda = \underline{1}$. We provide examples such that $\underline{1} \notin U_C$, hence the consideration of the class $\mathbb{G}_C^*$ is essential for this theory.

The main results of this paper were announced in [6]. Our results are of local nature in the framework of foliations (see also [15, 5, 7]). The classical local study of polar curves has been developed by several authors ([16, 13, 11, 12, 4, 10]). There are also related works for foliations from the global viewpoint [14, 8].

2 Strict adjoint curves

Before starting the study of polar curves, we describe some properties that can be deduced from the fact that a curve is a strict adjoint of another curve. We recall the notion of a strict adjoint curve:

Definition 1. *Assume that C has only non-singular branches. We say that Z is a* strict adjoint *of C if $m_p(Z) = m_p(C) - 1$ at each infinitely near point p of C and Z does not go through the corners of the desingularization of C.*

If Z is a strict adjoint of C, the properties above allow to give a decomposition of Z into bunches of branches in terms of the equisingularity data of C. Let us describe it using the dual graph $G(C)$ of C which is constructed from the minimal reduction of singularities $\pi_C : M \to (\mathbb{C}^2, 0)$ of C (see appendix A for all the notations concerning the dual graph of a curve). Given a divisor E of $\pi_C^{-1}(0)$, we denote by $\pi_E : M_E \to (\mathbb{C}^2, 0)$ the morphism reduction of π_C to E (see appendix A); recall that $\pi_C = \pi_E \circ \pi'_E$. Let $B(C)$ be the set of bifurcation divisors of $G(C)$. For any $E \in B(C)$, we define Z^E to be the union of the branches ζ of Z such that

- $\pi_E^* \zeta \cap \pi_E^* C = \emptyset$
- If $E' < E$, then $\pi_E^* \zeta \cap \pi_E'(E') = \emptyset$

where $\pi_E^* \zeta$ denotes the strict transform of ζ by π_E. Thus there is a unique decomposition $Z = \cup_{E \in B(C)} Z^E$ satisfying that:

d1. $m_0(Z^E) = b_E - 1$.

d2. $\pi_E^* Z^E \cap \pi_E^* C = \emptyset$.

d3. If $E' < E$ then $\pi_E^* Z^E \cap \pi_E'(E') = \emptyset$.

d4. If $E' > E$ then $\pi_{E'}^* Z^E \cap E'_{red} = \emptyset$.

In particular, if E is not a bifurcation divisor we have that $\pi_E^* Z \cap E_{red} = \pi_E^* C \cap E_{red}$. Moreover, the properties above imply the following ones which are stated in terms of the coincidences and of the data in $G(C)$. For each irreducible component ζ of Z^E we have that

(D-i) $C(C_i, \zeta) = v(E)$ if E belongs to the geodesic of C_i;

(D-ii) $C(C_j, \zeta) = C(C_j, C_i)$ if E belongs to the geodesic of C_i but not to the one of C_j.

(see appendix A for the definitions of b_E, $v(E)$ and *geodesic* of a curve in $G(C)$).

Consider now any curve C and let $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be any C-ramification (the reader can refer to appendix B for notations and general results concerning ramifications). If $\tilde{Z} = \rho^{-1} Z$ is a strict adjoint of $\tilde{C} = \rho^{-1} C$, then there is also a decomposition of Z in terms of the equisingularity data of C: for any bifurcation divisor E of $G(C)$, we define Z^E to be such that

$$\rho^{-1} Z^E = \bigcup_{i=1}^{n_E} \tilde{Z}^{\tilde{E}^j},$$

where $\{\tilde{E}^j\}_{j=1}^{n_E}$ are the divisors of $G(\tilde{C})$ associated to E in $G(\tilde{C})$ and $\tilde{Z} = \cup_{\tilde{E} \in G(\tilde{C})} \tilde{Z}^{\tilde{E}}$ is the decomposition of $\tilde{Z}$ described above. Hence, we get a decomposition $Z = \cup_{E \in B(C)} Z^E$ such that:

D1. $m_0(Z^E) = \begin{cases} \underline{n}_E n_E (b_E - 1), & \text{if } E \text{ does not belong to a dead arc;} \\ \underline{n}_E n_E (b_E - 1) - \underline{n}_E, & \text{otherwise.} \end{cases}$

D2. $\pi_E^* Z^E \cap \pi_E^* C = \emptyset$.

D3. If $E' < E$, then $\pi_E^* Z^E \cap \pi'_E(E') = \emptyset$.

D4. If $\pi_E^* Z^E \cap \pi'_E(E') \neq \emptyset$, then $\pi'_E(E') > E_{red}$.

D5. If $E' > E$ and E' does not belong to a dead arc joined to E, then $E'_{red} \cap \pi_{E'}^* Z^E = \emptyset$.

Moreover, properties (D-i) and (D-ii) also hold now for a branch ζ of Z^E.

It is clear that the properties above do not determine the equisingularity type of the curve Z even if C has only non-singular branches. Let us introduce a definition:

Definition 2. *Assume that C has only non-singular branches and let Z be a strict adjoint of C. We say that Z is a* perfect adjoint curve *of C if π_C gives a reduction of singularities of Z.*

Let us state a criterion to check if a curve Z is a perfect adjoint of C.

Proposition 1. *Let C be a curve with only non-singular branches. A strict adjoint curve Z of C is perfect adjoint curve of C if and only if the set*

$$\pi_E^* Z \cap E_{red} \setminus \pi_E^* C \cap E_{red}$$

has exactly $b_E - 1$ points for each irreducible component E of $\pi_C^{-1}(0)$.

Proof. Observe that the second part of the statement always holds when E is not a bifurcation divisor ($b_E = 1$) since $\pi_E^* Z \cap E_{red} = \pi_E^* C \cap E_{red}$ (see the properties of the decompositions above). Therefore we only need to prove the result for bifurcation divisors. Recall that there is a decomposition $Z = \cup_{E \in B(C)} Z^E$ such that $\pi_E^* Z \cap E_{red} \setminus \pi_E^* C \cap E_{red} = \pi_E^* Z^E \cap E_{red}$ by properties d2-d4.

Assume first that Z is a perfect adjoint curve of C. Then π_C is a reduction of singularities of $Z \cup C$. Hence the irreducible components of Z are non-singular and its number is equal to the multiplicity $m_0(Z)$. Moreover, the property d4. implies that π_E is a reduction of singularities of Z^E and the number of points of $\pi_E^* Z^E \cap E_{red}$ is equal to $m_0(Z^E) = b_E - 1$ since Z^E only cuts E_{red} by d3.

Reciprocally, assume that the set $\pi_E^* Z^E \cap E_{red}$ has exactly $b_E - 1$ points for each bifurcation divisor E of $G(C)$. This implies that Z^E has $b_E - 1$ irreducible components which are non-singular and that π_E is a reduction of singularities of Z^E. Then, from the equalities $\pi_C^* Z^E \cap E = \pi_C^* Z \cap E$ and $\pi_C^* Z^E \cap \pi_C^* C = \emptyset$, we deduce that π_C is a reduction of singularities of $Z \cup C$. $\square$

The next corollary gives a characterization of a perfect adjoint curve of a given curve C in terms of the equisingularity data of C, when C has only non-singular branches.

Corollary 1. *Consider a curve C with only non-singular branches and let $Z = \cup_{E \in B(C)} Z^E$ be the decomposition of a strict adjoint curve Z of C. The curve Z is perfect adjoint curve of C if and only if each curve Z^E is composed by $b_E - 1$ irreducible components $\{\zeta_i^E\}_{i=1}^{b_E-1}$ with $C(\zeta_i^E, \zeta_j^E) = v(E)$ for $i \neq j$.*

In particular, the corollary above implies that $G(C \cup Z)$ is obtained from $G(C)$ by adding $b_E - 1$ arrows to each bifurcation divisor E of $G(C)$ and this property characterizes the fact of Z being a perfect adjoint of C, when C has only non-singular branches. Hence, it is clear that $\epsilon(C \cup Z)$ does not depend on Z and we denote $\chi_C = \epsilon(C \cup Z)$.

In the general case of a curve C with singular branches, it is not possible to define χ_C in a compatible way with C-ramifications. Since this situation needs a more detailed treatment, we shall consider it in section 5.

3 Local invariants and polar curves

Let $\mathbb{F}$ be the space of singular foliations of $(\mathbb{C}^2, 0)$, that is, an element $\mathcal{F} \in \mathbb{F}$ is defined by a 1-form $\omega = 0$, with $\omega = Adx + Bdy$, $A, B \in \mathbb{C}\{x, y\}$ and $A(0) = B(0) = 0$. Given a plane curve $C \subset (\mathbb{C}^2, 0)$, we denote by $\mathbb{F}_C$ the subspace of $\mathbb{F}$ composed by the foliations which have C as a curve of separatrices.

For a direction $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$, the polar curve $\Gamma(\mathcal{F}; [a : b])$ is the curve

$$\Gamma = \{aA(x, y) + bB(x, y) = 0\}.$$

We denote by $\Gamma_{\mathcal{F}}$ a generic polar when the direction $[a : b]$ is not needed. Then the multiplicity $m_0(\Gamma_{\mathcal{F}})$ of $\Gamma_{\mathcal{F}}$ at the origin coincides with the multiplicity $\nu_0(\mathcal{F})$ of $\mathcal{F}$ at the origin. Recall that, if $\mathbb{G}$ is the space of generalized curve foliations of $(\mathbb{C}^2, 0)$ and $\mathbb{G}_C = \mathbb{F}_C \cap \mathbb{G}$, we have that $\nu_0(\mathcal{F}) = m_0(C) - 1$ for any $\mathcal{F} \in \mathbb{G}_C$.

The Newton polygon $\mathcal{N}(\mathcal{F}; x, y) = \mathcal{N}(\omega; x, y)$ of $\mathcal{F}$ is defined as the one of the ideal generated by xA and yB. More precisely, if we write $\omega = \sum_{i,j} \omega_{ij}$ with

$$\omega_{ij} = A_{ij} x^{i-1} y^j dx + B_{ij} x^i y^{j-1} dy, \tag{1}$$

and we put $\Delta(\omega) = \{(i, j) : \omega_{ij} \neq 0\}$, then $\mathcal{N}(\mathcal{F}; x, y)$ is the convex envelop of $\Delta(\omega) + \mathbb{R}^2_{\geq 0}$. In the case of an analytic function $f = \sum_{ij} f_{ij} x^i y^j$, we define $\Delta(f) = \{(i, j) : f_{ij} \neq 0\}$ and then the Newton polygon $\mathcal{N}(C; x, y)$ of the curve $C = (f = 0)$ is the convex envelop of $\Delta(f) + \mathbb{R}^2_{\geq 0}$. In particular, if $\mathcal{F} \in \mathbb{G}_C$, then $\mathcal{N}(\mathcal{F}; x, y)$ coincides with $\mathcal{N}(C; x, y) = \mathcal{N}(df; x, y)$.

From now on we will always assume that we chose coordinates (x, y) such that $x = 0$ is not tangent to the curve C of separatrices. In particular this implies that the first side of the Newton polygon $\mathcal{N}(\mathcal{F}; x, y)$ has slope greater or equal to -1.

Let us recall the relationship between Newton polygon and infinitely near points of a curve since it will be useful in the sequel. First we introduce some notations.

Notation. Let C be a curve with only non-singular branches and $\pi_C : M \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of C. Given an irreducible component E of $\pi_C^{-1}(0)$ with $v(E) = p$, the morphism $\pi_E : M_E \to (\mathbb{C}^2, 0)$ is a composition of p blowing-ups of points

$$\left(\mathbb{C}^2, 0\right) \xleftarrow{\sigma_1} \left(X_1, P_1\right) \leftarrow \cdots \leftarrow \left(X_{p-1}, P_{p-1}\right) \xleftarrow{\sigma_p} X_p = M_E.$$

If (x, y) are coordinates in $(\mathbb{C}^2, 0)$ there is a change of coordinates $(x, y) = (\tilde{x}, \tilde{y} + \varepsilon(\tilde{x}))$, with $\varepsilon(x) = a_1 x + \cdots + a_{p-1} x^{p-1}$, such that the blowing up σ_j is given by $x_{j-1} = x_j$, $y_{j-1} = x_j y_j$, for $j = 1, 2, \ldots, p$, where (x_j, y_j) are coordinates centered at P_j and $(x_0, y_0) = (\tilde{x}, \tilde{y})$. We say that $(\tilde{x}, \tilde{y})$ are *coordinates in* $(\mathbb{C}^2, 0)$ *adapted to* E.

Consider now a plane curve $\gamma \subset (\mathbb{C}^2, 0)$ with only non-singular irreducible components and let $\pi_\gamma : X \to (\mathbb{C}^2, 0)$ be its minimal reduction of singularities. Take E an irreducible component of $\pi_\gamma^{-1}(0)$ with $v(E) = p$ and choose (x, y) coordinates adapted to E. Assume that $\gamma = (f(x, y) = 0)$ with $f(x, y) = \sum_{i,j} f_{ij} x^i y^j \in \mathbb{C}\{x, y\}$. Since (x, y) are adapted to E, then there exists a side L of $\mathcal{N}(\gamma; x, y)$ with slope $-1/p$. Let $i + pj = k$ be the line which contains L and put

$$In_p(f; x, y) = \sum_{i+pj=k} f_{ij} x^i y^j.$$

Take now (x_p, y_p) coordinates in the first chart of E_{red} with $\pi_E(x_p, y_p) = (x_p, x_p^p y_p)$ and $E_{red} = (x_p = 0)$. Thus, a simple calculation shows that the points of $\pi_E^* \gamma \cap E_{red}$ are given by $x_p = 0$ and $\sum_{i+pj=k} f_{ij} y^j = 0$. We conclude that the points of $\pi_E^* \gamma \cap E_{red}$ are determined by $In_p(f; x, y)$ and reciprocally.

Consequently, the following result which describes the Newton polygon of a generic polar curve $\Gamma_{\mathcal{F}}$ will be useful to determine the infinitely near points of $\Gamma_{\mathcal{F}}$.

Lemma 1 ([5]). *Consider a foliation $\mathcal{F} \in \mathbb{F}$ and let L be a side of $\mathcal{N}(\mathcal{F}; x, y)$ with slope $-1/\mu$ where $\mu \in \mathbb{Q}$ and $\mu \geq 1$. If $i + \mu j = k$ is the equation of the line which contains L, then*

$$\mathcal{N}(\Gamma_{\mathcal{F}}; x, y) \subset \left\{(i, j) \ : \ i + \mu j \geq k - \mu\right\}.$$

More precisely, if $\mu > 1$ then $\Delta(B) \subset \{i + \mu j \geq k - \mu\}$ and $\Delta(A) \subset \{i + \mu j > k - \mu\}$.

However the result above does not provide enough information to obtain a description of the equisingularity type of $\Gamma_{\mathcal{F}}$. If we want to control the slopes of $\mathcal{N}(\Gamma_{\mathcal{F}}; x, y)$ we need to know the "contribution" in the points of the sides of $\mathcal{N}(\mathcal{F}; x, y)$. Recall that a point $(i, j) \in \Delta(\omega)$ is said to be a *contribution* of B if $B_{ij} \neq 0$ in the expression (1), i.e., if $(i, j) \in \Delta(yB)$.

Thus to get a more precise description of the Newton polygon $\mathcal{N}(\Gamma_{\mathcal{F}}; x, y)$ we need to consider foliations in $\mathbb{G}_C^*$ since the contributions on the sides of the Newton polygon of a foliation have a direct relationship with the values of the Camacho-Sad indices at the infinitely near points of $\mathcal{F}$ as it is explained in the next proposition.

Recall that, if $S = (y = 0)$ is a non-singular separatrix of $\mathcal{F}$, then the *Camacho-Sad index of $\mathcal{F}$ relative to S at the origin* is given by

$$I_0(\mathcal{F}, S) = -\mathrm{Res}_0 \frac{a(x, 0)}{b(x, 0)} \tag{2}$$

where the 1-form ω defining $\mathcal{F}$ is written as $\omega = ya(x, y)dx + b(x, y)dy$ (see [2]). Then we have the following result:

Proposition 2 [5]. *Consider a foliation $\mathcal{F} \in \mathbb{G}$ and take a side L of $\mathcal{N}(\mathcal{F})$ with slope $-1/p$, $p \in \mathbb{N}$. If L has no contribution of B in its highest vertex, then there is a corner in the reduction of singularities of $\mathcal{F}$ with Camacho-Sad index equal to -1.*

In particular, given a foliation $\mathcal{F} \in \mathbb{G}_C^*$ such that the curve C has only non-singular irreducible components, the result above implies that

> if $\mathcal{N}(\mathcal{F}; x, y)$ has s sides L_j with slopes $-1/p_j$, $p_j \in \mathbb{N}$, $j = 1, \ldots, s$ and $p_1 < p_2 < \cdots < p_s$, then the first $s - 1$ sides of $\mathcal{N}(\Gamma_{\mathcal{F}}; x, y)$ are obtained from the ones of $\mathcal{N}(\mathcal{F}; x, y)$ by a vertical translation of one unit and the other ones have slope $\geq -1/p_s$.

These results describing the Newton polygon of $\Gamma_{\mathcal{F}}$ are key in the proof of the decomposition theorem:

Theorem 1 (of decomposition [5]). *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and $\Gamma_{\mathcal{F}}$ a generic polar curve of $\mathcal{F}$. Given any C-ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, the curve $\rho^{-1}\Gamma_{\mathcal{F}}$ is a strict adjoint of $\rho^{-1}C$.*

By the results in section 2, we deduce that there is a unique decomposition $\rho^{-1}\Gamma_{\mathcal{F}} = \cup_{\tilde{E}\in B(\tilde{C})}\tilde{\Gamma}^{\tilde{E}}$, with $\tilde{C} = \rho^{-1}C$, satisfying the properties d1-d4, (D-i) and (D-ii). Moreover, the curve $\Gamma_{\mathcal{F}}$ can also be decomposed in unique way as

$$\Gamma_{\mathcal{F}} = \bigcup_{E\in B(C)} \Gamma^E$$

satisfying properties D1-D5, (D-i) and (D-ii) in section 2.

Observe now that the property of being a strict adjoint of a curve C does not determine the equisingularity type of the adjoint curve: for instance, if C is the union of 3 lines, then there are infinite many possible equisingularity types for its strict adjoint curves. However, the number of possible equisingularity types is finite when considering polar curves.

Theorem 2. *There exists a finite number of equisingularity types $\wp(\mathcal{F})$ for a foliation $\mathcal{F} \in \mathbb{G}_{C'}^*$ and any curve C' with $\epsilon(C') = \epsilon(C)$.*

Proof. Let $\mathcal{F}$ be a foliation in $\mathbb{G}_C^*$ and consider a generic polar curve $\Gamma = \Gamma_{\mathcal{F}}$ of $\mathcal{F}$. It is clear that the number of irreducible components of Γ is lower than or equal to the multiplicity $m_0(\Gamma) = m_0(C) - 1$.

Consider a ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ transversal to C and such that $\rho^{-1}C$ and $\rho^{-1}\Gamma$ have non-singular irreducible components. Let us prove that given any two irreducible components σ, σ' of $\rho^{-1}\Gamma$ the coincidence $C(\sigma, \sigma')$ is bounded in terms of the equisingularity data of $\rho^{-1}C$. In particular, this implies that there is only a finite number of possibilities for the characteristic exponents of the branches of Γ and for the coincidence between two branches of Γ once the equisingularity type of C is fixed (see appendix B). Hence, the number of possible equisingularity types for Γ is finite. Moreover, since the coincidences between the irreducible components of Γ and C are determined by $\epsilon(C)$, the result follows straightforward.

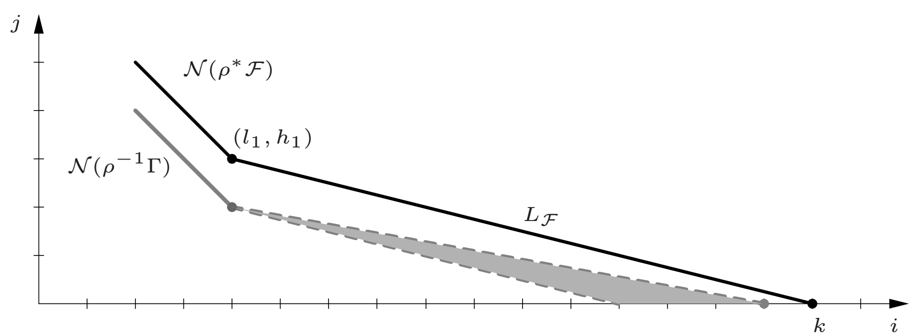
Let $p = \sup_{\sigma,\sigma'} C(\sigma, \sigma')$ where σ, σ' vary within the irreducible components of $\rho^{-1}\Gamma$; observe that $p \in \mathbb{N}$. If $p \leq \sup_{\alpha,\alpha'} C(\alpha, \alpha')$ for α, α' among the irreducible components of $\rho^{-1}C$ we finish. Otherwise let σ_0, σ_0' be two irreducible components of $\rho^{-1}\Gamma$ such that $C(\sigma_0, \sigma_0') = p$. In particular, by property (D-ii) of the decomposition of $\rho^{-1}\Gamma$, we have that $\mu = \sup_\alpha C(\sigma_0, \alpha) = \sup_\alpha C(\sigma_0', \alpha) < p$ where α varies within the irreducible components of $\rho^{-1}C$.

Take (x, y) coordinates in $(\mathbb{C}^2, 0)$ such that the coincidence of the axis $y = 0$ with the curves σ_0 and σ_0' is equal to p. This implies that the last side $L_{\tilde{\Gamma}}$ of the Newton polygon $\mathcal{N}(\rho^{-1}\Gamma; x, y)$ has a slope equal to $-1/p$. Moreover, the last side $L_{\mathcal{F}}$ of $\mathcal{N}(\rho^*\mathcal{F}; x, y)$ has a slope equal to $-1/\mu$.

Let $i + \mu j = k$ be the line which contains $L_{\mathcal{F}}$ and (l_1, h_1) be the highest vertex of $L_{\mathcal{F}}$ (note that $h_1 \geq 3$). The previous results concerning the behaviour of the Newton polygon $\mathcal{N}(\rho^{-1}\Gamma; x, y)$ imply that a point (i, j) on $L_{\tilde{\Gamma}}$ must verify the following conditions

$$\begin{cases} 0 \leq j \leq h_1 - 1 & \text{by prop. 2;} \\ i + \mu j \geq k - \mu & \text{by lemma 1;} \\ i + \frac{k-l_1-1}{h_1-1} j \leq k - 1 & \text{since } (l_1, h_1 - 1), (k-1, 0) \in \Delta(\rho^{-1}\Gamma). \end{cases}$$

Thus there exists only a finite number of possible values for p. Moreover, from the inequalities above we deduce that $\mu \leq p < 2\mu$. The next picture illustrates the situation: the side $L_{\tilde{\Gamma}}$ must be contained in the grey region with slope equal to $-1/p$, $p \in \mathbb{N}$.



□

Among all the possible equisingularity types $\wp(\mathcal{F}) = \epsilon(\Gamma_{\mathcal{F}} \cup C)$ for a fixed equisingularity type $\epsilon(C)$, there is one which can be considered as the "minimal" one satisfying the decomposition theorem. Next sections will be devoted to characterize foliations such that $\wp(\mathcal{F})$ is the minimal one.

4 Non-singular branches

In this section we consider a curve $C = \cup_{i=1}^r C_i$ with only non-singular irreducible components and we study under what conditions a generic polar curve $\Gamma_{\mathcal{F}}$ of a foliation $\mathcal{F} \in \mathbb{G}_C^*$ is a perfect adjoint of C. Denote by $\mathbb{G}_{C,\lambda}^*$ the space of foliations $\mathcal{F} \in \mathbb{G}_C^*$ such that $\lambda(\mathcal{F}) = \lambda$. Let $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ be the set defined by

$$\lambda \in U_C \text{ if there exists } \mathcal{F} \in \mathbb{G}_{C,\lambda}^* \text{ with } \wp(\mathcal{F}) = \chi_C.$$

Then we have

Theorem 3 (of genericity). *The set U_C is a non-empty Zariski open set. Moreover, for each $\mathcal{F} \in \mathbb{G}_{C,\lambda}^*$ with $\lambda \in U_C$ we have that $\wp(\mathcal{F}) = \chi_C$.*

Definition 3. *A foliation $\mathcal{F} \in \mathbb{G}_C^*$ is* Zariski-general *if $\lambda(\mathcal{F}) \in U_C$.*

Denote by $\mathcal{L}_\lambda$ a logarithmic foliation in $\mathbb{G}_C$ with $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{P}_{\mathbb{C}}^{r-1}$. We define the set

$$U_C^{\log} = \left\{\lambda \in \mathbb{P}_{\mathbb{C}}^{r-1} \ : \ \mathcal{L}_\lambda \in \mathbb{G}_C^* \text{ and } \wp(\mathcal{L}_\lambda) = \chi_C\right\}.$$

It is clear that $U_C^{\log} \subset U_C$. Let us prove the following result

Proposition 3. *The set $U_C^{\log}$ is a non-empty Zariski open set of $\mathbb{P}_{\mathbb{C}}^{r-1}$.*

Proof. We note first that the equisingularity type of a generic polar curve of a logarithmic foliation $\mathcal{L}_\lambda \in \mathbb{F}_C$ does not depend on the equations of $C = \cup_{i=1}^r C_i$ chosen to define $\mathcal{L}_\lambda$ (see prop. 3.8 of [5]). So we can assume that $\mathcal{L}_\lambda$ is defined by $\omega_\lambda = 0$ with

$$\omega_\lambda = \prod_{i=1}^r (y - \eta_i(x)) \sum_{i=1}^r \lambda_i \frac{d(y - \eta_i(x))}{(y - \eta_i(x))}, \tag{3}$$

where the curve C_i is defined by $(y - \eta_i(x) = 0)$ and $\eta_i(x) = \sum_{j=1}^\infty a_j^i x^j \in \mathbb{C}\{x\}$. Moreover, for a direction $[a : b] \in \mathbb{P}_{\mathbb{C}}^1$, the polar curve $\Gamma(\mathcal{L}_\lambda; [a : b])$ is given by

$$\sum_{i=1}^r \lambda_i \prod_{j \neq i} (y - \eta_j(x))(-a\eta_i'(x) + b) = 0 \tag{4}$$

and we denote by $\Gamma_{[a:b]}^\lambda$ a generic polar curve of $\mathcal{L}_\lambda$.

The first condition over λ to belong to $U_C^{\log}$ is that $\mathcal{L}_\lambda \in \mathbb{G}_C^*$ but this is equivalent to $\sum_{i=1}^r k_i \lambda_i \neq 0$ where $k \in R_{\epsilon(C)}$ and $R_{\epsilon(C)}$ is a finite set of resonances (see [5] for a detailed description of $R_{\epsilon(C)}$). Now, for each bifurcation divisor E of $G(C)$, we define U_C^E to be the set of $\lambda \in \mathbb{P}_{\mathbb{C}}^{r-1}$ such that

$$\pi_E^* \Gamma_{[a:b]}^\lambda \cap E_{red} \setminus \pi_E^* C \cap E_{red}$$

has exactly $b_E - 1$ different points, and we will prove that U_C^E is a non-empty Zariski open set. Using the criterion given in proposition 1, we obtain that $U_C^{\log}$ is equal to

$$U_C^{\log} = \left\{ \lambda \in \mathbb{P}_{\mathbb{C}}^{r-1} \ : \ \lambda \in \bigcap_{E \in B(C)} U_C^E \ \text{ and } \sum_{i=1}^{r} k_i \lambda_i \neq 0 \text{ for } k \in R_{\epsilon(C)} \right\}$$

which is a non-empty Zariski open set.

Take a bifurcation divisor E of $G(C)$ with $v(E) = p$ and let us prove that each U_C^E is a non-empty Zariski open set. Let $\pi_E : M_E \to (\mathbb{C}^2, 0)$ be the reduction of π_C to E. Since the equisingularity type of a generic polar curve of a foliation does not depend on the coordinates (see [5], §2), we can assume that the coordinates (x, y) are adapted to E. Take (x_p, y_p) coordinates in the first chart of $E_{red} \subset M_E$ such that $\pi_E(x_p, y_p) = (x_p, x_p^p y_p)$ and $E_{red} = (x_p = 0)$. If the strict transform $\pi_E^* \mathcal{L}_\lambda$ of $\mathcal{L}_\lambda$ is defined by $\omega_\lambda^E = 0$ with

$$\omega_\lambda^E = A_\lambda^E\big(x_p, y_p\big) dx_p + x_p B_\lambda^E\big(x_p, y_p\big) dy_p,$$

then the singular points of $\pi_E^* \mathcal{L}_\lambda$ in the first chart of E_{red} are given by $x_p = 0$ and $A_\lambda^E(0, y_p) = 0$. Let us compute the polynomials $A_\lambda^E(0, y)$ and $B_\lambda^E(0, y)$.

We consider two situations: E being the first bifurcation divisor of $G(C)$ or not. If E is the first bifurcation divisor, then E belongs to the geodesic of all the irreducible components of C. Let $\{R_1^E, \dots, R_{b_E}^E\}$ be the singular points of $\pi_E^* \mathcal{L}_\lambda$ in the first chart of E_{red} where $R_i^E = (0, c_i^E)$ in the coordinates (x_p, y_p).

Compute the strict transform of ω_λ by π_E using the expression in (3) and the fact that $\{R_1^E, \dots, R_{b_E}^E\} = \pi_E^* C \cap E_{red}$, thus we get that

$$A_\lambda^E(0, y) = p \cdot \left(\sum_{i=1}^{r} \lambda_i \right) \prod_{i=1}^{r} \big(y - a_p^i\big) = p \left(\sum_{i=1}^{r} \lambda_i \right) \prod_{l=1}^{b_E} \big(y - c_l^E\big)^{r_l}$$

$$B_\lambda^E(0, y) = \sum_{i=1}^{r} \lambda_i \prod_{j \neq i} \big(y - a_p^j\big)$$

where $r_l = m_{R_l^E}(\pi_E^* C)$; note that also $r_l = \sharp\{j \ : \pi_E^* C_j \cap E_{red} = \{R_l^E\}\}$.

Let us now compute the strict transform of $\Gamma_{[a:b]}^\lambda$ by π_E. By the equation of $\Gamma_{[a:b]}^\lambda$ given in (4) and lemma 1, we obtain that the points of the set $\pi_E^* \Gamma_{[a:b]}^\lambda \cap E_{red}$ are given by $x_p = 0$ and

$$\left\{ \begin{array}{ll} B_\lambda^E(0, y_p) = 0, & \text{if } p > 1; \\ a A_\lambda^{r-1}(1, y_p) + b B_\lambda^{r-1}(1, y_p) = 0, & \text{if } p = 1, \end{array} \right. \tag{5}$$

where $A_\lambda^{r-1}(x,y)dx + B_\lambda^{r-1}(x,y)dy$ is the jet of order $\nu_0(\mathcal{L}_\lambda) = r-1$ of ω_λ. Hence we shall consider the two cases: $p > 1$ and $p = 1$ to describe the set $\pi_E^*\Gamma_{[a:b]}^\lambda \cap E_{red} \setminus \pi_E^* C \cap E_{red}$.

By theorem 1, we know that $m_{R_i^E}(\pi_E^*\Gamma_{[a:b]}^\lambda) = r_l - 1$ and consequently, the polynomial $\prod_{l=1}^{b_E}(y - c_l^E)^{r_l - 1}$ divides the polynomials in (5). In particular, the points of $\pi_E^*\Gamma_{[a:b]}^\lambda \cap E_{red} \setminus \pi_E^* C \cap E_{red}$ are given by $x_p = 0$ and $H_\lambda^E(y_p) = 0$ with

$$H_\lambda^E(y) = \begin{cases} B_\lambda^E(0,y)/\prod_{l=1}^{b_E}(y-c_l^E)^{r_l-1}, & \text{if } p>1; \\ (aA_\lambda^{r-1}(1,y) + bB_\lambda^{r-1}(1,y))/\prod_{l=1}^{b_E}(y-c_l^E)^{r_l-1}, & \text{if } p=1. \end{cases}$$

The degree of $H_\lambda^E(y)$ as a polynomial in y is equal to $b_E - 1$ and its coefficients depend linearly on the λ_i. Let us study the two cases $p > 1$ and $p = 1$.

Case $p > 1$: Let $D^E(\lambda)$ be the discriminant of $H_\lambda^E(y)$ as a polynomial in y. Thus, the polynomial $H_\lambda^E(y)$ has $b_E - 1$ different roots if and only if $D^E(\lambda) \neq 0$. Note that $D^E(\lambda) \not\equiv 0$ since $D^E(1,0,\ldots,0) \neq 0$. Thus, the set U_C^E is equal to the non-empty Zariski open set $\mathbb{P}_{\mathbb{C}}^{r-1} \setminus \{D^E = 0\}$.

Case $p = 1$: The exceptional divisor E coincides with E_1 and the coordinates (x,y) are adapted to E_1. From (3) we get that

$$A_\lambda^{r-1}(1,y) = -\sum_{i=1}^{r}\lambda_i a_1^i \prod_{j\neq i}(y - a_1^j)$$

$$B_\lambda^{r-1}(1,y) = B_\lambda^{E_1}(0,y) = \sum_{i=1}^{r}\lambda_i \prod_{j\neq i}(y - a_1^j).$$

Thus the polynomial $H_\lambda^{E_1}(y)$ can be written as follows

$$H_\lambda^{E_1}(y) = \frac{aA_\lambda^{r-1}(1,y) + bB_\lambda^{r-1}(1,y)}{\prod_{l=1}^{b}(y - c_l^{E_1})^{r_l-1}} = aA_\lambda^\natural(y) + bB_\lambda^\natural(y).$$

Let us show that $H_\lambda^{E_1}(y)$ has $b_{E_1} - 1$ different roots. It is clear that

$$A_\lambda^{E_1}(0,y) = A_\lambda^{r-1}(1,y) + yB_\lambda^{r-1}(1,y) = \left(\sum_{i=1}^{r}\lambda_i\right)\prod_{l=1}^{b_{E_1}}\left(y - c_l^{E_1}\right)^{r_l}$$

and then

$$A_\lambda^\natural(y) + yB_\lambda^\natural(y) = \left(\sum_{i=1}^{r} \lambda_i\right) \prod_{l=1}^{b_{E_1}} \left(y - c_l^{E_1}\right).$$

In particular, we deduce that $A_\lambda^\natural(y)$ and $B_\lambda^\natural(y)$ do not have common roots. In fact, the only possible common roots are the elements of the set $\{c_l^{E_1}\}_{l=1}^{b_{E_1}}$, but if $c_l^{E_1}$ is a common root of both polynomials then it is also a root of $H_\lambda^{E_1}(y)$ in contradiction with theorem 1. Thus for a, b generic, the polynomial $H_\lambda^{E_1}(y)$ has $b_{E_1} - 1$ different roots and hence $U_C^{E_1} = \mathbb{P}_{\mathbb{C}}^{r-1}$.

We consider now the case of E being any bifurcation divisor. Put $I = \{1, 2, \dots, r\}$ and $I^E = \{i \in I : E$ belongs to the geodesic of $C_i\}$. We can write $\omega_\lambda = \omega_\lambda^* + \omega_\lambda^{**}$ where

$$\omega_\lambda^* = \prod_{i \in I^E} (y - \eta_i(x)) \sum_{j \in I \setminus I^E} \lambda_j \prod_{\substack{l \in I \setminus I^E \\ l \neq j}} (y - \eta_l(x))(-\eta_j'(x)dx + dy),$$

$$\omega_\lambda^{**} = \prod_{i \in I \setminus I^E} (y - \eta_i(x)) \sum_{j \in I^E} \lambda_j \prod_{\substack{l \in I^E \\ l \neq j}} (y - \eta_l(x))(-\eta_j'(x)dx + dy).$$

If we compute the strict transform ω_λ^E of ω_λ by π_E, we get that the polynomials $A_\lambda^E(0, y)$ and $B_\lambda^E(0, y)$ are given by

$$A_\lambda^E(0, y) = C \cdot \prod_{i \in I^E} \left(y - a_p^i\right); \quad B_\lambda^E(0, y) = C' \cdot \sum_{i \in I^E} \lambda_i \prod_{\substack{j \in I^E \\ j \neq i}} \left(y - a_p^j\right)$$

where C, C' are non-zero constants. Thus the set U_C^E is defined in a similar way to the case of E being the first bifurcation divisor with $p > 1$.

We conclude that $U_C^{\log}$ is a non-empty Zariski open set because it is a finite intersection of non-empty Zariski open sets. $\square$

The next lemma concerns the infinitely near points of generic polar curves and, in particular, it allows to show the equality of the sets U_C and $U_C^{\log}$.

Lemma 2. *Consider two foliations $\mathcal{F}, \mathcal{L}_\lambda \in \mathbb{G}_{C,\lambda}^*$. Let $\Gamma_{[a:b]}^{\mathcal{F}}$ and $\Gamma_{[a:b]}^{\mathcal{L}_\lambda}$ be generic polar curves of $\mathcal{F}$ and $\mathcal{L}_\lambda$ respectively. Then, for each irreducible component E of $\pi_C^{-1}(0)$, we have that*

$$\pi_E^* \Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red} = \pi_E^* \Gamma_{[a:b]}^{\mathcal{L}_\lambda} \cap E_{red}$$

and the multiplicities satisfy that $m_P(\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}}) = m_P(\pi_E^*\Gamma_{[a:b]}^{\mathcal{L}_\lambda})$ *at each point* $P \in \pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red}$*. Moreover, if* $E \neq E_1$*, the sets above does not depend on* $[a : b]$*, that is,*

$$\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red} = \pi_E^*\Gamma_{[a':b']}^{\mathcal{F}} \cap E_{red} = \pi_E^*\Gamma_{[a:b]}^{\mathcal{L}_\lambda} \cap E_{red} = \pi_E^*\Gamma_{[a':b']}^{\mathcal{L}_\lambda} \cap E_{red}$$

for all $[a : b]$, $[a' : b']$ *generic.*

Proof. Take an irreducible component E of $\pi_C^{-1}(0)$ and let $\pi_E : M_E \to (\mathbb{C}^2, 0)$ be the reduction of π_C to E. If E is not a bifurcation divisor, then $\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red}$ and $\pi_E^*\Gamma_{[a:b]}^{\mathcal{L}_\lambda} \cap E_{red}$ coincide with $\pi_E^* C \cap E_{red}$ because $\Gamma_{[a:b]}^{\mathcal{F}}$ and $\Gamma_{[a:b]}^{\mathcal{L}_\lambda}$ are strict adjoint curves of C; in particular, the points of the set $\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red}$ does not depend on $[a : b]$. Moreover, $m_P(\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}}) = m_P(\pi_E^*\Gamma_{[a:b]}^{\mathcal{L}_\lambda}) = m_P(\pi_E^* C) - 1$ at each point $P \in \pi_E^* C \cap E_{red}$ by theorem 1.

Assume now that E is a bifurcation divisor with $v(E) = p$. In order to simplify notations, we suppose that E is the first bifurcation divisor and that the coordinates (x, y) are adapted to E; otherwise we work in a similar way as in the proof of proposition 3. Consider two 1-forms $\omega_{\mathcal{F}} = A_{\mathcal{F}}(x, y)dx + B_{\mathcal{F}}(x, y)dy$ and $\omega_{\mathcal{L}} = A_{\mathcal{L}}(x, y)dx + B_{\mathcal{L}}(x, y)dy$ such that $\mathcal{F}$ and $\mathcal{L} = \mathcal{L}_\lambda$ are defined by $\omega_{\mathcal{F}} = 0$ and $\omega_{\mathcal{L}} = 0$ respectively.

Take (x_p, y_p) coordinates in the first chart of E_{red} such that $\pi_E(x_p, y_p) = (x_p, x_p^p y_p)$ and $E_{red} = (x_p = 0)$. Let $\omega_{\mathcal{F}}^E$ and $\omega_{\mathcal{L}}^E$ be the strict transforms of $\omega_{\mathcal{F}}$ and $\omega_{\mathcal{L}}$ by π_E with

$$\omega_{\mathcal{F}}^E = A_{\mathcal{F}}^E\big(x_p, y_p\big)dx_p + x_p B_{\mathcal{F}}^E\big(x_p, y_p\big)dy_p, \tag{6}$$

$$\omega_{\mathcal{L}}^E = A_{\mathcal{L}}^E\big(x_p, y_p\big)dx_p + x_p B_{\mathcal{L}}^E\big(x_p, y_p\big)dy_p. \tag{7}$$

Denote by $\{R_1^E, \ldots, R_{b_E}^E\}$ the points of the set $\pi_E^* C \cap E_{red}$ and assume that each point $R_l^E = (0, c_l^E)$ in the coordinates (x_p, y_p). The singular points of $\pi_E^*\mathcal{F}$ and $\pi_E^*\mathcal{L}$ in the first chart of E_{red} coincide with the points of $\pi_E^* C \cap E_{red}$ since $\mathcal{F}$ and $\mathcal{L}$ belong to $\mathbb{G}_C$. Moreover, $m_{R_i^E}(\pi_E^*\mathcal{F}) = m_{R_i^E}(\pi_E^*\mathcal{L}) = m_{R_i^E}(\pi_E^* C)$. Thus, up to divide $\omega_{\mathcal{F}}^E$ and $\omega_{\mathcal{L}}^E$ by a constant, we have that

$$A_{\mathcal{F}}^E(0, y) = A_{\mathcal{L}}^E(0, y) = \prod_{l=1}^{b_E}\big(y - c_l^E\big)^{r_l} \tag{8}$$

with $r_l = m_{R_l^E}(\pi_E^* C)$. By theorem 1, we also have that $m_{R_l^E}(\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}}) = m_{R_l^E}(\pi_E^*\Gamma_{[a:b]}^{\mathcal{L}}) = m_{R_l^E}(\pi_E^* C) - 1$. Thus we only need to show that the sets

$\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red} \setminus \pi_E^* C \cap E_{red}$ and $\pi_E^*\Gamma_{[a:b]}^{\mathcal{L}} \cap E_{red} \setminus \pi_E^* C \cap E_{red}$ coincide. Using similar arguments as in the proof of proposition 3, we obtain that the points of $\pi_E^*\Gamma_{[a:b]}^{\mathcal{F}} \cap E_{red} \setminus \pi_E^* C \cap E_{red}$ are given by $x_p = 0$ and $H_{\mathcal{F}}^E(y_p) = 0$ where

$$H_{\mathcal{F}}^E(y) = \begin{cases} B_{\mathcal{F}}^E(0, y) \big/ \prod_{l=1}^{b_E} \left(y - c_l^E\right)^{r_l - 1}, & \text{if } p > 1; \\ (a A_{\mathcal{F}}^{r-1}(1, y) + b B_{\mathcal{F}}^{r-1}(1, y)) \big/ \prod_{l=1}^{b_{E_1}} \left(y - c_l^{E_1}\right)^{r_l - 1}, & \text{if } p = 1, \end{cases}$$

and $A_{\mathcal{F}}^{r-1}(x, y)dx + B_{\mathcal{F}}^{r-1}(x, y)dy$ is the jet of order $\nu_0(\mathcal{F}) = r - 1$ of $\omega_{\mathcal{F}}$. We obtain in a similar way a polynomial $H_{\mathcal{L}}^E(y)$ for the foliation $\mathcal{L}$. In order to prove the lemma we only need to show that the polynomials $H_{\mathcal{F}}^E(y)$ and $H_{\mathcal{L}}^E(y)$ coincide.

Taking into account that $\mathcal{L}$ is a logarithmic model of $\mathcal{F}$, we get that the Camacho-Sad indices $I_{R_l^E}(\pi_E^*\mathcal{F}, E_{red})$ and $I_{R_l^E}(\pi_E^*\mathcal{L}, E_{red})$ are equal for $l = 1, \ldots, b_E$. From the definition of the Camacho-Sad index given in (2) and equations (6), (7) we obtain that

$$I_{R_l^E}\left(\pi_E^*\mathcal{F}, E_{red}\right) = \mathrm{Res}_{y=c_l^E} \frac{-B_{\mathcal{F}}^E(0, y)}{A_{\mathcal{F}}^E(0, y)};$$

$$I_{R_l^E}\left(\pi_E^*\mathcal{L}, E_{red}\right) = \mathrm{Res}_{y=c_l^E} \frac{-B_{\mathcal{L}}^E(0, y)}{A_{\mathcal{L}}^E(0, y)}.$$

If $p > 1$, the computation of the indices gives that

$$I_{R_l^E}\left(\pi_E^*\mathcal{F}, E_{red}\right) = \frac{-H_{\mathcal{F}}^E\left(c_l^E\right)}{\prod_{\substack{j=1 \\ j \neq l}}^{b_E} \left(c_l^E - c_j^E\right)};$$

$$I_{R_l^E}\left(\pi_E^*\mathcal{L}, E_{red}\right) = \frac{-H_{\mathcal{L}}^E\left(c_l^E\right)}{\prod_{\substack{j=1 \\ j \neq l}}^{b_E} \left(c_l^E - c_j^E\right)}$$

and hence $H_{\mathcal{F}}^E(c_l^E) = H_{\mathcal{L}}^E(c_l^E)$ for $l = 1, 2, \ldots, b_E$. Consequently, we deduce that the polynomials $H_{\mathcal{F}}^E(y)$ and $H_{\mathcal{L}}^E(y)$ are equal.

Consider now the case $p = 1$ which corresponds to $E = E_1$. We can write

$$H_{\mathcal{F}}^{E_1}(y) = a A_{\mathcal{F}}^{\natural}(y) + b B_{\mathcal{F}}^{\natural}(y); \quad H_{\mathcal{L}}^{E_1}(y) = a A_{\mathcal{L}}^{\natural}(y) + b B_{\mathcal{L}}^{\natural}(y)$$

with $A_{-}^{\natural}(y), B_{-}^{\natural}(y) \in \mathbb{C}[y]$. Since π_{E_1} is the blowing-up of the origin, it is easy to see that

$$A_{\mathcal{F}}^{E_1}(0, y) = A_{\mathcal{F}}^{r-1}(1, y) + y B_{\mathcal{F}}^{r-1}(1, y); \quad B_{\mathcal{F}}^{E_1}(0, y) = B_{\mathcal{F}}^{r-1}(1, y)$$

and similar equalities hold for the foliation $\mathcal{L}$. Thus, from equation (8), we deduce that

$$A^{\natural}_{\mathcal{F}}(y) + yB^{\natural}_{\mathcal{F}}(y) = A^{\natural}_{\mathcal{L}}(y) + yB^{\natural}_{\mathcal{L}}(y) = \prod_{l=1}^{b_{E_1}}(y - c_l^{E_1}).$$

Furthermore, the equality of the Camacho-Sad indices implies that $B^{\natural}_{\mathcal{F}}(y) = B^{\natural}_{\mathcal{L}}(y)$ and consequently $A^{\natural}_{\mathcal{F}}(y) = A^{\natural}_{\mathcal{L}}(y)$. We conclude that $H^{E_1}_{\mathcal{F}}(y) = H^{E_1}_{\mathcal{L}}(y)$ and this finish the proof of the lemma. □

Proof of theorem 3. From the previous lemma we deduce that $\lambda \in U_C^{\log}$ if and only if, each foliation $\mathcal{F} \in \mathbb{G}^*_{C,\lambda}$ is Zariski-general. This implies that $U_C = U_C^{\log}$ and the theorem follows straightforward. □

Remark 1. Note that there are non Zariski-general foliations, even hamiltonian ones. For instance, take $f = y(y - x^2)(2y - (1 + \sqrt{-3})x^2)$ and $\omega = df$; a generic polar curve of $\omega = 0$ is irreducible with one Puiseux pair equal to $(5, 2)$ and hence the reduction of singularities of $f = 0$ is not a reduction of singularities of a generic polar curve. Moreover, in this example $(1, 1, 1) \notin U_C$ whereas for $g = y(y - x^2)(y + x^2)$ a generic polar curve of $dg = 0$ has two branches with coincidence equal to two and hence $(1, 1, 1) \in U_C$. This shows that the set U_C depends on the analytic type of the curve C.

Corollary 2. *If $\mathcal{F} \in \mathbb{G}^*_{C,\lambda}$ is a Zariski-general foliation, then the curves $C \cup \Gamma_{\mathcal{F}}$ and $C \cup \Gamma_{\mathcal{L}_\lambda}$ are equisingular.*

Observe that the reciprocal of the corollary above is not true. Consider $\mathcal{F}$ defined by $\omega = 0$ with $\omega = (4ixy^2 + 2x^6y)dx + (y^2 - 2ix^2y - x^4 - x^7)dy$. The foliation $\mathcal{F}$ belongs to $\mathbb{G}^*_{C,\lambda}$ with $C = (y(y-x^2)(y+x^2) = 0)$ and $\lambda = (1, -i, i)$. The curves $\Gamma_{\mathcal{F}}$ and $\Gamma_{\mathcal{L}_\lambda}$ are both irreducible with one Puiseux pair equal to $(5, 2)$. Hence $C \cup \Gamma_{\mathcal{F}}$ and $C \cup \Gamma_{\mathcal{L}_\lambda}$ are equisingular. However, π_C is not a reduction of singularities of any of the generic polar curves and then $\lambda \notin U_C$. We also remark that $\mathcal{F}$ belongs to $\mathbb{G}^*_C$ although $(1, -i, i)$ is resonant.

5 Kind equisingularity type

Let us consider a curve $C \subset (\mathbb{C}^2, 0)$ which can have singular branches and take $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ any C-ramification. The existence of a curve Z such that $\rho^{-1}Z$ is a perfect adjoint curve of $\rho^{-1}C$ can not be assured in general. We look for conditions over C that guarantee the existence of perfect adjoint curves of $\rho^{-1}C$ and, in this case, we also define the equisingularity type χ_C.

Definition 4. *We say that a curve C has a* kind equisingularity type *if for each dead arc of $G(C)$ with bifurcation divisor E_b and terminal divisor E_t we have that $m(E_b) = 2m(E_t)$.*

Let us explain what having a kind equisingularity type means in terms of the equisingularity type of C. If E_b is a bifurcation divisor of $G(C)$ belonging to a dead arc with terminal divisor E_t, then $m(E_b) = n_{E_b} m(E_t)$ by appendix A. Hence, the curve C has a kind equisingularity type if, and only if, $n_{E_b} = 2$ for each bifurcation divisor E_b of $G(C)$ which belongs to a dead arc. In particular, this implies that each dead arc in $G(C)$ has only two vertices: the bifurcation divisor and the terminal divisor. Observe that this property does not characterize the fact of having a kind equisingularity type; it is enough to consider the curve $y^3 - x^5 = 0$ which does not have kind equisingularity type. We have the following result of characterization for kind equisingularity types:

Proposition 4. *The following statements are equivalent:*

- *The equisingularity type $\epsilon(C)$ is kind.*
- *There is a germ of curve $Z \subset (\mathbb{C}^2, 0)$ such that $\rho^{-1}Z$ is a perfect adjoint of $\rho^{-1}C$ for any C-ramification ρ.*

Moreover $\epsilon(C \cup Z)$ does not depend on the choice of Z.

Proof. Let $C \subset (\mathbb{C}^2, 0)$ be a plane curve and consider $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ any C-ramification.

Assume first that there is a curve Z such that $\rho^{-1}Z$ is a perfect adjoint curve of $\rho^{-1}C$. Take any bifurcation divisor E of $G(C)$ which belongs to a dead arc with terminal divisor E_t. Then E is a Puiseux divisor and $m(E) = \underline{n}_E n_E$ with $n_E \geq 2$ and $m(E_t) = \underline{n}_E$. Let us prove that $n_E = 2$.

Let $\{\tilde{E}^j\}_{j=1}^{\underline{n}_E}$ be the divisors associated to E in $G(\rho^{-1}C)$. We have that

$$b_{\tilde{E}^j} = (b_E - 1)n_E \quad \text{for all} \quad j = 1, \ldots, \underline{n}_E. \tag{9}$$

Let us denote by $b^*_{\tilde{E}^j}$ the number of edges and arrows which leave from $\tilde{E}^j$ in $G(\rho^{-1}C \cup \rho^{-1}Z)$. Taking into account that $\rho^{-1}Z$ is a perfect adjoint of $\rho^{-1}C$, from corollary 1 we have that

$$b^*_{\tilde{E}^j} = 2b_{\tilde{E}^j} - 1 \quad \text{for all} \quad j = 1, \ldots, \underline{n}_E. \tag{10}$$

Moreover, using the relationship between $G(C \cup Z)$ and $G(\rho^{-1}C \cup \rho^{-1}Z)$, we can compute $b^*_{\tilde{E}^j}$ in terms of b^*_E, where b^*_E is the number of edges and arrows which leave from E in $G(C \cup Z)$. In fact, note that E is also a Puiseux divisor in $G(C \cup Z)$ and then there are two possibilities:

$$b^*_{\tilde{E}^j} = \begin{cases} \big(b^*_E - 1\big) n_E, & \text{if } E \text{ belong to a dead arc in } G(C \cup Z); \\ \big(b^*_E - 1\big) n_E + 1, & \text{otherwise.} \end{cases}$$

The first situation is not possible, because the equality $b^*_{\tilde{E}^j} = (b^*_E - 1)n_E$ and equations (9), (10) would imply that $2n_E(b_E - 1) - 1 = (b^*_E - 1)n_E$ and hence $n_E = 1$ against the hypothesis. Then the second situation holds so $b^*_{\tilde{E}^j} = (b^*_E - 1)n_E + 1$. Using again equations (9) and (10), we get that $(2b_E - b^*_E - 1)n_E = 2$. Thus the only possible values are $n_E = 2$ and $b^*_E = 2b_E - 2$.

Assume now that C has a kind equisingularity type. Let Z be a plane curve such that π_C gives a reduction of singularities of $Z \cup C$ and that $G(C \cup Z)$ is obtained by adding to each divisor E of $G(C)$ the following number of arrows:

$$\begin{cases} b_E - 1, & \text{if } E \text{ is a bifurcation divisor which does not belong to a dead arc in } G(C); \\ b_E - 2, & \text{if } E \text{ is a bifurcation divisor which belongs to a dead arc in } G(C); \\ 1, & \text{if } E \text{ is the terminal divisor of a dead arc in } G(C); \\ 0, & \text{in any other case.} \end{cases}$$

Let us show that $\rho^{-1}Z$ is a perfect adjoint curve of $\rho^{-1}C$. By the description of the reduction of singularities of Z given above, it is clear that $\rho^{-1}Z$ is composed only by non-singular branches. We first prove that $\pi_{\rho^{-1}C}$ gives a reduction of singularities of $\rho^{-1}C \cup \rho^{-1}Z$. Take any branch γ of Z and consider the divisor E of $G(C)$ such that $\pi^*_C\gamma \cap E \neq \emptyset$. Let us see that $\pi_{\rho^{-1}C}$ desingularizes $\rho^{-1}\gamma$. There are three possible situations:

- E is a contact divisor with associated divisors $\{\tilde{E}^j\}_{j=1}^{n_E}$. Then $\rho^{-1}\gamma$ is composed by $\underline{n}_E$ non-singular branches and each of them cuts one and only one divisor $\tilde{E}^j$.
- E is a Puiseux divisor with associated divisors $\{\tilde{E}^j\}_{j=1}^{n_E}$. Then $\rho^{-1}\gamma$ is composed by $\underline{n}_E n_E$ non-singular branches and there are exactly n_E branches of $\rho^{-1}\gamma$ which cut each $\tilde{E}^j$ in n_E different points (see appendix B).
- E is the extremity of a dead arc with bifurcation divisor E_b. Let $\{\tilde{E}^j_b\}_{j=1}^{n_{E_b}}$ be the divisors associated to E_b. Then $\rho^{-1}\gamma$ is composed by $\underline{n}_{E_b} = m(E)$ branches and each of them cuts one and only one of the divisors $\tilde{E}^j_b$.

Moreover, $\pi_{\rho^{-1}C}$ is a reduction of singularities of $\rho^{-1}Z$. In fact, consider two branches γ and γ' of Z which cut the same divisor E and let σ and σ' be two branches of $\rho^{-1}\gamma$ and $\rho^{-1}\gamma'$ respectively, such that they cut the same divisor $\tilde{E}^j$. Then σ and σ' cut $\tilde{E}^j$ in different points since otherwise the coincidence between γ and γ' would be greater than $v(E)$. A similar argument proves that $\pi_{\rho^{-1}C}$ is the minimal reduction of singularities of $\rho^{-1}C \cup \rho^{-1}Z$.

In order to assure that $\rho^{-1}Z$ is a perfect adjoint of $\rho^{-1}C$ we also need to check if $b^*_{\tilde{E}} = 2b_{\tilde{E}} - 1$ for each bifurcation divisor $\tilde{E}$ of $G(\rho^{-1}C)$. Let E be the bifurcation divisor of $G(C)$ which $\tilde{E}$ is associated to. Let us consider the three possible cases for E:

- E is a contact divisor in $G(C)$ and we have that $b_{\tilde{E}} = b_E$ and $b^*_E = 2b_E - 1$. But E is also a contact divisor in $G(C \cup Z)$ and hence $b^*_{\tilde{E}} = b^*_E$. We deduce that $b^*_{\tilde{E}} = 2b_{\tilde{E}} - 1$.
- E is a Puiseux divisor belonging to a dead arc in $G(C)$ and hence $b_{\tilde{E}} = (b_E - 1)n_E$ and $b^*_E = 2b_E - 2$. In this case, E is a Puiseux divisor without dead arc in $G(C \cup Z)$ and we have that $b^*_{\tilde{E}} = (b^*_E - 1)n_E + 1$. We deduce that $b^*_{\tilde{E}} = 2b_{\tilde{E}} - n_E + 1$ and the result follows since by hypothesis $n_E = 2$.
- E is a Puiseux divisor without a dead arc in $G(C)$, thus $b_{\tilde{E}} = (b_E - 1)n_E + 1$ and $b^*_E = 2b_E - 1$. The divisor E is also a Puiseux divisor without a dead arc in $G(C \cup Z)$, so $b^*_{\tilde{E}} = (b^*_E - 1)n_E + 1$. Hence we conclude that $b^*_{\tilde{E}} = 2b_{\tilde{E}} - 1$.

It is clear that the equisingularity type $\epsilon(C \cup Z)$ does not depend on the choice of the curve Z. □

If C is a curve with kind equisingularity type, we say that Z is a *perfect adjoint curve* of C if $\rho^{-1}Z$ is a perfect adjoint curve of $\rho^{-1}C$, for any C-ramification ρ. We are interested in the description of the equisingularity type $\chi_C = \epsilon(C \cup Z)$. A first result in this direction is the following lemma:

Lemma 3. *Consider a curve C with kind equisingularity type and let Z be a perfect adjoint curve of C with $Z = \cup_{E \in B(C)} Z^E$. Then $C(\zeta^E, \xi^E) = v(E)$ for any two branches ζ^E, ξ^E of Z^E.*

Proof. The result follows from corollary 1 and equation (12). □

The next proposition gives a completely description of $\chi_C = \epsilon(C \cup Z)$ in terms of $\epsilon(C)$:

Proposition 5. *Let C be a curve with kind equisingularity type and Z a perfect adjoint curve of C. Then π_C gives a reduction of singularities of $Z \cup C$. Moreover, the branches of Z intersect an irreducible component E of the exceptional divisor of π_C as follows:*

- *If E is a bifurcation divisor of $G(C)$, the number of branches of Z cutting E equals to $b_E - 2$ if E is in a dead arc and to $b_E - 1$ otherwise.*
- *If E is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of Z through E.*
- *Otherwise, no branches of Z intersect E.*

Remark that the fact that "π_C gives a reduction of singularities of $C \cup Z$" does not imply that $\pi_{\rho^{-1}C}$ desingularizes $\rho^{-1}C \cup \rho^{-1}Z$. However, the description of the dual graph $G(C \cup Z)$ given in proposition 5 characterizes the fact of Z being a perfect adjoint curve of C whenever C has a kind equisingularity type. In fact, in proposition 4 we have already proved that, if C has a kind equisingularity type, a curve Z such that $G(C \cup Z)$ is as described in proposition 5 is a perfect adjoint curve of C and the proof of proposition 5 will show the reciprocal.

In order to prove proposition 5 we first describe the equisingularity type of the irreducible components of Z in terms of the equisingularity data of $C = \cup_{i=1}^{r} C_i$. Given an irreducible component C_i of C we denote by $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ its characteristic exponents, $\{(m_j^i, n_j^i)\}_{j=1}^{g_i}$ the Puiseux pairs of C_i and n^i is the multiplicity $m_0(C_i)$ at the origin. We use the notations introduced in appendix A for the dual graph $G(C)$.

Lemma 4. *Consider a curve C with kind equisingularity type and let Z be perfect adjoint curve of C with decomposition $Z = \cup_{E \in B(C)} Z^E$. Then, for each $E \in B(C)$, we have that*

(i) *If E is a contact divisor, then the curve Z^E has $b_E - 1$ irreducible components. Each irreducible component ζ of Z with characteristic exponents $\{\nu_0^\zeta, \nu_1^\zeta, \ldots, \nu_{k_E}^\zeta\}$ given by*

$$\nu_0^\zeta = m_0(\zeta) = \underline{n}_E, \quad \nu_l^\zeta = \underline{n}_E \beta_l^i / n^i \ \text{ for } \ l = 1, 2, \ldots, k_E,$$

for any $i \in I_E$.

(ii) *If E is a Puiseux divisor which belongs to a dead arc, the curve Z^E has one irreducible component ζ_0 with characteristic exponents $\{\nu_0^{\zeta_0}, \nu_1^{\zeta_0}, \ldots, \nu_{k_E}^{\zeta_0}\}$ given by*

$$\nu_0^{\zeta_0} = m_0(\zeta_0) = \underline{n}_E, \quad \nu_l^{\zeta_0} = \underline{n}_E \beta_l^i / n^i \ \text{ for } \ l = 1, 2, \ldots, k_E,$$

and $b_E - 2$ irreducible components such that each branch $\zeta \subset Z^E \setminus \zeta_0$ has characteristic exponents $\{\nu_0^\zeta, \nu_1^\zeta, \ldots, \nu_{k_E}^\zeta, \nu_{k_E+1}^\zeta\}$ given by

$$\nu_0^\zeta = m_0(\zeta) = \underline{n}_E n_E, \quad \nu_l^\zeta = \underline{n}_E n_E \beta_l^i / n^i \ \ \textit{for} \ \ l = 1, 2, \ldots, k_E + 1,$$

for any $i \in I_E^*$.

(iii) *If E is a bifurcation divisor which does not belong to a dead arc, then Z^E has $b_E - 1$ irreducible components. Each irreducible component ζ of Z with characteristic exponents $\{\nu_0^\zeta, \nu_1^\zeta, \ldots, \nu_{k_E}^\zeta, \nu_{k_E+1}^\zeta\}$ given by*

$$\nu_0^\zeta = m_0(\zeta) = \underline{n}_E n_E, \quad \nu_l^\zeta = \underline{n}_E n_E \beta_l^i / n^i \ \ \textit{for} \ \ l = 1, 2, \ldots, k_E + 1,$$

for any $i \in I_E^*$.

Proof. Consider any C-ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and denote $\tilde{C} = \rho^{-1}C$. Let $\{\tilde{E}^l\}_{l=1}^{n_E}$ be the divisors of $G(\tilde{C})$ associated to a divisor E of $G(C)$. By the results in section 2, we have that $\rho^{-1}Z^E = \cup_{j=1}^{n_E} \tilde{Z}^{\tilde{E}^j}$ where $\tilde{Z} = \cup_{\tilde{E} \in B(\tilde{C})} \tilde{Z}^{\tilde{E}}$ is the decomposition of $\tilde{Z} = \rho^{-1}Z$. Let us study the different possibilities for E:

(i) *E is a contact divisor:* then $v(E) = m_E/\underline{n}_E$ with $m_E > m_{k_E}^i$ and $\underline{n}_E = n_1^i \cdots n_{k_E}^i$ for any $i \in I_E$. Consequently, the k_E first Puiseux pairs of an irreducible component ζ^E of Z^E coincide with the ones of C_i, for any $i \in I_E$, since $C(\zeta^E, C_i) = v(E)$. Thus, a Puiseux series of ζ^E is given by

$$\varphi_\zeta(x) = \sum_{i<\tau} a_i x^i + a_\tau x^\tau + \cdots + a^\zeta x^{v(E)} + \cdots,$$

where $\tau = m_{k_E}^i / \underline{n}_E$ and $a_\tau \neq 0$. This implies that $m_0(\zeta^E) = d \cdot \underline{n}_E$. Let us show that $m_0(\zeta^E) = \underline{n}_E$.

We have that $\tilde{\zeta}^E = \rho^{-1}\zeta^E \subset \cup_{l=1}^{n_E} \tilde{Z}^{\tilde{E}_l}$, and if we write $\tilde{\zeta}^E = \cup_{l=1}^{n_E} \tilde{\zeta}^{\tilde{E}_l}$ with $\tilde{\zeta}^{\tilde{E}_l} \subset \tilde{Z}^{\tilde{E}_l}$, then $m_0(\tilde{\zeta}^{\tilde{E}_l}) \geq 1$. By corollary 1, each curve $\tilde{\zeta}^{\tilde{E}_l}$ has $m_0(\tilde{\zeta}^{\tilde{E}_l})$ non-singular irreducible components and the coincidence between two of them is equal to $v(\tilde{E}_l)$. Moreover, the irreducible components of $\tilde{\zeta}^E$ are in bijective correspondence with the Puiseux series of ζ^E. Then, if $a^\zeta \neq 0$, the coefficients of $x^{v(E)}$ in the different Puiseux series of ζ^E are given by $a^\zeta \xi^{v(E) m_0(\zeta^E)}$ with $\xi^{m_0(\zeta^E)} = 1$. But since

$$v(E) \cdot m_0\big(\zeta^E\big) = \frac{m_E}{\underline{n}_E} \cdot m_0\big(\zeta^E\big) = m_E \cdot d$$

then $a^{\zeta}\xi^{v(E)m_0(\zeta^E)}$ takes at most $\underline{n}_E$ different values and hence $d = 1$. If $a^{\zeta} = 0$, then $m_0(\zeta^E) = \underline{n}_E$ since otherwise one of the curves $\tilde{\zeta}^{\tilde{E}_l}$ has at least two irreducible components with coincidence greater than $v(\tilde{E}_l)$.

We deduce that each irreducible component ζ^E of Z^E has multiplicity equal to $\underline{n}_E$. Since $m_0(Z^E) = \underline{n}_E(b_E - 1)$, then Z^E has exactly $b_E - 1$ irreducible components with multiplicity $\underline{n}_E$. Moreover, the Puiseux pairs of each irreducible component ζ of Z^E coincide with the k_E first Puiseux pairs of C_i for $i \in I_E$ and the characteristic exponents $\{\nu_0^{\zeta}, \nu_1^{\zeta}, \ldots, \nu_{k_E}^{\zeta}\}$ of ζ are given by $\nu_l^{\zeta} = \underline{n}_E \beta_l^i / n^i$ for $l = 0, 1, \ldots, k_E$.

(ii) *E is a Puiseux divisor which belongs to a dead arc:* we have that $v(E) = m_E/\underline{n}_E n_E$ with $n_E = 2$ because C has a kind equisingularity type and then $m_0(Z^E) = \underline{n}_E(n_E(b_E - 1) - 1) = \underline{n}_E n_E(b_E - 2) + \underline{n}_E$.

An irreducible component ζ^E of Z^E has at least the k_E first Puiseux pairs equal to the ones of C_i with $i \in I_E$. Thus $m_0(\zeta^E) \geq \underline{n}_E$. A Puiseux series $\varphi_{\zeta}(x)$ of ζ^E is given by

$$\varphi_{\zeta}(x) = \sum_{l < v(E)} a_l x^l + a^{\zeta} x^{v(E)} + \ldots,$$

but since $\underline{n}_E n_E$ does not divide $m_0(Z^E)$, then there is at least one irreducible component ζ_0^E of Z^E such that the coefficient a^{ζ_0} of $x^{v(E)}$ is zero. Moreover, ζ_0^E must be unique because the existence of another irreducible component δ_0^E of Z^E with $a^{\delta_0} = 0$ would imply that $C(\zeta_0^E, \delta_0^E) > v(E)$ in contradiction with lemma 3. Let us show that $m_0(\zeta_0^E) = \underline{n}_E$. In fact, $m_0(\zeta_0^E) = d \cdot \underline{n}_E$ with $d \in \mathbb{N}$. Consider the curve $\tilde{\zeta}_0^E = \rho^{-1}\zeta_0^E$ and write

$$\tilde{\zeta}_0^E = \bigcup_{l=1}^{\underline{n}_E} \tilde{\zeta}_0^{\tilde{E}_l} \quad \text{with} \quad \tilde{\zeta}_0^{\tilde{E}_l} \subset \tilde{Z}^{\tilde{E}_l}.$$

By corollary 1, the number of irreducible components of $\tilde{Z}^{\tilde{E}_l}$ is equal to its multiplicity, hence $m_0(\tilde{\zeta}_0^{\tilde{E}_l}) = 1$ since otherwise the coincidence between two branches of $\tilde{\zeta}_0^{\tilde{E}_l}$ will be greater than $v(\tilde{E}_l)$. Hence $m_0(\zeta_0^E) = \underline{n}_E$. Consequently, we have that

$$m_0(Z^E \setminus \zeta_0^E) = \underline{n}_E n_E(b_E - 2).$$

Consider now an irreducible component ζ^E of $Z^E \setminus \zeta_0^E$. The coefficient a^{ζ} in $\varphi_{\zeta}(x)$ must be non-zero and thus $m_0(\zeta^E) \geq \underline{n}_E n_E$. With similar arguments as above, we show that $m_0(\zeta^E) = \underline{n}_E n_E$.

We have proved that Z^E has one irreducible component ζ_0^E with multiplicity $\underline{n}_E$ and $b_E - 2$ irreducible components with multiplicity $\underline{n}_E n_E$. The characteristic exponents $\{\nu_0^{\zeta_0}, \nu_1^{\zeta_0}, \ldots, \nu_{k_E}^{\zeta_0}\}$ of ζ_0^E are given by $\nu_l^{\zeta_0} = \underline{n}_E \beta_l^i / n^i$, for $l = 1, \ldots, k_E$, and the characteristic exponents $\{\nu_0^{\zeta}, \nu_1^{\zeta}, \ldots, \nu_{k_E+1}^{\zeta}\}$ of a branch ζ^E of $Z^E \setminus \zeta_0^E$ are given by $\nu_l^{\zeta} = \underline{n}_E n_E \beta_l^i / n^i$ for $l = 0, 1, \ldots, k_E + 1$ and $i \in I_E$.

(iii) *E is a Puiseux divisor which does not belong to a dead arc:* we have that $v(E) = m_E / \underline{n}_E n_E$ with $n_E > 1$. Take any irreducible component ζ^E of Z^E. Let us see that $m_0(\zeta^E) = \underline{n}_E n_E$. Consider

$$\varphi_\zeta(x) = \sum_{l < v(E)} a_l x^l + a^\zeta x^{v(E)} + \cdots$$

a Puiseux series of ζ^E. The hypothesis over E imply that (m_E, n_E) is not a Puiseux pair of C_j if $j \in I_E \setminus I_E^*$, or equivalently, the coefficient of $x^{v(E)}$ in the Puiseux series of C_j is zero. In particular, we deduce that $a^\zeta \neq 0$ for all irreducible components ζ^E of Z^E since $C(C_j, \zeta^E) = v(E)$. Consequently, (m_E, n_E) is a Puiseux pair of ζ^E and the $k_E + 1$ Puiseux pairs of ζ^E coincide with the ones of C_i with $i \in I_E^*$. With similar arguments as in case (i) we prove that $m_0(\zeta^E) = \underline{n}_E n_E$.

From the fact that $m_0(Z^E) = \underline{n}_E n_E (b_E - 1)$, we deduce that Z^E has exactly $b_E - 1$ irreducible components, each of them with multiplicity $\underline{n}_E n_E$. Hence, the characteristic exponents $\{\nu_0^{\zeta}, \nu_1^{\zeta}, \ldots, \nu_{k_E+1}^{\zeta}\}$ of a branch ζ^E of Z^E are given by $\nu_l^{\zeta} = \underline{n}_E n_E \beta_l^i / n^i$ for $l = 1, \ldots, k_E + 1$ and $i \in I_E^*$. □

The previous description of the equisingularity type of the irreducible components of Z^E will be useful in the proof of proposition 5.

Proof of proposition 5. Let C be a curve with kind equisingularity type and let $\pi_C : M \to (\mathbb{C}^2, 0)$ be its minimal reduction of singularities. Consider Z a perfect adjoint curve of Z with decomposition $Z = \cup_{E \in B(C)} Z^E$ satisfying properties D1.-D5. in section 2. It is clear that the points of $\pi_C^* Z \cap \pi_C^{-1}(0)$ coincide with the union of the sets $\pi_C^* Z^E \cap \pi_C^{-1}(0)$ for $E \in B(C)$. We deduce that if Z cuts a divisor E, then E is either a bifurcation divisor or it belongs to a dead arc, but since each dead arc of $G(C)$ has only to vertices, then E is either a bifurcation or a terminal divisor.

Assume first that E is a bifurcation divisor without a dead arc attached to it. Then properties D3.-D5. of the decomposition of Z imply that each irreducible component ζ^E of Z^E cuts E, i.e., $\pi_E^* \zeta^E \cap E_{red} \neq \emptyset$. Moreover, the number of points of $\pi_E^* Z^E \cap E_{red}$ is equal to the number of irreducible components of Z^E. In fact, if $\pi_E^* \zeta^E \cap E_{red} = \pi_E^* \xi^E \cap E_{red}$ then $C(\zeta^E, \xi^E) > v(E)$ in contradiction with lemma 3. The present hypothesis correspond to the cases (i) and (iii) of lemma 4, hence the number of points of $\pi_E^* Z^E \cap E_{red}$ is equal to $b_E - 1$. It is clear that π_E is a reduction of singularities of each irreducible component ζ^E of Z^E since each curve $\pi_E^* \zeta^E$ is an E_{red}-curvette by lemma 4.

Assume now that E is a bifurcation divisor which belong to a dead with terminal divisor E_t. By properties D3.-D5. of the decomposition of Z, we have that either $\pi_E^* \zeta^E \cap E_{red} \neq \emptyset$ or $\pi_E^* \zeta^E \cap \pi_E'(E_t) \neq \emptyset$ for an irreducible component ζ^E of Z^E. By lemma 4, there is an irreducible component ζ_0^E of Z^E with multiplicity $\underline{n}_E$, thus $\pi_E^* \zeta_0^E \cap \pi_E'(E_t) \neq \emptyset$ since each curve γ with $\pi_E^* \gamma \cap E_{red} \neq \emptyset$ must have multiplicity $\geq m(E) = \underline{n}_E n_E$. Moreover, ζ_0^E is the only irreducible component of Z^E which cuts E_t because the existence of another one ξ_0^E would imply that $C(\zeta_0^E, \xi_0^E) \geq v(E_t) > v(E)$ in contradiction with lemma 3. Finally, the number of points of $\pi_E^* Z^E \cap E_{red}$ coincides with the number of irreducible components of $Z^E \setminus \zeta_0^E$ which is $b_E - 2$. We also have that π_E is a reduction of singularities of Z^E since ζ_0^E is a $\pi_E'(E_t)$-curvette and ζ^E is an E_{red}-curvette for each $\zeta^E \subset Z^E \setminus \zeta_0^E$ by lemma 4.

The fact that π_C gives a reduction of singularities of $C \cup Z$ follows using property D2. and the result is proved. $\square$

6 Proof of the main theorem

Consider a curve $C = \cup_{i=1}^r C_i$ which can have singular branches. Let U_C be the set of $\lambda \in \mathbb{P}_{\mathbb{C}}^{r-1}$ such that there exists $\mathcal{F} \in \mathbb{G}_{C,\lambda}^*$ with $\rho^{-1}\Gamma_{\mathcal{F}}$ a perfect adjoint curve of $\rho^{-1}C$, for any C-ramification ρ. This section is devoted to prove the following result:

Theorem 4. *The set U_C is a non-empty Zariski open set if and only if C has a kind equisingularity type. Moreover, in this case $\wp(\mathcal{F}) = \chi_C$ for any $\mathcal{F} \in \mathbb{G}_C^*$ with $\lambda(\mathcal{F}) \in U_C$.*

Take any C-ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $x = u^n, y = v$. Consider a foliation $\mathcal{F} \in \mathbb{G}_{C,\lambda}^*$, then the transform $\rho^*\mathcal{F}$ belongs to $G_{\rho^{-1}C,\lambda^*}^*$ where $\lambda^* = \lambda(\rho^*\mathcal{F}) \in \mathbb{P}_{\mathbb{C}}^{m-1}$ and $m = m_0(C)$ is the multiplicity of C at the origin. We denote by $\Gamma_{\mathcal{F}}$ and $\Gamma_{\rho^*\mathcal{F}}$ two generic polar curves of $\mathcal{F}$ and $\rho^*\mathcal{F}$ respectively.

It is clear that the foliation $\rho^*\mathcal{F}$ has a curve of separatrices with only non-singular branches. Consequently, by the results of section 4, $\Gamma_{\rho^*\mathcal{F}}$ is a perfect adjoint curve of $\rho^{-1}C$ if and only if $\lambda^* \in U_{\rho^{-1}C}$ and in that case, $\epsilon(\Gamma_{\rho^*\mathcal{F}} \cup \rho^{-1}C) = \chi_{\rho^{-1}C}$. However, in general, $\rho^{-1}\Gamma_{\mathcal{F}}$ and $\Gamma_{\rho^*\mathcal{F}}$ are not equisingular (see [5]). Consider the following properties:

$$(A):\ \epsilon(\Gamma_{\rho^*\mathcal{F}} \cap \rho^{-1}C) = \chi_{\rho^{-1}C}$$
$$(B):\ \epsilon(\rho^{-1}\Gamma_{\mathcal{F}} \cap \rho^{-1}C) = \chi_{\rho^{-1}C}$$

Proposition 6. *Property (A) implies (B). Moreover, both properties are equivalent if the curve C has at most two different tangent lines.*

Observe that properties (A) and (B) above do not depend on the choice of the C-ramification ρ.

Definition 5. *We say that $\mathcal{F}$ is a* Zariski-general foliation *when property (B) holds.*

Notation. In this section, we denote by $\tilde{C}$ and $\tilde{\Gamma}$ the curves $\rho^{-1}C$ and $\rho^{-1}\Gamma_{\mathcal{F}}$ respectively; the transform of the polar $\rho^{-1}\Gamma(\mathcal{F};[a:b])$ will be denoted by $\tilde{\Gamma}_{[a:b]}$ or $\tilde{\Gamma}_{\mathcal{F}}$ when the explicit direction of polarity or the foliation are needed. If $\pi_{\tilde{C}} : \tilde{M} \to (\mathbb{C}^2, 0)$ is the minimal reduction of singularities of $\tilde{C}$, we denote by $\tilde{E}$ an irreducible component of $\pi_{\tilde{C}}^{-1}(0)$ and by $\pi_{\tilde{E}} : \tilde{M}_{\tilde{E}} \to (\mathbb{C}^2, 0)$ the morphism reduction of $\pi_{\tilde{C}}$ to $\tilde{E}$. The reader could refer to appendix B for a detailed description of the ramification tools.

Let us state two lemmas concerning the infinitely near points of $\tilde{\Gamma}$ and $\Gamma_{\rho^*\mathcal{F}}$.

Lemma 5. *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and let $\tilde{E}_1$ be the irreducible component of $\pi_{\tilde{C}}^{-1}(0)$ with $v(\tilde{E}_1) = n$. Then the set*

$$\pi_{\tilde{E}_1}^* \tilde{\Gamma}_{[a:b]} \cap \tilde{E}_{1,red} \setminus \pi_{\tilde{E}_1}^* \tilde{C} \cap \tilde{E}_{1,red}$$

has exactly $b_{\tilde{E}_1} - 1$ points which depend on $[a:b]$.

Proof. Observe that the divisor $\tilde{E}_1$ of $\pi_{\tilde{C}}^{-1}(0)$ is associated to the divisor E_1 of $\pi_C^{-1}(0)$ and hence the coordinates (x, y) and (u, v) are adapted to E_1 and $\tilde{E}_1$, respectively. Let $\omega = A(x, y)dx + B(x, y)dy$ be a 1-form defining $\mathcal{F}$. Then $\Gamma_{[a:b]}$ is defined by $aA(x, y) + bB(x, y) = 0$ and $\tilde{\Gamma}_{[a:b]}$ is given by $aA(u^n, v) + bB(u^n, v) = 0$. Take coordinates $(\tilde{u}, \tilde{v})$ in the first chart of $\tilde{E}_1$ such that $\pi_{\tilde{E}_1}(\tilde{u},\tilde{v}) = (\tilde{u},\tilde{u}^n\tilde{v})$ and $\tilde{E}_1 = (\tilde{u}=0)$. The strict transform $\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{[a:b]}$ is given by

$$\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{[a:b]} = \{aA_\nu(1,\tilde{v}) + bB_\nu(1,\tilde{v}) + \tilde{u}(\cdots) = 0\},$$

where $\nu = \nu_0(\mathcal{F})$ and $A_\nu(x,y)dx + B_\nu(x,y)dy$ is the ν-jet of ω. Then the points of $\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{[a:b]} \cap \tilde{E}_{1,red}$ are defined by $\tilde{u}=0$ and $aA_\nu(1,\tilde{v}) + bB_\nu(1,\tilde{v}) = 0$. Taking into account that $\tilde{\Gamma}_{[a:b]}$ is a strict adjoint of $\tilde{C}$ and using similar arguments as in the proof of proposition 3 case $p=1$, we get that the points of $\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{[a:b]} \cap \tilde{E}_{1,red} \setminus \pi^*_{\tilde{E}_1}\tilde{C} \cap \tilde{E}_{1,red}$ are given by $\tilde{u}=0$ and $H^{\tilde{E}_1}(\tilde{v}) = 0$ with

$$H^{\tilde{E}_1}(v) = aA^\natural_\nu(v) + bB^\natural_\nu(v),$$

where $A^\natural_\nu(v)$ and $B^\natural_\nu(v)$ do not have common roots. Thus the result follows straightforward. □

Corollary 3. *Given a foliation $\mathcal{F} \in \mathbb{G}^*_C$, the set $\pi^*_{E_1}\Gamma^{\mathcal{F}}_{[a:b]} \cap E_{1,red} \setminus \pi^*_{E_1}C \cap E_{1,red}$ has exactly $b_{E_1} - 1$ points which depend on $[a:b]$.*

Proof. The result follows from the fact that there is a bijection between the points in $E_{1,red}$ and the ones in $\tilde{E}_{1,red}$ (see lemma 8). □

Lemma 6. *Consider a foliation $\mathcal{F} \in \mathbb{G}^*_C$. Then we have that*

$$\pi^*_{\tilde{E}}\tilde{\Gamma} \cap \tilde{E}_{red} = \pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{F}} \cap \tilde{E}_{red}$$

for each irreducible component $\tilde{E}$ of $\pi^{-1}_{\tilde{C}}(0)$ with $v(\tilde{E}) > n$. Moreover, $m_P(\pi^*_{\tilde{E}}\tilde{\Gamma}) = m_P(\pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{F}})$ for each $P \in \pi^*_{\tilde{E}}\tilde{\Gamma} \cap \tilde{E}_{red}$.

Proof. Let $\omega = A(x,y)dx + B(x,y)dy$ be a 1-form defining $\mathcal{F}$. Then the curves $\tilde{\Gamma}$ and $\Gamma_{\rho^*\mathcal{F}}$ are given by

$$\tilde{\Gamma} = \{aA(u^n,v) + bB(u^n,v) = 0\};$$
$$\Gamma_{\rho^*\mathcal{F}} = \{aA(u^n,v)nu^{n-1} + bB(u^n,v) = 0\}$$

Take any irreducible component $\tilde{E}$ of $\pi^{-1}_{\tilde{C}}(0)$ with $v(\tilde{E}) = p > n$ and assume that (u,v) are coordinates adapted to $\tilde{E}$. By the results in section 3, it is enough to prove that

$$In_p(aA^* + b\tilde{B}; u, v) = In_p(a\tilde{A} + b\tilde{B}; u, v) = In_p(b\tilde{B}; u, v) \tag{11}$$

where $\tilde{A}(u,v) = nu^{n-1}A(u^n,v)$, $\tilde{B}(u,v) = B(u^n,v)$ and $A^*(u,v) = A(u^n,v)$.

Let $i + pj = k$ be the equation of the line which contains the side of $\mathcal{N}(\rho^*\mathcal{F}; u, v)$ with slope equal to $-1/p$. Then it is clear that $\Delta(\rho^*\omega) \subset \{(i,j) \in \mathbb{R}^2 : i+pj \geq k\}$. Moreover, $\Delta(a\tilde{A}+b\tilde{B}) \subset \{(i,j) : i+pj \geq k-p\}$ by lemma 1. Let us prove that $\Delta(\tilde{A})$ and $\Delta(A^*)$ are contained in $\{(i,j) : i+pj > k-p\}$. Consider two cases:

- If $(i,j) \in \Delta(\tilde{A})$ then $(i+1,j) \in \Delta(\rho^*\omega)$ and hence $i + pj \geq k-1 > k-p$.
- If $(i,j) \in \Delta(A^*)$ then $(i+n,j) \in \Delta(\rho^*\omega)$ and consequently $i + pj \geq k-n > k-p$.

Thus the equalities in (11) hold and the lemma is proved. □

Let us show now that being a Zariski-general foliation only depends on $\lambda(\mathcal{F})$.

Proposition 7. *A foliation $\mathcal{F} \in \mathbb{G}^*_{C,\lambda}$ is Zariski-general if and only if $\mathcal{L}_\lambda$ is a Zariski-general foliation.*

Proof. Let $\Gamma_{\mathcal{F}}$ and $\Gamma_{\mathcal{L}}$ be generic polar curves of $\mathcal{F}$ and $\mathcal{L} = \mathcal{L}_\lambda$, respectively, and put $\tilde{\Gamma}_{\mathcal{F}} = \rho^{-1}\Gamma_{\mathcal{F}}$ and $\tilde{\Gamma}_{\mathcal{L}} = \rho^{-1}\Gamma_{\mathcal{L}}$. Let us prove that the infinitely near points of $\tilde{\Gamma}_{\mathcal{F}}$ and $\tilde{\Gamma}_{\mathcal{L}}$ coincide at each irreducible component $\tilde{E}$ of $\pi_{\tilde{C}}^{-1}(0)$, $\tilde{E} \neq \tilde{E}_1$. In fact, by lemma 2, we have that $\pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{F}} \cap \tilde{E}_{red} = \pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{L}} \cap \tilde{E}_{red}$ for each irreducible component $\tilde{E}$ of $\pi_{\tilde{C}}^{-1}(0)$, and from lemma 6, we deduce that

$$\pi^*_{\tilde{E}}\tilde{\Gamma}_{\mathcal{F}} \cap \tilde{E}_{red} = \pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{F}} \cap \tilde{E}_{red}; \quad \pi^*_{\tilde{E}}\tilde{\Gamma}_{\mathcal{L}} \cap \tilde{E}_{red} = \pi^*_{\tilde{E}}\Gamma_{\rho^*\mathcal{L}} \cap \tilde{E}_{red}$$

if $\tilde{E} \neq \tilde{E}_1$. Consequently, $\pi^*_{\tilde{E}}\tilde{\Gamma}_{\mathcal{F}} \cap \tilde{E}_{red} = \pi^*_{\tilde{E}}\tilde{\Gamma}_{\mathcal{L}} \cap \tilde{E}_{red}$ provided that $\tilde{E} \neq \tilde{E}_1$.

Moreover, the sets $\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{\mathcal{F}} \cap \tilde{E}_{1,red} \setminus \pi^*_{\tilde{E}_1}\tilde{C} \cap \tilde{E}_{1,red}$ and $\pi^*_{\tilde{E}_1}\tilde{\Gamma}_{\mathcal{L}} \cap \tilde{E}_{1,red} \setminus \pi^*_{\tilde{E}_1}\tilde{C} \cap \tilde{E}_{1,red}$ have always $b_{\tilde{E}_1} - 1$ different points by lemma 5. Then the result follows straightforward applying the criterion given in proposition 1. □

Now we are ready to prove proposition 6:

Proof of proposition 6. Let $\mathcal{F}$ be a foliation in $\mathbb{G}^*_C$. By the results of section 4, it is clear that

$$\epsilon(\Gamma_{\rho^*\mathcal{F}} \cap \rho^{-1}C) = \chi_{\rho^{-1}C} \text{ if and only if, } \lambda^* = \lambda(\rho^*\mathcal{F}) \in U_{\rho^{-1}C} = \bigcap_{\tilde{E} \in B(\tilde{C})} U^{\tilde{E}}_{\tilde{C}},$$

where $B(\tilde{C})$ is the set of bifurcation divisors of $\pi_{\tilde{C}}^{-1}(0)$ and $U_{\tilde{C}}^{\tilde{E}} \subset \mathbb{P}_{\mathbb{C}}^{m-1}$ are the Zariski-open sets defined in section 4. From lemmas 5 and 6 we deduce that

$$\epsilon(\rho^{-1}\Gamma_{\mathcal{F}} \cap \rho^{-1}C) = \chi_{\rho^{-1}C} \text{ if and only if } \lambda^* \in \bigcap_{\tilde{E} \in B(\tilde{C}) \setminus \{\tilde{E}_1\}} U_{\tilde{C}}^{\tilde{E}}.$$

Consequently property (A) implies (B).

Assume now that C has at most two different tangent lines, i.e., $b_{E_1} = b_{\tilde{E}_1} \leq 2$. If $b_{\tilde{E}_1} = 1$, then $\tilde{E}_1$ is not a bifurcation divisor. If $b_{\tilde{E}_1} = 2$, we can see that $U_{\tilde{C}}^{\tilde{E}_1} = \mathbb{P}_{\mathbb{C}}^{m_0(C)-1}$ (see its definition in section 4). It follows that (A) and (B) are equivalent when C has at most two different tangent lines. $\square$

The set U_C is equal to the set of λ such that each $\mathcal{F} \in \mathbb{G}_{C,\lambda}^*$ is a Zariski-general foliation. It is an open subset of $\mathbb{P}_{\mathbb{C}}^{r-1}$ but it could be empty. In fact, remark that $\lambda = (\lambda_1, \ldots, \lambda_r) \in U_C$ if and only if,

$$\lambda^* = (\overbrace{\lambda_1, \ldots, \lambda_1}^{n^1}, \ldots, \overbrace{\lambda_r, \ldots, \lambda_r}^{n^r}) \in \bigcap_{\substack{\tilde{E} \in B(\tilde{C}) \\ \tilde{E} \neq \tilde{E}_1}} U_{\tilde{C}}^{\tilde{E}} \subset \mathbb{P}_{\mathbb{C}}^{m_0(C)-1}$$

where $n^i = m_0(C_i)$ for $i = 1, \ldots, r$. The theorem 4 characterizes the equisingularity types $\epsilon(C)$ such that $U_C \neq \emptyset$.

Proof of theorem 4. Let us see that, for each bifurcation divisor E of $G(C)$, we can construct an open set $U_C^E \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ such that

$$\lambda \in U_C \text{ if and only if } \lambda \in \bigcap_{E \in B(C)} U_C^E \text{ and } \sum_{i=1}^{r} k_i \lambda_i \neq 0 \text{ for } k \in R_{\epsilon(C)}.$$

Moreover, we prove that a necessary and sufficient condition to assure that each U_C^E is non-empty is that C has a kind equisingularity type.

Consider a logarithmic foliation $\mathcal{L}_\lambda \in \mathbb{G}_C^*$. Denote by Γ_λ a generic polar curve of $\mathcal{L}_\lambda$ and put $\tilde{\Gamma}_\lambda = \rho^{-1}\Gamma_\lambda$. Take a bifurcation divisor E of $G(C)$ and let $\tilde{E}$ be any bifurcation divisor of $G(\tilde{C})$ associated to E. Let us determine the conditions over λ which are equivalent to the fact that the set $\pi_{\tilde{E}}^* \tilde{\Gamma}_\lambda \cap \tilde{E}_{red} \setminus \pi_{\tilde{E}}^* \tilde{C} \cap \tilde{E}_{red}$ has exactly $b_{\tilde{E}} - 1$ different points. By lemma 5, we only need to check this condition for $\tilde{E} \neq \tilde{E}_1$ and hence, by lemma 6, we have that

$$\pi_{\tilde{E}}^* \tilde{\Gamma}_\lambda \cap \tilde{E}_{red} \setminus \pi_{\tilde{E}}^* \tilde{C} \cap \tilde{E}_{red} = \pi_{\tilde{E}}^* \Gamma_{\lambda^*} \cap \tilde{E}_{red} \setminus \pi_{\tilde{E}}^* \tilde{C} \cap \tilde{E}_{red}$$

where Γ_{λ^*} is a generic polar curve of $\mathcal{L}_{\lambda^*} = \rho^*\mathcal{L}_\lambda$.

Up to a coordinate change, we can assume that (u, v) are coordinates adapted to $\tilde{E}$. Let $\pi_{\tilde{E}} : \tilde{M}_{\tilde{E}} \to (\mathbb{C}^2, 0)$ be the morphism reduction of $\pi_{\tilde{C}}$ to $\tilde{E}$ and take coordinates (u_p, v_p) in the first chart of $\tilde{E}_{red}$ such that $\tilde{E}_{red} = (u_p = 0)$ and $\pi_{\tilde{E}}(u_p, v_p) = (u_p, u_p^p v_p)$. Consider the 1-form

$$\omega_{\lambda^*}^{\tilde{E}} = A_{\lambda^*}^{\tilde{E}}(u_p, v_p)du_p + u_p B_{\lambda^*}^{\tilde{E}}(u_p, v_p)dv_p$$

such that the strict transform $\pi_{\tilde{E}}^*\mathcal{L}_{\lambda^*}$ is defined by $\omega_{\lambda^*}^{\tilde{E}} = 0$. By the results of section 4, we know that the singular points of $\pi_{\tilde{E}}^*\mathcal{L}_{\lambda^*}$ in the first chart of $\tilde{E}_{red}$ are given by $u_p = 0$ and $A_{\lambda^*}^{\tilde{E}}(0, v_p) = 0$ and the points of $\pi_{\tilde{E}}^*\Gamma_{\lambda^*} \cap \tilde{E}_{red}$ are given by $u_p = 0$ and $B_{\lambda^*}^{\tilde{E}}(0, v_p) = 0$. Denote by $\{R_1^{\tilde{E}}, \ldots, R_{b_{\tilde{E}}}^{\tilde{E}}\}$ the points of the set $\pi_{\tilde{E}}^*\tilde{C} \cap \tilde{E}_{red}$ with $R_i^{\tilde{E}} = (0, c_i^{\tilde{E}})$ in the coordinates (u_p, v_p). Note that these points are also the singular points of $\pi_{\tilde{E}}^*\mathcal{L}_{\lambda^*}$ in the first chart of $\tilde{E}_{red}$. We deduce that, up to divide by a constant, we have that

$$A_{\lambda^*}^{\tilde{E}}(0, v) = \prod_{i=1}^{b_{\tilde{E}}}(v - c_i^{\tilde{E}})^{r_i},$$

where $r_i = m_{R_i^{\tilde{E}}}(\pi_{\tilde{E}}^*\tilde{C})$. We put $A^{\tilde{E}}(v) = A_{\lambda^*}^{\tilde{E}}(0, v)$. Moreover, the points of the set $\pi_{\tilde{E}}^*\Gamma_{\lambda^*} \cap \tilde{E}_{red} \setminus \pi_{\tilde{E}}^*\tilde{C} \cap \tilde{E}_{red}$ are given by $u_p = 0$ and $H_{\lambda^*}^{\tilde{E}}(v_p) = 0$ with

$$H_{\lambda^*}^{\tilde{E}}(v) = \frac{B_{\lambda^*}^{\tilde{E}}(0, v)}{\prod_{i=1}^{b_{\tilde{E}}}(v - c_i^{\tilde{E}})^{r_i - 1}}.$$

The polynomial $H_{\lambda^*}^{\tilde{E}}(v)$ has degree $b_{\tilde{E}} - 1$ as a polynomial in v and its coefficients depend linearly on λ; we denote $H_\lambda^{\tilde{E}}(v) = H_{\lambda^*}^{\tilde{E}}(v)$. Let $D^{\tilde{E}}(\lambda)$ be the discriminant of $H_\lambda^{\tilde{E}}(v)$ as a polynomial in v and we define U_C^E to be the set of λ such that $D^{\tilde{E}}(\lambda) \neq 0$ for all divisor $\tilde{E} \in B(\tilde{C})$ associated to E. Let us show that each set U_C^E is a non-empty Zariski open set if and only if C has a kind equisingularity type.

First we compute the polynomials above in terms of the Puiseux series of the branches of C. The expression of the polynomials $A^{\tilde{E}}(v)$ and $B_{\lambda^*}^{\tilde{E}}(0, v)$ for a logarithmic foliation with only non-singular separatrices in terms of the parameterizations of its separatrices was described in the proof of proposition 3. To compute these polynomials in our situation we must take into account that the curve $\tilde{C}$ is obtained by ramification from $C = \cup_{i=1}^r C_i$. Consider a Puiseux series $y^i(x) = \sum_{s\geq n^i} a_s^i x^{s/n^i}$ for each curve C_i where $n^i = m_0(C_i)$. Thus all the Puiseux series of C_i are given by

$$y_j^i(x) = \sum_{s\geq n^i} a_s^i (\varepsilon_i)^{sj} x^{s/n^i}, \quad \text{for } j = 1, 2, \ldots, n^i,$$

where ε_i is a primitive n^i-root of the unity. Put $v_j^i(u) = y_j^i(u^n)$. Then $\rho^{-1}C_i = \{\sigma_j^i\}_{j=1}^{n^i}$ where $\sigma_j^i = (v - v_j^i(u) = 0)$.

Let $\{\tilde{E}^i\}_{i=1}^{n_E}$ be the vertices of $G(\tilde{C})$ associated to E and assume that $\tilde{E} = \tilde{E}^l$ for a certain $l \in \{1, \ldots, \underline{n}_E\}$. By the results of appendix B, we know that the choice of a vertex $\tilde{E}^l$ is equivalent to the choice of a $\underline{n}_E$-th root ξ_l of the unity. Given any $i \in I_E$, we denote $e_E^i = n^i/\underline{n}_E$ and we consider $\{\zeta_{ilt}\}_{t=1}^{e_E^i}$ the e_E^i-th roots of ξ_l. Thus, if we denote by $\{\sigma_{lt}^i\}_{t=1}^{e_E^i}$ the branches of $\rho^{-1}C_i$ such that $\tilde{E}^l$ belongs to their geodesics, then $\sigma_{lt}^i = (v - \eta_{lt}^i(u) = 0)$ where

$$\eta_{lt}^i(u) = \sum_{s\geq n^i} a_s^i (\zeta_{ilt})^s u^{sn/n^i}, \text{ for } t = 1, \ldots, e_E^i.$$

The use of the expressions above to compute the polynomials $A^{\tilde{E}^l}(v)$ and $B_\lambda^{\tilde{E}^l}(v) = B_{\lambda^*}^{\tilde{E}^l}(0, v)$ gives that

$$A^{\tilde{E}^l}(v) = \prod_{i\in I_E} \prod_{t=1}^{e_E^i} \left(v - a^i_{n^i v(E)} (\zeta_{ilt})^{n^i v(E)}\right) \tag{$*_1$}$$

$$B_\lambda^{\tilde{E}^l}(v) = \sum_{i\in I_E} \lambda_i \prod_{\substack{j\in I_E\\ j\neq i}} \prod_{t=1}^{e_E^j} \left(v - a^j_{n^j v(E)} \zeta_{jlt}^{n^j v(E)}\right) \sum_{t=1}^{e_E^i} \prod_{\substack{k=1\\ k\neq t}}^{e_E^i} \left(v - a^i_{n^i v(E)} \zeta_{ilk}^{n^i v(E)}\right) \tag{$*_2$}$$

Since both polynomials only depend on the invariants associated to E, we consider the three possibilities for a divisor E of $G(C)$ in order to obtain a more precisely expression of them:

(i) *E is a contact divisor:* we have that $v(E) = m_E/\underline{n}_E$ and $n_E = 1$. Then $n^i v(E) = e_E^i m_E$ for each $i \in I_E$ and consequently $(\zeta_{ilt})^{n^i v(E)} = \xi_l^{m_E}$ for each $t \in \{1, \ldots, e_E^i\}$. Thus we have that

$$A^{\tilde{E}^l}(v) = \prod_{i\in I_E} \left(v - a^i_{n^i v(E)} \xi_l^{m_E}\right)^{e_E^i}$$

$$B_\lambda^{\tilde{E}^l}(v) = \prod_{j\in I_E} \left(v - a^j_{n^j v(E)} \xi_l^{m_E}\right)^{e_E^j - 1} \sum_{i\in I_E} \lambda_i e_E^i \prod_{\substack{j\in I_E\\ j\neq i}} \left(v - a^j_{n^j v(E)} \xi_l^{m_E}\right)$$

Denote by $I^s_{\tilde{E}^l} = \left\{i \in I_E : a^i_{n^i v(E)} \xi_l^{m_E} = c_s^{\tilde{E}^l}\right\}$ for $s = 1, \ldots, b_{\tilde{E}^l}$. Thus $r_s = \sum_{i \in I^s_{\tilde{E}^l}} e^i_E$ and we have that

$$H_\lambda^{\tilde{E}^l}(v) = \sum_{i=1}^{b_{\tilde{E}^l}} \Bigg(\sum_{s \in I^i_{\tilde{E}^l}} \lambda_s e^s_E \Bigg) \prod_{\substack{j=1 \\ j \neq i}}^{b_{\tilde{E}^l}} \left(v - c_j^{\tilde{E}^l}\right)$$

which is a polynomial of degree $b_{\tilde{E}^l} - 1$ in v. Observe that $b_{\tilde{E}^l} = b_E$. The discriminant $D^{\tilde{E}^l}(\lambda)$ of $H_\lambda^{\tilde{E}^l}(v)$ as a polynomial in v is a non-zero polynomial. Hence, the set $U_C^E = \{\lambda : D^{\tilde{E}^l}(\lambda) \neq 0 \text{ for } l = 1, \ldots, \underline{n}_E\}$ is a non-empty Zariski open set.

(ii) *E is a Puiseux divisor with a dead arc:* we have that $v(E) = m_E/\underline{n}_E n_E$ with $n_E > 1$ and $(m^i_{k_E+1}, n^i_{k_E+1}) = (m_E, n_E)$ for each $i \in I_E$. It follows that $n^i v(E) = e^i_E m_E / n_E$ and the set $\{\zeta_{ilt}^{n^i v(E)}\}_{t=1}^{e^i_E}$ has n_E different values which coincide with the n_E-th roots $\{\theta_{lt}\}_{t=1}^{n_E}$ of $\xi_l^{m_E}$. Moreover, we have that

$$\prod_{s=1}^{n_E} \left(v - a^i_{n^i v(E)} \theta_{ls}\right) = v^{n_E} - \alpha^i_{\tilde{E}^l} \text{ with } \alpha^i_{\tilde{E}^l} = \left(a^i_{n^i v(E)}\right)^{n_E} \xi_l^{m_E}$$

and

$$\sum_{t=1}^{n_E} \prod_{\substack{p=1 \\ p \neq t}}^{n_E} \left(v - a^i_{n^i v(E)} \theta_{lp}\right) = n_E v^{n_E - 1}.$$

Thus the expressions $(*_1)$ and $(*_2)$ become

$$A^{\tilde{E}^l}(v) = \prod_{i \in I_E} \left(v^{n_E} - \alpha^i_{\tilde{E}^l}\right)^{e^i_E/n_E}$$

$$B_\lambda^{\tilde{E}^l}(v) = n_E v^{n_E - 1} \prod_{i \in I_E} \left(v^{n_E} - \alpha^i_{\tilde{E}^l}\right)^{\frac{e^i_E}{n_E} - 1} \sum_{i \in I_E} \lambda_i \frac{e^i_E}{n_E} \prod_{\substack{j \in I_E \\ j \neq i}} \left(v^{n_E} - \alpha^j_{\tilde{E}^l}\right)$$

In this case we have that $b_{\tilde{E}^l} = n_E(b_E - 1)$ and hence there are exactly $b_E - 1$ different values $\{\phi_s^{\tilde{E}_l}\}_{s=1}^{b_E - 1}$ in the set $\{\alpha^i_{\tilde{E}_l}\}_{i \in I_E}$. Denote $I^s_{\tilde{E}^l} = \{i \in I_E : \alpha^i_{\tilde{E}^l} = \phi_s^{\tilde{E}^l}\}$ and $r_s = \sum_{i \in I^s_{\tilde{E}^l}} e^i_E / n_E$. Then we have that

$$A^{\tilde{E}^l}(v) = \prod_{s=1}^{b_E - 1} \left(v^{n_E} - \phi_s^{\tilde{E}^l}\right)^{r_s}$$

and

$$H_\lambda^{\tilde{E}^l}(v) = v^{n_E - 1} \sum_{s=1}^{b_E - 1} \Big(\sum_{i \in I^s_{\tilde{E}^l}} \lambda_i e^i_E \Big) \prod_{\substack{j=1 \\ j \neq s}}^{b_E - 1} \big(v^{n_E} - \phi_j^{\tilde{E}^l}\big).$$

In this situation $D^{\tilde{E}^l}(\lambda) \not\equiv 0$ if and only if $n_E = 2$. Hence, we conclude that U_C^E is a non-empty Zariski open set if and only if C has a kind equisingularity type.

(iii) *E is a Puiseux divisor without a dead arc:* we have that $v(E) = m_E / \underline{n}_E n_E$ with $n_E > 1$ and $b_{\tilde{E}^l} = 1 + n_E(b_E - 1)$. We know that $(m_E, n_E) = (m^i_{k_E+1}, n^i_{k_E+1})$ for each $i \in I_E^*$ and $a^j_{n^i v(E)} = 0$ for $i \in I_E \setminus I_E^*$ (see appendix A). Denote by $r_0 = \sharp(I_E \setminus I_E^*)$. With similar arguments and notations as in case (ii), we get that

$$\begin{aligned} A^{\tilde{E}^l}(v) &= v^{r_0} \prod_{i \in I_E^*} \big(v^{n_E} - \alpha^i_{\tilde{E}^l}\big)^{e^i_E / n_E} \\ B_\lambda^{\tilde{E}^l}(v) &= v^{r_0 - 1} \prod_{i \in I_E^*} \big(v^{n_E} - \alpha^i_{\tilde{E}^l}\big)^{\frac{e^i_E}{n_E} - 1} \Big\{ v^{n_E} \sum_{i \in I_E^*} \lambda_i e^i_E \prod_{\substack{j \in I_E^* \\ j \neq i}} \big(v^{n_E} - \alpha^j_{\tilde{E}^l}\big) \\ &\quad + \prod_{j \in I_E^*} \big(v^{n_E} - \alpha^j_{\tilde{E}^l}\big) \Big(\sum_{i \in I_E \setminus I_E^*} \lambda_i \Big) \Big\} \end{aligned}$$

Let $\{\phi_s^{\tilde{E}_l}\}_{s=1}^{b_E - 1}$ be the $b_E - 1$ different values in the set $\{\alpha_i^{\tilde{E}^l}\}_{i \in I_E^*}$. Denote $I^s_{\tilde{E}^l} = \{i \in I_E^* : \alpha^i_{\tilde{E}^l} = \phi_s^{\tilde{E}^l}\}$ and $r_s = \sum_{i \in I^s_{\tilde{E}^l}} e^i_E / n_E$. Thus we have that

$$\begin{aligned} A^{\tilde{E}^l}(v) &= v^{r_0} \prod_{i=1}^{b_E - 1} \big(v^{n_E} - \phi_s^{\tilde{E}^l}\big)^{r_i} \\ H_\lambda^{\tilde{E}^l}(v) &= v^{n_E} \sum_{s=1}^{b_E - 1} \Big(\sum_{i \in I^s_{\tilde{E}^l}} \lambda_i e^i_E \Big) \prod_{\substack{j=1 \\ j \neq s}}^{b_E - 1} \big(v^{n_E} - \phi_j^{\tilde{E}^l}\big) \\ &\quad + \Big(\sum_{j \in I_E \setminus I_E^*} \lambda_j \Big) \prod_{s=1}^{b_E - 1} \big(v^{n_E} - \phi_s^{\tilde{E}^l}\big). \end{aligned}$$

It is clear that in this case $D^{\tilde{E}^l}(\lambda) \not\equiv 0$ for each $l = 1, \ldots, \underline{n}_E$. Consequently, U_C^E is a non-empty Zariski open set.

We conclude that a necessary and sufficient condition to assure that all the sets U_C^E are non-empty Zariski open sets is that C has a kind equisingularity type and the result follows straightforward. $\square$

With similar arguments to the ones in the proof above we can show that:

Corollary 4. *The following statements are equivalent:*

- *The curve C has a kind equisingularity type;*
- *There exists a foliation $\mathcal{F} \in \mathbb{G}^*_C$ such that $\rho^*\mathcal{F}$ is Zariski-general.*

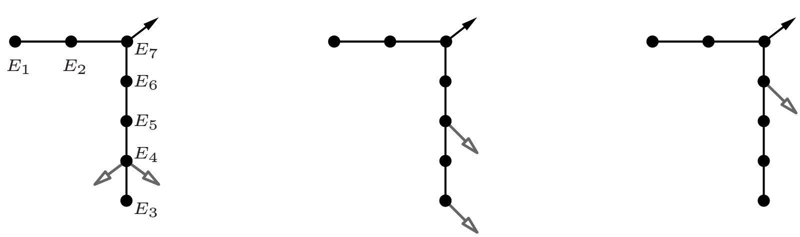
In particular, if $\mathcal{F} \in \mathbb{G}^*_{C,\lambda}$ with $\lambda \in U_C$, the equisingularity type of a generic polar curve $\Gamma_{\mathcal{F}}$ is completely determined in terms of C and π_C gives a reduction of singularities of $C \cup \Gamma_{\mathcal{F}}$. Moreover, we get that the irreducible components of $\Gamma_{\mathcal{F}}$ cut the exceptional divisor $\pi_C^{-1}(0)$ as described in proposition 5; we get a more specific description than the one of Lê-Michel-Weber in [12].

Observe that the property "π_C gives a reduction of singularities of $\Gamma_{\mathcal{F}} \cup C$" does not imply that $\mathcal{F}$ is a Zariski-general foliation. Moreover, this property does not determine the equisingularity type of $\Gamma_{\mathcal{F}} \cup C$ even if we fix λ.

Example 1. Consider the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ given by $\omega_i = 0$ with

$$\begin{aligned}
\omega_1 &= -11x^{10}dx + 5y^4dy; \\
\omega_2 &= 11\big(-x^{10} + y^2x^6\big)dx + 5(y^4 - x^7y)dy; \\
\omega_3 &= 11\big(-x^{10} + yx^8\big)dx + 5\big(y^4 - x^9\big)dy
\end{aligned}$$

respectively. All the foliations have the same separatrix $C = (y^5 - x^{11} = 0)$ which does not have a kind type of equisingularity, therefore $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ cannot be Zariski-general foliations. The generic polar curves $\Gamma_{\mathcal{F}_1}$, $\Gamma_{\mathcal{F}_2}$ and $\Gamma_{\mathcal{F}_3}$ are not equisingular but the minimal reduction of singularities of C is also a reduction of singularities of the curves $\Gamma_{\mathcal{F}_1}$, $\Gamma_{\mathcal{F}_2}$ and $\Gamma_{\mathcal{F}_3}$.



$G(C \cup \Gamma_{\mathcal{F}_1})$ $G(C \cup \Gamma_{\mathcal{F}_2})$ $G(C \cup \Gamma_{\mathcal{F}_3})$

Appendix A. Equisingularity data: the dual graph

Let us recall the construction of the dual graph which is one of the different ways to represent the equisingularity data of a plane curve (see [1] for more details). Let $C \subset (\mathbb{C}^2, 0)$ be a plane curve and $\pi_C : M \to (\mathbb{C}^2, 0)$ be its minimal reduction of singularities. The *dual graph* $G(C)$ is constructed as follows: each irreducible component E of $\pi_C^{-1}(0)$ is represented by a vertex which we also call E (we identify a divisor and its associated vertex in the dual graph). Two vertices are joined by an edge if and only if the associated divisors intersect. Each irreducible component of C is represented by an arrow joined to the only divisor which meets the strict transform of C by π_C. If we give a weight to each vertex E of $G(C)$ equal to the self-intersection of the divisor $E \subset M$, this weighted dual graph is equivalent to the equisingularity data of C.

We denote by E_1 the irreducible component of $\pi_C^{-1}(0)$ obtained by the blowing-up of the origin. Thus the first divisor E_1 gives an orientation to the graph $G(C)$. The *geodesic* of a divisor E is the path which joins the first divisor E_1 with the divisor E. The geodesic of a curve is the geodesic of the divisor that meets the transform strict of the curve. In this way, there is a partial order in the set of vertices of $G(C)$ given by $E < E'$ if and only if the geodesic of E' goes through E.

Let us introduce some notations concerning the dual graph of a curve. Given a vertex E of $G(C)$ we define the number b_E as follows: $b_E + 1$ is the valence of E if $E \neq E_1$ and b_{E_1} is the valence of E_1. Observe that b_{E_1} is the number of different lines in the tangent cone of C. We say that E is a *bifurcation divisor* if $b_E \geq 2$ and E is a *terminal divisor* if $b_E = 0$. A *dead arc* in $G(C)$ is an arc which joins a bifurcation divisor with a terminal one without passing through other bifurcation divisors. Observe that a bifurcation divisor can belong only to one dead arc.

A *curvette* $\tilde{\gamma}$ of a divisor E is a non-singular curve transversal to E at a non-singular point of $\pi_C^{-1}(0)$. The projection $\gamma = \pi_C(\tilde{\gamma})$ is a germ of plane curve in $(\mathbb{C}^2, 0)$ and we say that γ is an E-curvette. We denote by $m(E)$ the multiplicity at the origin of any E-curvette. Take $\tilde{\gamma}, \tilde{\gamma}'$ two curvettes of E which intersect E in two different points, we denote by $v(E)$ the coincidence $C(\pi_C(\tilde{\gamma}), \pi_C(\tilde{\gamma}'))$; then $v(E) < v(E')$ if $E < E'$. Recall that the *coincidence* $C(\gamma, \delta)$ between two irreducible curves γ and δ is defined as

$$C(\gamma, \delta) = \sup_{\substack{1 \leq i \leq m_0(\gamma) \\ 1 \leq j \leq m_0(\delta)}} \left\{ ord_x(y_i^\gamma(x) - y_j^\delta(x)) \right\}$$

where $\{y_i^\gamma(x)\}_{i=1}^{m_0(\gamma)}$, $\{y_j^\delta(x)\}_{j=1}^{m_0(\delta)}$ are the Puiseux series of γ and δ respectively.

Given any irreducible component E of the exceptional divisor $\pi_C^{-1}(0)$, we denote by $\pi_E : M_E \to (\mathbb{C}^2, 0)$ the *reduction of* π_C *to* E, that is, the morphism which satisfies that

- there is a factorization $\pi_C = \pi'_E \circ \pi_E$ where π'_E and π_E are composition of punctual blow-ups;
- the divisor E is the strict transform by π'_E of an irreducible component E_{red} of $\pi_E^{-1}(0)$ and $E_{red} \subset M_E$ is the only component of $\pi_E^{-1}(0)$ with self-intersection equal to -1.

It is clear that π_E is obtained from π_C by blowing-down successively the divisors which are different from E and whose self-intersection is equal to -1. Take any curvette $\tilde{\gamma}_E$ of E, then $\pi'_E(\tilde{\gamma}_E)$ is also a curvette of $E_{red} \subset M_E$. Let $\{\beta_0^E, \beta_1^E, \ldots, \beta_{g(E)}^E\}$ be the characteristic exponents of $\gamma_E = \pi_C(\tilde{\gamma}_E)$. It is clear that $m(E) = \beta_0^E = m_0(\gamma_E)$ and there are two possibilities for the value $v(E)$:

1. either π_E is the minimal reduction of singularities of γ_E and then $v(E) = \beta_{g(E)}^E/\beta_0^E$. We say that E is a *Puiseux divisor* for π_C.
2. or π_E is obtained by blowing-up $q \geq 1$ times after the minimal reduction of singularities of γ_E and in this situation $v(E) = (\beta_{g(E)}^E + q\beta_0^E)/\beta_0^E$. We say that E is a *contact divisor* for π_C.

Observe that $m(E) = m(E_{red})$ and $v(E) = v(E_{red})$. Moreover, E can belong to a dead arc only if it is a Puiseux divisor.

Consider a bifurcation divisor E of $G(C)$ and let $\{(m_1^E, n_1^E), (m_2^E, n_2^E), \ldots, (m_{g(E)}^E, n_{g(E)}^E)\}$ be the Puiseux pairs of an E-curvette γ_E, we denote

$$n_E = \begin{cases} n_{g(E)}, & \text{if } E \text{ is a Puiseux divisor;} \\ 1, & \text{otherwise,} \end{cases}$$

and $\underline{n}_E = m(E)/n_E$. Observe that, if E belongs to a dead arc with terminal divisor F, then $m(F) = \underline{n}_E$. We define k_E to be

$$k_E = \begin{cases} g(E) - 1, & \text{if } E \text{ is a Puiseux divisor;} \\ g(E), & \text{if } E \text{ is a contact divisor.} \end{cases}$$

Let us explain these notations in terms of the equisingularity data of the curve $C = \cup_{i=1}^r C_i$. Denote by $\{(m_l^i, n_l^i)\}_{l=1}^{g_i}$ the Puiseux pairs of C_i and by $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ its characteristic exponents. Denote $I = \{1, 2, \ldots, r\}$ and let I_E be the set of indices $i \in I$ such that E belong to the geodesic of C_i. Take $i \in I_E$. There are several possibilities for the value of $v(E)$ depending on E:

(i) If E is a contact divisor, then there exists $j \in I_E$ such that $v(E) = C(C_i, C_j)$.

(ii) If E is a Puiseux divisor which belongs to a dead arc, then $v(E) = \beta^i_{k_E+1}/\beta^i_0$.

(iii) If E is a Puiseux divisor which does not belong to a dead arc, we denote by I^*_E the set of indices $i \in I_E$ such that $v(E) = \beta^i_{k_E+1}/\beta^i_0$. Then $C(C_i, C_j) = v(E)$ for $i \in I^*_E$ and $j \in I_E \setminus I^*_E$. Moreover, $C(C_j, C_l) > v(E)$ if $j, l \in I_E \setminus I^*_E$.

Consequently, we have that $(m^i_l, n^i_l) = (m^E_l, n^E_l)$, for $l = 1, \ldots, k_E$, and $\underline{n}_E = n^i_1 \cdots n^i_{k_E}$ for any $i \in I_E$.

Appendix B. Ramification

Consider a plane curve $C = \cup^r_{i=1} C_i \subset (\mathbb{C}^2, 0)$. Let $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be any C-ramification, that is, ρ is transversal to C and $\tilde{C} = \rho^{-1}C$ has only non-singular irreducible components. Assume that the ramification is given by $x = u^n, y = v$.

Denote by $\{(m^i_l, n^i_l)\}^{g_i}_{l=1}$ the Puiseux pairs of C_i and by $\{\beta^i_0, \beta^i_1, \ldots, \beta^i_{g_i}\}$ the characteristic exponents of C_i. If $n^i = m_0(C_i)$, then it is necessary that $n \equiv 0$ mod $(n^1, n^2, \ldots, n^r)$ in order to have that $\tilde{C}$ has only non-singular irreducible components. Moreover, the number of irreducible components of $\tilde{C}$ is equal to $m_0(C) = n^1 + \cdots + n^r$. More precisely, each curve $\rho^{-1}C_i$ has exactly n^i irreducible components. In fact, let $y^i(x) = \sum_{l \geq n^i} a^i_l x^{l/n^i}$ be a Puiseux series of C_i, thus all its Puiseux series are given by

$$y^i_j(x) = \sum_{l \geq n^i} a^i_l \varepsilon^{lj}_i x^{l/n^i} \quad \text{for } j = 1, 2, \ldots, n^i,$$

where ε_i is a primitive n^i-root of the unity. Then

$$f_i(x, y) = \prod_{l=1}^{n^i} \left(y - y^i_l(x)\right)$$

is a reduced equation of C_i. If we put $v^i_j(u) = y^i_j(u^n)$, then $v^i_j(u) \in \mathbb{C}\{u\}$ since $n/n^i \in \mathbb{N}$. It is clear that the curve $\sigma^i_j = (v - v^i_j(u) = 0)$ is non-singular and it is one of the irreducible components of $\rho^{-1}C_i$. Then

$$g_i(u, v) = f_i(u^n, v) = \prod_{l=1}^{n^i} \left(v - v^i_l(u)\right)$$

is an equation of $\rho^{-1}C_i$. We conclude that the irreducible components $\{\sigma_j^i\}_{j=1}^{n^i}$ of $\rho^{-1}C_i$ are in bijection with the Puiseux series of C_i.

It is well-known that the equisingularity type of a curve C is determined by the characteristic exponents $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}_{i=1}^r$ of its irreducible components and the intersection multiplicities $\{(C_i, C_j)_0\}_{i\neq j}$. Let us show that we can obtain all this information from $\rho^{-1}C$. The next lemma states the relationship between the intersection multiplicity $(\gamma, \delta)_0$ and the coincidence $C(\gamma, \delta)$ (see Zariski [17], prop. 6.1 or Merle [13], prop. 2.4):

Lemma 7. *Let γ and δ be two germs of irreducible plane curves of $(\mathbb{C}^2, 0)$. If $\{\beta_0, \beta_1, \ldots, \beta_g\}$ are the characteristic exponents of γ and α is a rational number such that $\beta_q \leq \alpha < \beta_{q+1}$ ($\beta_{g+1} = \infty$), then the following statements are equivalent:*

1. $C(\gamma, \delta) = \dfrac{\alpha}{m_0(\gamma)}$
2. $\dfrac{(\gamma, \delta)_0}{m_0(\delta)} = \dfrac{\bar{\beta}_q}{n_1 \cdots n_{q-1}} + \dfrac{\alpha - \beta_q}{n_1 \cdots n_q}$

where $\{(m_i, n_i)\}_{i=1}^g$ are the Puiseux pairs of γ ($n_0 = 1$) and $\{\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_q\}$ is a minimal system of generators of the semigroup $S(\gamma)$ of γ.

In particular, the equisingularity type of C is also determined by the characteristic exponents of each C_i and the coincidences $\{C(C_i, C_j)\}_{i\neq j}$. Let us show that these data could be obtained from $\rho^{-1}C$. Given an irreducible component σ of $\rho^{-1}C$, we take an equation $(v - v^\sigma(u) = 0)$ of σ with

$$v^\sigma(u) = \sum_{l\geq 1} a_l^\sigma u^l \in \mathbb{C}\{u\}.$$

Given two irreducible components σ, σ' of $\rho^{-1}C$, we say that they are equivalent $\sigma \sim \sigma'$ if and only if $(a_j^\sigma)^n = (a_j^{\sigma'})^n$ for all $j \in \mathbb{N}$. Denote by $[\sigma]$ the equivalence classes of a curve σ. Thus the number of irreducible components r of C is equal to the number of equivalence classes for the irreducible components of $\rho^{-1}C$. Let $[\sigma^1], \ldots, [\sigma^r]$ be these equivalence classes. Up to reorder, we can assume that $[\sigma^i]$ corresponds to $\rho^{-1}C_i$, for $i = 1, \ldots, r$. Thus the multiplicity n^i of $\rho^{-1}C_i$ is equal to the number of elements in the equivalence class $[\sigma^i]$. We put $\rho^{-1}C_i = \{\sigma_l^i\}_{l=1}^{n^i}$. Hence $\beta_0^i = n^i$ and the other characteristic exponents of C_i are obtained from the computation of the coincidences among the curves in the equivalence class $[\sigma^i]$ since

$$\left\{C(\sigma_j^i, \sigma_l^i) \;:\; j \neq l\right\} = \left\{\beta_1^i, \ldots, \beta_{g_i}^i\right\}.$$

Thus we only need to compute the coincidences between any two branches C_i and C_j. But they are obtained from the following equality

$$C(C_i, C_j) = \frac{1}{n} \sup_{\substack{1 \le l \le n^i \\ 1 \le s \le n^j}} \left\{ C(\sigma_l^i, \sigma_s^j) \right\}, \tag{12}$$

which is true for any two irreducible curves. Hence we conclude that the equisingularity data of C can be recovered from $\rho^{-1}C$.

Ramification of the dual graph. Let $\pi_C : M \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of C and denote by $\pi_{\tilde{C}} : \tilde{M} \to (\mathbb{C}^2, 0)$ the minimal reduction of singularities of $\tilde{C} = \rho^{-1}C$. Let us explain the relationship between $G(C)$ and $G(\tilde{C})$.

Let K_i be the geodesic in $G(C)$ of a branch C_i of C and let $\tilde{K}_i$ be the sub-graph of $G(\tilde{C})$ corresponding to the geodesics of the irreducible components $\{\sigma_l^i\}_{l=1}^{n^i}$ of $\rho^{-1}C_i$. Let us see how to construct $\tilde{K}_i$ from K_i. Observe first that, if $\tilde{E}$ and $\tilde{E}'$ are two consecutive vertices of $G(\tilde{C})$ with $\tilde{E} < \tilde{E}'$, then $v(\tilde{E}') = v(\tilde{E}) + 1$. Thus, $G(\tilde{C})$ is completely determined once we know the bifurcation divisors, the order relations among them and the number of edges which leave from each bifurcation divisor. Denote by $B(\tilde{K}_i)$ and $B(K_i)$ the bifurcation vertices of $\tilde{K}_i$ and K_i respectively. We say that a vertex $\tilde{E}$ of $B(\tilde{K}_i)$ is associated to a vertex E of $B(K_i)$ if $v(\tilde{E}) = nv(E)$.

Let E be a vertex of $B(K_i)$. Assume first that E is the first bifurcation divisor of $B(K_i)$ and take E' its consecutive vertex in $B(K_i)$. Then E has only one associated vertex $\tilde{E}$ in $B(\tilde{K}_i)$ and there are two possibilities for the number of edges which leave from it:

- If E is a Puiseux divisor, then there are n_1^i edges which leave from $\tilde{E}$ in $\tilde{K}_i$; then E' has n_1^i associated vertices in $B(\tilde{K}_i)$.
- If E is a contact divisor, then there is only one edge which leave from $\tilde{E}$ in $\tilde{K}_i$ and thus E' has only one vertex associated in $B(\tilde{K}_i)$.

Take now any vertex E of $B(K_i)$ and assume that we know the part of $\tilde{K}_i$ corresponding to the vertices of K_i with valuation $\le v(E)$. Then there are $\underline{n}_E = n_1^i \cdots n_{k_E}^i$ vertices $\{\tilde{E}^l\}_{l=1}^{\underline{n}_E}$ associated to E and

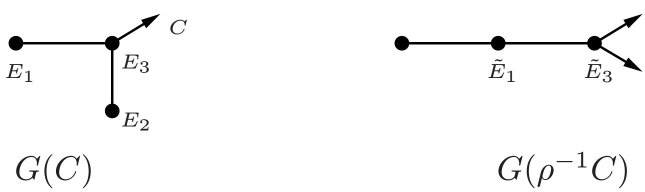
- If E is a Puiseux divisor, then there are n_{k_E+1} edges which leave from each vertex $\tilde{E}_l$ in $\tilde{K}_i$.
- If E is a contact divisor, then there is only one edge which leaves from each vertex $\bar{E}_l$ in $\tilde{K}_i$.

The dual graph $G(\tilde{C})$ is constructed in the natural way by gluing the graphs $\tilde{K}_i$. From the construction described above, we deduced that

$$b_{\tilde{E}} = \begin{cases} b_E, & \text{if } E \text{ is a contact divisor;} \\ (b_E - 1)n_E, & \text{if } E \text{ is a bifurcation divisor which belong} \\ & \text{to a dead arc;} \\ (b_E - 1)n_E + 1, & \text{if } E \text{ is a bifurcarion divisor which does not} \\ & \text{belong to a dead arc.} \end{cases}$$

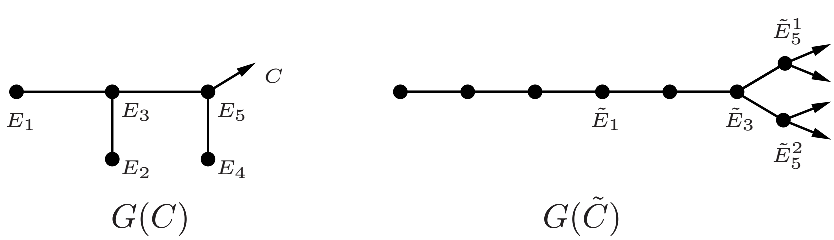
Observe that, in general, non-bifurcation divisors of $G(C)$ have no associated divisors in $G(\tilde{C})$. Let us illustrate with some examples the relationship between $G(C)$ and $G(\tilde{C})$:

Example 2. Consider the curve $C = (y^2 - x^3 = 0)$ and the ramification $\rho(u, v) = (u^2, v)$. Then $\tilde{C}$ has two irreducible components given by $v - u^3 = 0$ and $v + u^3 = 0$. The next figure represents the dual graphs of C and $\rho^{-1}C$:



where $\tilde{E}_1$, $\tilde{E}_3$ are the vertices associated to E_1 and E_3 respectively.

Consider now a curve C with characteristic exponents $\{4, 6, 7\}$. Take ρ the ramification given by $\rho(u, v) = (u^4, v)$ and put $\tilde{C} = \rho^{-1}C$. Then we have that



Note that E_3 has one associated vertex $\tilde{E}_3$ and that E_5 has two associated vertices $\tilde{E}_5^1$ and $\tilde{E}_5^2$ in $G(\tilde{C})$.

Remark 2. Let us denote by $\tilde{E}_1$ the divisor of $G(\tilde{C})$ with $v(\tilde{E}_1) = n$. It is unique since it precedes all the other bifurcation divisors and it could be or not a bifurcation divisor. Moreover, $\tilde{E}_1$ is a bifurcation divisor of $G(\tilde{C})$ if and only if E_1 is a bifurcation divisor of $G(C)$ and $b_{\tilde{E}_1} = b_{E_1}$. Then, the divisor E_1 of $G(C)$ has always a unique divisor, denoted by $\tilde{E}_1$, which is associated to it in $G(\tilde{C})$ even if $E_1 \notin B(C)$. Recall that E_1 is a bifurcation divisor if and only if the number of different tangent lines in the tangent cone of C is ≥ 2.

We have seen that there is a bijection between the Puiseux series of C_i and the irreducible components of $\rho^{-1}C_i$. In particular, this implies that the choice of a vertex $\tilde{E}^l \in B(\tilde{K}_i)$ associated to a bifurcation divisor E is equivalent to the choice of a $\underline{n}_E$-th root of the unity ξ_l. Thus there are $e^i_E = n^i/\underline{n}_E$ irreducible components $\{\sigma^i_{lt}\}_{t=1}^{e^i_E}$ of $\rho^{-1}C_i$ such that $\tilde{E}^l$ belongs to their geodesics. Moreover, the curve σ^i_{lt} is given by $(v - \eta^i_{lt}(u) = 0)$ where

$$\eta^i_{lt}(u) = \sum_{s \geq n^i} a^i_s (\zeta_{ilt})^s u^{sn/n^i}, \quad \text{for } t = 1, \ldots, e^i_E.$$

and $\{\zeta_{ilt}\}_{t=1}^{e^i_E}$ are the e^i_E-th roots of ξ_l. Additionally , if γ_E is an E-curvette of a bifurcation divisor E of $G(C)$, the curve $\rho^{-1}\gamma_E$ has $m(E) = \underline{n}_E n_E$ irreducible components which are all non-singular and each divisor $\tilde{E}^l$ belongs to the geodesic of exactly n_E branches of $\rho^{-1}\gamma_E$ which are curvettes of $\tilde{E}^l$ in different points. In particular, we can prove the following result

Lemma 8. *Let E be either a bifurcation divisor of $G(C)$ or $E = E_1$ and consider any of its associated divisors $\tilde{E}$ in $G(\tilde{C})$. Then there exists a morphism $\rho_{\tilde{E},E} : \tilde{E}_{red} \to E_{red}$ which is a ramification of order n_E.*

Proof. Consider a C-ramification $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $x = u^n$, $y = v$. Let $\pi_{\tilde{E}} : \tilde{M}_{\tilde{E}} \to (\mathbb{C}^2, 0)$ be the reduction of $\pi_{\tilde{C}}$ to $\tilde{E}$ and $\pi_E : M_E \to (\mathbb{C}^2, 0)$ be the reduction of π_C to E. Let us define the map $\rho_{\tilde{E},E} : \tilde{E}_{red} \to E_{red}$. The map $\rho_{\tilde{E},E}$ sends the "infinity point" of $\tilde{E}_{red}$ (that is, the origin of the second chart of $\tilde{E}_{red}$) into the "infinity point" of E_{red}. For any other point P of $\tilde{E}_{red}$, we consider an $\tilde{E}$-curvette $\gamma^P_{\tilde{E}} = (v - \psi^P_{\tilde{E}}(u) = 0)$ with

$$\psi^P_{\tilde{E}}(u) = \sum_{i=1}^{v(\tilde{E})-1} a^{\tilde{E}}_i u^i + a^P_{v(\tilde{E})} u^{v(\tilde{E})},$$

and such that $\pi^*_{\tilde{E}}\gamma^P_{\tilde{E}} \cap \tilde{E}_{red} = \{P\}$. Let γ^P_E be the curve given by the Puiseux series

$$y^P(x) = \sum_{i=1}^{v(\tilde{E})-1} a^{\tilde{E}}_i x^{i/m(E)} + a^P_{v(\tilde{E})} x^{v(\tilde{E})/m(E)}.$$

Thus γ_E^P is an E-curvette and we define $\rho_{\tilde{E},E}(P)$ to be the only point $\pi_E^* \gamma_E^P \cap E_{red}$. From the properties of ρ we deduce that $\rho_{\tilde{E},E}$ is a ramification of order n_E. □

Remark also that, if γ_{E_t} is a curvette of a terminal divisor E_t of a dead arc with bifurcation divisor E, then $\rho^{-1}\gamma_{E_t}$ is composed by $m(E_t) = \underline{n}_E$ non-singular irreducible components and each divisor $\tilde{E}^l$ belongs to the geodesic of exactly one branch of $\rho^{-1}\gamma_{E_t}$, where $\{\tilde{E}^l\}_{l=1}^{\underline{n}_E}$ are the divisors associated to E in $G(\tilde{C})$.

For more results concerning foliations, ramifications and blow-ups, the reader can refer to [9].

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