

Characterizing equivalent discrete Morse functions

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Abstract. We get a characterization theorem for equivalent discrete Morse functions defined on simplicial complexes in terms of their gradient vector field. As a consequence, we also characterize them in the 1-dimensional case by using critical elements.

Keywords: simplicial complex, discrete Morse function, critical element.

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1 Introduction

Morse theory is the study of the relationships between functions defined on a space and the shape of this space. More precisely, these relations can be settled by obtaining information of the shape of the space from the information about the critical points of the function. Therefore, the main goal of this theory lies in how the critical points of a function defined on a space affect the topological shape of such space and, conversely, how the shape of a space controls the distribution of the critical points of a function.

R. Forman [5, 6] introduced the notion of discrete Morse function defined on a finite regular cw -complex and, in this combinatorial context, he developed a discrete Morse theory as a purely combinatorial tool for studying the topology of the considered complex by means, for example, either of its homotopy type or its homology. Moreover, this theory has many other applications [9]. On the other hand, as Forman stated in [7], one area in which much work remains to be done is the investigation of discrete Morse functions defined on infinite simplicial complexes. Recently, the authors have obtained some results in this area [1, 2, 3, 11].

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In this context, it would be more useful to know the behaviour of a particular discrete Morse function f instead of which its values are. Thus, the notion of discrete gradient vector field induced by f , which was introduced by Forman in [5] for the finite case and extended by the authors to the infinite case, is specially appropriated because it describes the way how f increases and decreases. It turns out that, in some situations, it is helpful to use it and not the explicit definition of f .

As Forman pointed out in [7], the ideas of his discrete Morse theory can be applied with no modification at all to any regular cw -complex. So, for simplicity, we confine our attention to simplicial complexes. In this paper, we prove that discrete Morse functions defined either on finite or infinite simplicial complexes which induce the same gradient vector field are essentially the same, that is, they are equivalent in the Forman's sense (namely, they have the same monotonous behaviour in the same simplices) [8].

In Section 2 we compile all notions and results for later use. Essentially, we introduce the concepts of discrete Morse function, critical simplex, decreasing ray, gradient path and some basic results about them.

Section 3 is devoted to state the announced characterization result, that is, the characterization of equivalent discrete Morse functions attending its induced gradient fields. Moreover, we also show that given a gradient vector field there are infinitely many different discrete Morse functions which induce it. Using a well known term from the differentiable case, we can say that such functions are “potential functions” associated to a discrete vector field and the procedure to get a potential function can be called “integration” of the gradient vector field.

Finally, we study the relationship between the notion of equivalence on discrete Morse functions and the critical simplices of the complex with respect to this function. To this end, we prove that equivalent discrete Morse functions have the same critical simplices and we provide examples which show that the converse does not hold in general for dimension greater or equal than 2 and for dimension 1 in the infinite case. However, by introducing the concept of critical element, which includes critical simplices and decreasing 1-rays, we prove, as a consequence of the above result, that these critical elements characterize equivalent discrete Morse functions on any 1-dimensional simplicial complex.

2 Preliminaries

Through all this paper, we consider infinite simplicial complexes which are locally finite. For terminology and background concerning these objects, we refer to [10].

Let us start with the notion of discrete Morse function which was introduced for the finite case by R. Forman in [4, 5, 6]. A *discrete Morse function* defined on a (finite or infinite) simplicial complex M is a proper function $f : M \rightarrow \mathbf{R}$ such that, for any simplex $\sigma^{(p)} \in M$ (where (p) , when needed, is indicating the dimension of the simplex):

$$(M1) \quad \text{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} \leq 1.$$

$$(M2) \quad \text{card}\{v^{(p-1)} < \sigma / f(v) \geq f(\sigma)\} \leq 1.$$

Associated to the concept of discrete Morse function we have the notion of critical simplex. A p -simplex $\sigma \in M$ is said to be *critical* with respect to f if:

$$(C1) \quad \text{card}\{\tau^{(p+1)} > \sigma / f(\tau) \leq f(\sigma)\} = 0.$$

$$(C2) \quad \text{card}\{v^{(p-1)} < \sigma / f(v) \geq f(\sigma)\} = 0.$$

It can be deduced from the above definitions that $\sigma^{(p)}$ is a non-critical simplex if and only if it verifies one of the following conditions:

$$(NC1) \quad \text{There exists a simplex } \tau^{(p+1)} > \sigma^{(p)} \text{ such that } f(\tau^{(p+1)}) \leq f(\sigma^{(p)}).$$

$$(NC2) \quad \text{There exists a simplex } v^{(p-1)} < \sigma^{(p)} \text{ such that } f(v^{(p-1)}) \geq f(\sigma^{(p)}).$$

It is important to point out that both conditions can not be fulfilled simultaneously by a non-critical simplex.

Note that, in the 1-dimensional case, by applying (C1) and (C2) we get that every critical vertex (0-simplex) is a local minimum of f and every critical edge (1-simplex) is a local maximum of f .

Given a simplicial complex M , a *discrete vector field* V defined on M is a collection of pairs $(\alpha^{(p)} < \beta^{(p+1)})$ of simplices of M such that every simplex is in, at most, one pair of V . We can visualize discrete vector fields in low dimensional complexes by considering arrows in the following way (see Figure 1) where the first arrow is indicating that the vertex v and the edge e verify that $(v, e) \in V$ and analogously, the second arrow is indicating that the edge e' and the triangle t verify that $(e', t) \in V$.

Since (NC1) and (NC2) can not be verified simultaneously by a non critical simplex, it can be deduced that every discrete Morse function $f : M \rightarrow \mathbf{R}$ induces a discrete vector field on M . Namely, given a simplex $\sigma^{(p)}$ of M , it can be critical or not. If it is not, there is a unique simplex τ of consecutive dimension such that $\tau^{(p-1)} < \sigma^{(p)}$ and $f(\tau) \geq f(\sigma)$ or $\tau^{(p+1)} > \sigma^{(p)}$ and $f(\tau) \leq f(\sigma)$. So we can consider the pair $(\tau < \sigma)$ or $(\sigma < \tau)$ depending on

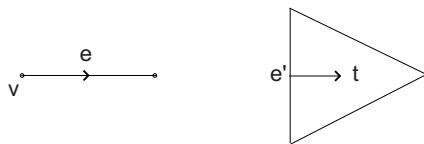


Figure 1

the case. If σ is critical there is not any simplex in M paired in this way with it. Thus, each simplex of M is either the first simplex of a pair or the second simplex of a pair or it is not in any pair. Hence, this set of pairs verifies the definition of discrete vector field on M . Essentially, a pair $\tau^{(p-1)} < \sigma^{(p)}$ is in this vector field if and only if $f(\sigma^{(p)}) \leq f(\tau^{(p-1)})$. This vector field is called the *gradient vector field induced by f* .

Next, let M be a simplicial complex and let V a discrete vectorial field defined on M . A V -path is a (finite or infinite) sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \dots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}, \dots$$

such that for all i , the pair

$$(\alpha_i^{(p)} < \beta_i^{(p+1)}) \in V \quad \text{and} \quad \beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}.$$

In the particular case in which V is a gradient field induced by a discrete Morse function, we give a special name for the V -paths: we call them *gradient paths*.

Given a finite V -path, that is, a finite sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \dots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

we say that it is a *closed path* if $r \geq 0$ and $\alpha_0^{(p)} = \alpha_{r+1}^{(p)}$.

An infinite sequence of simplices of an infinite simplicial complex M ,

$$\alpha_0^{(i-1)}, \beta_0^{(i)}, \alpha_1^{(i-1)}, \beta_1^{(i)}, \dots, \beta_r^{(i)}, \alpha_{r+1}^{(i-1)}, \dots$$

is called a i -ray ($i = 1, 2$) if it verifies that the $(i-1)$ -simplices $\alpha_{n-1}^{(i-1)}$ and $\alpha_n^{(i-1)}$ are faces of the i -simplex $\beta_n^{(i)}$, for any $n \in \mathbb{N}$.

Given two i -rays r and $\hat{r}$ contained in the same complex,

$$\begin{aligned} r &= \alpha_0^{(i-1)}, \beta_0^{(i)}, \alpha_1^{(i-1)}, \beta_1^{(i)}, \dots, \beta_n^{(i)}, \alpha_{n+1}^{(i-1)}, \dots, \\ \hat{r} &= \hat{\alpha}_0^{(i-1)}, \hat{\beta}_0^{(i)}, \hat{\alpha}_1^{(i-1)}, \hat{\beta}_1^{(i)}, \dots, \hat{\beta}_m^{(i)}, \hat{\alpha}_{m+1}^{(i-1)}, \dots, \end{aligned}$$

we say they are *equivalent* or *cofinal* if there are $n, m \in N$ such that

$$\alpha_n^{(i-1)} = \hat{\alpha}_m^{(i-1)}, \beta_n^{(i)} = \hat{\beta}_m^{(i)}, \dots, \alpha_{n+k}^{(i-1)} = \hat{\alpha}_{m+k}^{(i-1)}, \beta_{n+k}^{(i)} = \hat{\beta}_{m+k}^{(i)}, \dots$$

In other words, the two i -rays are cofinal if they coincide from a common $(i-1)$ -simplex or, equivalently, if they are the same one outside a finite compact sub-complex.

Now, given a discrete Morse function defined on M , we say that an i -ray,

$$\alpha_0^{(i-1)}, \beta_0^{(i)}, \alpha_1^{(i-1)}, \beta_1^{(i)}, \dots, \beta_r^{(i)}, \alpha_{r+1}^{(i-1)}, \dots$$

is a *decreasing i -ray* if it verifies that:

$$\begin{aligned} f(\alpha_0^{(i-1)}) &\geq f(\beta_0^{(i)}) > f(\alpha_1^{(i-1)}) \geq f(\beta_1^{(i)}) > \dots \\ &\geq f(\beta_r^{(i)}) > f(\alpha_{r+1}^{(i-1)}) \geq \dots \end{aligned}$$

Notice that, if V is the gradient vector field induced by a discrete Morse function defined on an infinite complex, then infinite V -paths and decreasing rays are the same objects.

R. Forman proved that, in the finite case, gradient paths are indicating the decreasing V -paths of f [7]. Hence, a gradient vector field does not contain any closed gradient path. Moreover, he proved that the converse is true, characterizing discrete vector fields V on finite complexes induced by a discrete Morse function as those which do not contain closed V -paths. It is easy to show that the same argument can be used in the infinite case.

3 Characterizing discrete Morse functions by the gradient vector field

As we have stated before, we are going to prove that two discrete Morse functions with the same gradient vector field are equivalent. This equivalence notion was introduced by Forman in [8] in the following sense: two discrete Morse functions f and g defined on a simplicial complex M are said to be *equivalent* if every pair of simplices $\alpha^{(p)}$ and $\beta^{(p+1)}$ in M such that $\alpha^{(p)} < \beta^{(p+1)}$ verify that,

$$f(\alpha) < f(\beta) \quad \text{if and only if} \quad g(\alpha) < g(\beta)$$

Then, we have:

Theorem 3.1. *Two discrete Morse functions f and g defined on a simplicial complex M are equivalent if and only if f and g induce the same gradient vector field.*

Proof. Let us suppose that f and g are equivalent discrete Morse functions defined on M . Thus, given any p -simplex α which is a face of a $(p+1)$ -simplex $\beta^{(p+1)}$, it holds that $f(\alpha) < f(\beta)$ if and only if $g(\alpha) < g(\beta)$. Moreover, it is equivalent to state that $f(\alpha) \geq f(\beta)$ if and only if $g(\alpha) \geq g(\beta)$ and, by using the definition of gradient vector field, it is equivalent to state that $(\alpha, \beta) \in V_f$ if and only if $(\alpha, \beta) \in V_g$.

Conversely, let us suppose that f and g induce the same gradient field $V = V_f = V_g$. Note that, since (NC1) and (NC2) can not be satisfied simultaneously by a non critical simplex, we get that any non critical simplex has to be exactly in one pair of V and any critical simplex is not in any pair of V . Then, given any two different simplices $\alpha^{(p)}$ and $\beta^{(p+1)}$, such that $\alpha < \beta$, we have to consider the following cases:

- Both simplices are in the same pair of V , that is, $(\alpha, \beta) \in V$. Then, from the definition of gradient vector field, this implies that $f(\alpha) \geq f(\beta)$ and $g(\alpha) \geq g(\beta)$.
- The simplex α is not in any pair of V and β is in one pair of V . Thus, we get that $f(\alpha) < f(\beta)$ and $g(\alpha) < g(\beta)$, since α is a critical p -simplex of both functions.
- The simplex α is in one pair of V and β is not in any pair of V . Thus, $f(\alpha) < f(\beta)$ and $g(\alpha) < g(\beta)$, since β is a critical $(p+1)$ -simplex of both functions.
- Finally, none of both simplices is in any pair of V . Then, $f(\alpha) < f(\beta)$ and $g(\alpha) < g(\beta)$, since both simplices are critical simplices of both functions.

So we can conclude that $f(\alpha) < f(\beta)$ if and only if $g(\alpha) < g(\beta)$ and, hence, f and g are equivalent. $\square$

A direct consequence of the above theorem is that, in practice, we can use a gradient vector field induced by a discrete Morse function instead of using the function explicitly. Note that we can check if a discrete vector field is the gradient vector field induced by a certain (eventually unknown) discrete Morse function by means of Forman's characterization (Theorem 3.5 of [7]) in the finite case and its extension to the infinite case. A natural question is to know how many discrete Morse functions induce the same gradient vector field. In this sense, following any integration procedure, we can get a discrete Morse function such that its induced gradient vector field is a previously given one V . Now, we can consider a perturbation in the values assigned to such a function by means of

adding to all these values a fixed number in order to get all the functions which induce the discrete vector field V . It shall provide us of infinite (non countable) different discrete Morse functions, whose gradient vector field is V .

On the other hand, we have the following general proposition:

Proposition 3.2. *Any two equivalent discrete Morse functions f and g defined on a simplicial complex M have the same critical simplices.*

Proof. Let $\alpha^{(p)}$ be a critical simplex of f on M . From (C1) and (C2) we have that every simplices $\gamma^{(p-1)} < \alpha^{(p)} < \beta^{(p+1)}$ verify that $f(\gamma^{(p-1)}) < f(\alpha^{(p)}) < f(\beta^{(p+1)})$. Since f and g are equivalent, then $g(\gamma^{(p-1)}) < g(\alpha^{(p)}) < g(\beta^{(p+1)})$ and hence, $\alpha^{(p)}$ is a critical simplex of f on M . Conversely, by using the equivalence of f and g and the definition of critical simplex, we have that any critical simplex of g on M is a critical simplex of f on M . $\square$

The converse of the above proposition does not hold in general. In order to justify it, let us consider the following three examples.

Example 1 (Infinite 1-dimensional case). Consider the following infinite 1-simplicial complex in which we have defined two different gradient vector fields (Figs. 2 and 3) and so, we get two non-equivalent discrete Morse functions. Moreover, both functions have not any critical simplex.

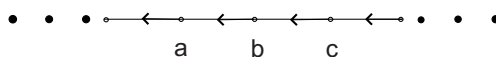


Figure 2

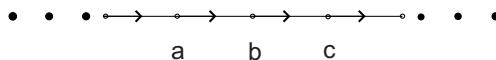


Figure 3

Example 2 (Finite 2-dimensional case). Consider the following finite 2-simplicial complex in which we have defined two different gradient vector fields (Figs. 4 and 5) and so, by Theorem 3.1, we get two non-equivalent discrete Morse functions. However, it is easy to show that they have the same critical simplices, marked by small circles.

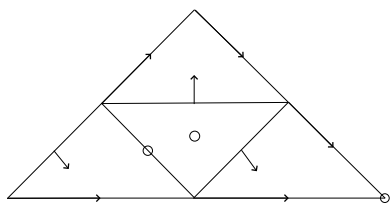


Figure 4

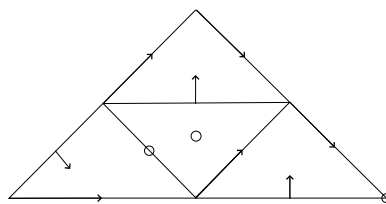


Figure 5

Example 3 (Infinite 2-dimensional case). Consider the following two gradient vector fields (Figs. 6 and 7) defined in the same infinite triangulated cylinder. As we did in the first example, we can check that both fields are induced by two non equivalent discrete Morse functions which have the same critical simplices (none in particular).

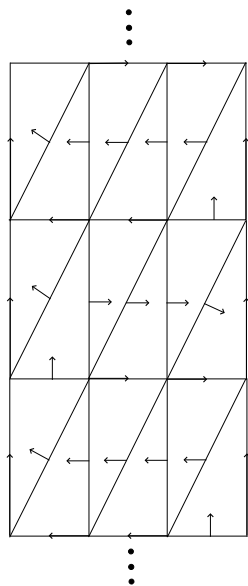


Figure 6

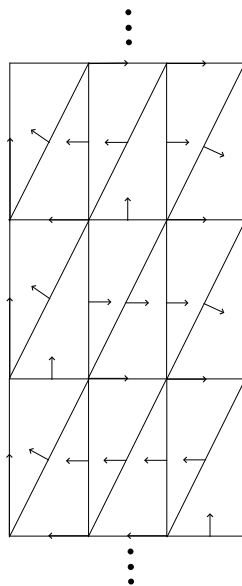


Figure 7

Remark 3.3. However, it is easy to show that the converse of Proposition 3.2 holds for finite 1-dimensional simplicial complexes since it is well known that every critical vertex or edge is either a local minimum or a local maximum [2] and so, by fixing critical simplices, we define the corresponding gradient vector field since it determines a unique increasing way from critical vertices to critical edges.

However, the situation is quite different in the infinite case, as Example 1 shows. To study it, we need to introduce the notion of critical element on an infinite simplicial complex.

Definition 3.4. *Let M be an infinite n -dimensional simplicial complex and let f be a discrete Morse function defined on M . A critical element of f on M is either a critical simplex of f on M or a decreasing i -ray with $1 \leq i \leq n$, up to cofinality.*

Now, we can state the following characterization of discrete Morse functions defined on any 1-dimensional simplicial complex by using its critical elements.

Theorem 3.5. *Two discrete Morse functions f and g defined in a 1-dimensional simplicial complex M are equivalent if and only if f and g have the same critical elements.*

Proof. First, let us suppose that M is finite. From Proposition 3.2, we get that f and g have the same critical simplices and hence, the same critical elements (since M is finite, it does not contain decreasing 1-rays). The converse is the above Remark 3.3.

If M is infinite, we can apply Proposition 3.2 to get that equivalent discrete Morse functions have the same critical simplices. Now, we have to prove that both functions have the same decreasing 1-rays too. If this is not the case, we can find one 1-ray in M which is decreasing for f and not decreasing for g , that is, this 1-ray can be either increasing or such that it contains critical simplices of g which were not critical simplices of f . Obviously, the second option is impossible and so, in this 1-ray g has to be increasing. Now, let us consider any edge e of this 1-ray with its two bounding vertices v and w . It holds that $f(v) \geq f(e) > f(w)$ and $g(v) < g(e) \leq g(w)$ and hence f and g are not equivalent.

Conversely, let us suppose that f and g have the same critical elements. Reasoning as we did for the finite case, by fixing critical elements we define the corresponding gradient vector field since it determines a unique increasing way either from critical vertices to critical edges or from decreasing rays to critical edges. $\square$

Notice that for two discrete Morse functions to be equivalent is not enough that they have the same number of critical elements, but these critical elements also have to be exactly the same. For instance, Figures 2 and 3 of Example 1 show two non-equivalent discrete Morse functions with the same number of critical

elements: they have no any critical simplices and they have one decreasing 1-ray. However, these decreasing rays are not cofinal and so, they are different critical elements.

On the other hand, we want to point out that the proof of the converse part of above theorem (either in finite or infinite case) is based on the fact that, in the 1-dimensional case, to fix the gradient vector field is equivalent to fix the critical elements, but this does not hold for dimension greater or equal than 2 as we have checked in the Examples 2 and 3. In particular, Example 2 shows, in the finite 2-dimensional case, that having the same critical simplices is not enough to get the same gradient vector field and Example 3 illustrates, for the infinite 2-dimensional case, that having the same critical elements does not imply to have equal gradient vector fields.

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