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On exposed faces and smoothness

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Abstract. In this paper we study two types of Banach spaces: those whose unit ball exposed faces are pairwise disjoint and those that are smooth. We present original characterizations of both types that easily lead to local and global results on smoothness.

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1 Introduction

Along this paper we will discuss and make use of the following concepts. A convex subset C of the unit sphere S_X of a real Banach space X is said to be

(1) a face of B_X if it verifies the extremal condition: for every $x, y \in \mathsf{B}_X$ and every $t \in (0, 1)$ such that $tx + (1-t)y \in C$, we have that $x, y \in C$;

(2) an exposed face of B_X if there exists $f \in \mathsf{S}_{X^*}$ such that $C = f^{-1}(1) \cap \mathsf{B}_X$.

(3) a maximal face of B_X if it is a maximal element of the set of proper faces of B_X ordered by the inclusion.

If C is a convex subset of S_{X^*} then we say that C is a ω^*-exposed face of B_{X^*} if there exists $x \in \mathsf{S}_X$ such that $C = x^{-1}(1) \cap \mathsf{B}_{X^*}$. If $c \in \mathsf{S}_X$ (S_{X^*}), then we will say that c is a (ω^*-)exposed point of B_X (B_{X^*}) if $\{c\}$ is a (ω^*-)exposed face of B_X (B_{X^*}). And $c \in \mathsf{S}_X$ is said to be a smooth point of B_X if there is only one $f \in \mathsf{S}_{X^*}$ such that $f(c) = 1$. We refer the reader to [1] and [5] for a wider perspective about the above concepts. Sometimes, Banach spaces whose unit ball exposed faces are pairwise disjoint appear as auxiliary hypothesis to partially answer relevant questions. For instance, the problem of proving whether

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a transitive and separable space is rotund is an old and open problem. In [2] it is proved the following result:

Theorem 1.1 (Aizpuru and García-Pacheco, 2006)**.** *Let X be Banach space. If X is transitive, has normal structure, and the exposed faces of B_X are pairwise disjoint, then X is rotund.*

In the next section we will go through the following three classes of Banach spaces:

(1) Spaces whose unit ball maximal faces are pairwise disjoint.

(2) Spaces whose unit ball exposed faces are pairwise disjoint.

(3) Spaces whose unit ball faces are pairwise disjoint.

Observe that, since every maximal face is an exposed face and every exposed face is a face, we have that the first class of Banach spaces contains the second class, which in fact contains the third one. We will show that these inclusions are strict. Notice also that the class of smooth spaces is contained in the second class, and the class of rotund spaces is contained in the third class. We will also show that the inclusion involving smoothness is strict and the one involving rotundity is, actually, an equality.

2 Results

At first, we will characterize the Banach spaces whose unit ball maximal faces are pairwise disjoint. Afterwards, we will prove that that class strictly contains the class of the Banach spaces whose unit ball exposed faces are pairwise disjoint.

Theorem 2.1. *Let X be a Banach space. The following conditions are equivalent:*

(1) *The maximal faces of B_X are pairwise disjoint.*

(2) *For every $x \in \mathsf{S}_X$, the set $\left\{y \in \mathsf{S}_X : [y, x] \subset \mathsf{S}_X\right\}$ is convex.*

Proof. Assume first that the maximal faces of B_X are pairwise disjoint. Let $x \in \mathsf{S}_X$. By hypothesis, there is only one maximal face C containing x. Clearly, $C \subseteq \left\{y \in \mathsf{S}_X : [y, x] \subset \mathsf{S}_X\right\}$. Take now $y \in \mathsf{S}_X$ such that $[y, x] \subset \mathsf{S}_X$. By the Zorn's Lemma, there exists a maximal face containing the segment $[y, x]$, which has to be C. This proves that $C = \left\{y \in \mathsf{S}_X : [y, x] \subset \mathsf{S}_X\right\}$. Assume now that (2) holds. Let C and D be maximal faces of B_X such that $C \cap D \neq \varnothing$. Then, we can find $x \in C \cap D$, so again we have that $C, D \subseteq \left\{y \in \mathsf{S}_X : [y, x] \subset \mathsf{S}_X\right\}$. Since the last set is convex, it has to be contained in a maximal face according to the Zorn's Lemma. By maximality, C and D must be equal to that maximal face and hence they are equal to each other. □

Theorem 2.2. *Let X be a Banach space. If the exposed faces of B_X are pairwise disjoint, then the maximal faces of S_X are also pairwise disjoint. The converse does not hold.*

Proof. To show that the converse does not hold it suffices to consider the 2-dimensional real Banach space whose unit ball is the intersection of the unit circle $x^2 + y^2 \leq 1$ and the strip $-\frac{1}{2} \leq x \leq \frac{1}{2}$. □

It is the turn now of Banach spaces whose unit ball exposed faces are pairwise disjoint. In order to characterize these spaces we will make use of the following fact and of the next lemma.

Fact 2.3. *Let X be Banach space. Let $g, h \in \mathsf{S}_{X^*}$ be norm-attaining and $t \in (0, 1)$. Then, $tg+(1-t)\,h$ is a norm-attaining functional of norm 1 if and only if $\left(g^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(h^{-1}(1) \cap \mathsf{S}_X\right) \neq \varnothing$. In this situation, $(tg + (1-t)\,h)^{-1}(1) \cap \mathsf{B}_X = \left(g^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(h^{-1}(1) \cap \mathsf{B}_X\right)$.*

Remark 2.4. Given a Banach space X, we will consider the relation of equivalence on S_{X^*} given by

$$\mathcal{R} = \left\{(f, g) \in \mathsf{S}_{X^*} \times \mathsf{S}_{X^*} : f^{-1}(1) \cap \mathsf{B}_X = g^{-1}(1) \cap \mathsf{B}_X\right\}.$$

The elements of the quotient set $\mathsf{S}_{X^*}/\mathcal{R}$ which will be denoted by $[f]_{\mathcal{R}}$ with $f \in \mathsf{S}_{X^*}$.

Lemma 2.5. *Let X be a Banach space. Let $f \in \mathsf{S}_{X^*}$. Then:*

(1) *f is norm-attaining if and only if $[f]_{\mathcal{R}}$ is convex.*

(2) *f is not norm-attaining if and only if $[f]_{\mathcal{R}}$ is symmetric.*

(3) *Assume that f is norm-attaining. The following are equivalent:*

   (a) *$f^{-1}(1) \cap \mathsf{B}_X$ is a maximal face of B_X.*
   (b) *$[f]_{\mathcal{R}}$ is a face of B_{X^*}.*
   (c) *$[f]_{\mathcal{R}} = \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}$.*
   (d) *$[f]_{\mathcal{R}}$ is ω^*-closed.*
   (e) *$[f]_{\mathcal{R}}$ is closed.*

(4) *Assume that f is not norm-attaining. Then,*

$$\mathrm{cl}\left([f]_{\mathcal{R}}\right) \supseteq \left\{C \subset \mathsf{S}_{X^*} : C \text{ is convex and } C \cap [f]_{\mathcal{R}} \neq \varnothing\right\}.$$

Proof.

(1) Assume first that f is norm-attaining. Let $g, h \in [f]_{\mathcal{R}}$ and $t \in (0,1)$. By applying Fact 2.3, we have that

$$\begin{aligned}(tg + (1-t)\,h)^{-1}(1) \cap \mathsf{B}_X &= \left(g^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(h^{-1}(1) \cap \mathsf{B}_X\right) \\ &= f^{-1}(1) \cap \mathsf{B}_X,\end{aligned}$$

that is, $tg + (1-t)\,h \in [f]_{\mathcal{R}}$. Conversely, assume that $[f]_{\mathcal{R}}$ is convex. If f is not norm-attaining, then $-f$ is not either, therefore $-f \in [f]_{\mathcal{R}}$. Since $[f]_{\mathcal{R}}$ is convex, we deduce that $0 \in [f]_{\mathcal{R}} \subseteq \mathsf{S}_{X^*}$, which is impossible.

(2) Assume first that f is not norm-attaining. Then, $-f$ is not either, so $-f \in [f]_{\mathcal{R}}$. Conversely, assume that $[f]_{\mathcal{R}}$ is symmetric. If f is norm-attaining, then by paragraph (1) we have that $[f]_{\mathcal{R}}$ is convex, which means the contradiction that $0 \in [f]_{\mathcal{R}} \subseteq \mathsf{S}_{X^*}$.

(3) Assume first that $f^{-1}(1) \cap \mathsf{B}_X$ is a maximal face of B_X. Let $t \in (0,1)$ and $g, h \in \mathsf{S}_{X^*}$ such that $tg + (1-t)\,h \in [f]_{\mathcal{R}}$. By Fact 2.3, $f^{-1}(1) \cap \mathsf{B}_X \subseteq g^{-1}(1) \cap \mathsf{B}_X, h^{-1}(1) \cap \mathsf{B}_X$. By maximality, both $g, h \in [f]_{\mathcal{R}}$. Secondly, assume that $[f]_{\mathcal{R}}$ is a face of B_{X^*}. On the one hand, if $g \in [f]_{\mathcal{R}}$, then $g^{-1}(1) \cap \mathsf{B}_X = f^{-1}(1) \cap \mathsf{B}_X$, therefore $g \in x^{-1}(1) \cap \mathsf{B}_{X^*}$ for every $x \in f^{-1}(1) \cap \mathsf{B}_X$. As a consequence,

$$[f]_{\mathcal{R}} \subseteq \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}.$$

On the other hand, if $g \in \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}$, then $f^{-1}(1) \cap \mathsf{B}_X \subseteq g^{-1}(1) \cap \mathsf{B}_X$, therefore $\frac{f+g}{2} \in [f]_{\mathcal{R}}$ in accordance to Fact 2.3. Since $[f]_{\mathcal{R}}$ is a face of B_{X^*}, we deduce that $g \in [f]_{\mathcal{R}}$. As a consequence,

$$[f]_{\mathcal{R}} = \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}.$$

Thirdly, if $[f]_{\mathcal{R}} = \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}$, then $[f]_{\mathcal{R}}$ is an intersection of ω^*-closed sets, so it is ω^*-closed and thus closed. Finally, suppose that $[f]_{\mathcal{R}}$ is closed. Let $g \in \mathsf{S}_{X^*}$ be such that $f^{-1}(1) \cap \mathsf{B}_X \subseteq g^{-1}(1) \cap \mathsf{B}_X$. Then, by Fact 2.3, we have that, for every $n > 1$,

$$\begin{aligned}\left(\frac{1}{n} f + \left(1 - \frac{1}{n}\right) g\right)^{-1} (1) \cap \mathsf{B}_X &= \left(f^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(g^{-1}(1) \cap \mathsf{B}_X\right) \\ &= f^{-1}(1) \cap \mathsf{B}_X.\end{aligned}$$

Then, $\left(\frac{1}{n}f+\left(1-\frac{1}{n}\right)g\right)_{n>1}$ is a sequence in $[f]_{\mathcal{R}}$ that converges to g. By hypothesis, $g \in [f]_{\mathcal{R}}$ and this proves that $f^{-1}(1) \cap \mathsf{B}_X$ is a maximal face.

(4) Let $g \in C$ where C is a convex subset of S_{X^*} such that $C \cap [f]_{\mathcal{R}} \neq \varnothing$. Let $h \in C \cap [f]_{\mathcal{R}}$. By Fact 2.3, we have that $\left(\frac{1}{n}h+\left(1-\frac{1}{n}\right)g\right)_{n>1}$ is a sequence of $[f]_{\mathcal{R}}$ that converges to g.

□

As we will see, Lemma 2.5 will have several consequences on smoothness. The first one is a local characterization. The hypothesis of separability is strictly necessary.

Theorem 2.6. *Let X be a separable Banach space. Let $f \in \mathsf{S}_{X^*}$ be a norm-attaining functional. The following conditions are equivalent:*

(1) *$[f]_{\mathcal{R}}$ contains a ω^*-exposed point of B_{X^*}.*

(2) *$f^{-1}(1) \cap \mathsf{B}_X$ contains a smooth point of B_X.*

(3) *$[f]_{\mathcal{R}} = \{f\}$.*

Proof. First, suppose that there exists $g \in [f]_{\mathcal{R}}$ that is a ω^*-exposed point of B_{X^*}. Let then $x \in \mathsf{S}_X$ such that $x^{-1}(1) \cap \mathsf{B}_{X^*} = \{g\}$. We clearly have that $x \in f^{-1}(1) \cap \mathsf{B}_X$ and x is a smooth point of B_X. Second, suppose that $f^{-1}(1) \cap \mathsf{B}_X$ contains a smooth point x of B_X. Then, $g(x) = 1$ for all $g \in [f]_{\mathcal{R}}$, so $[f]_{\mathcal{R}} = \{f\}$. Last, suppose that $[f]_{\mathcal{R}} = \{f\}$. We have that $[f]_{\mathcal{R}}$ is obviously closed, thus, by Lemma 2.5, we have that

$$[f]_{\mathcal{R}} = \bigcap_{x \in f^{-1}(1) \cap \mathsf{B}_X} x^{-1}(1) \cap \mathsf{B}_{X^*}.$$

Since X is separable, we deduce by [3, Lemma 2.4] that f is a ω^*-exposed point of B_{X^*}. □

Observe that, in the previous theorem, (1) and (2) are always equivalent and both imply (3). However, (3) does not necessarily imply (1) or (2) as shown as follows.

Theorem 2.7. *Let K be a Hausdorff, compact topological space. Let $t \in K$. Then:*

(1) *$[\delta_t]_{\mathcal{R}} = \{\delta_t\}$.*

(2) *If $K \setminus \{t\}$ is pseudo-compact but not compact, then $\delta_t^{-1}(1) \cap \mathsf{B}_{C(K)}$ is free of smooth points.*

Proof.

(1) Let μ be a finitely additive regular measure on $\widehat{L}$ of bounded variation equal to 1 such that $\mu \in [\delta_t]_{\mathcal{R}}$. Notice that, since the constant function equal to 1 belongs to $\delta_t^{-1}(1) \cap \mathsf{B}_{C(K)}$, we have that μ is, in fact, a probability measure. We will show that $\mu(\{t\}) = 1$, which directly implies that $\mu = \delta_t$. In order to do that, since μ is regular, we will show that, for any open neighborhood U of t, $\mu(U) = 1$. So, assume that there exists an open neighborhood U of t such that $\mu(U) < 1$. Now, the continuous function

$$\begin{array}{rrcl} g: & \{t\} \cup K \setminus U & \to & [-1,1] \\ & k & \mapsto & g(k) = \begin{cases} 1 & \text{if } k = t, \\ 0 & \text{if } k \in \widehat{L} \setminus U, \end{cases} \end{array}$$

can be continuously extended to a function $f : K \to [-1,1]$. Clearly, we have that $f \in C$, but the inequalities

$$\begin{aligned} 1 &= \left| \int_K f d\mu \right| \\ &\leq \int_K |f|\, d\mu \\ &\leq \sup |f(U)|\, \mu(U) + \sup |f(K \setminus U)|\, \mu\left(\widehat{L} \setminus U\right) \\ &= \mu(U) \\ &< 1 \end{aligned}$$

give us a contradiction.

(2) Let $f \in \delta_t^{-1}(1) \cap \mathsf{B}_{C(K)}$. On the one hand, since $K \setminus \{t\}$ is not compact, we have that it is not closed, therefore $\mathrm{cl}(K \setminus \{t\}) = K$. This means that $\sup f(K \setminus \{t\}) = 1$. On the other hand, $K \setminus \{t\}$ is pseudo-compact, therefore there exists $s \in K \setminus \{t\}$ such that $f(s) = 1$. Now, $\delta_s \neq \delta_t$ and $\delta_s(f) = 1 = \delta_t(f)$. This proves that f is not a smooth point of $\mathsf{B}_{C(K)}$.

□

Remark 2.8. The space Ω_1 of all countable ordinals with the order topology is a pseudo-compact, Hausdorff, locally compact topological space that is not compact. Let $\widehat{\Omega_1} := \Omega_1 \cup \{\infty\}$ denote the one-point compactification of Ω_1. According to the previous theorem, $[\delta_\infty]_{\mathcal{R}} = \{\delta_\infty\}$ but $\delta_\infty^{-1}(1) \cap \mathsf{B}_{C(\widehat{\Omega_1})}$ is free of smooth points.

The second consequence of Lemma 2.5 is a global characterization of smoothness. This time we do not precise of any additional hypothesis like separability, etc.

Theorem 2.9. *Let X be a Banach space. The following conditions are equivalent:*

(1) *X is smooth.*

(2) *$[f]_{\mathcal{R}} = \{f\}$ for every norm-attaining functional $f \in \mathsf{S}_{X^*}$.*

Proof. By definition, if X is smooth then $[f]_{\mathcal{R}} = \{f\}$ for every norm-attaining functional f of S_{X^*}. Conversely, suppose that $x \in \mathsf{S}_X$ and f and g are functionals of S_{X^*} such that $f(x) = g(x) = 1$. Then, according to Fact 2.3,

$$(\alpha f + (1-\alpha) g)^{-1}(1) \cap \mathsf{B}_X = \left(f^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(g^{-1}(1) \cap \mathsf{B}_X\right)$$

for every $\alpha \in (0,1)$. In other words, for every $\alpha \neq \beta \in (0,1)$, $\{\alpha f + (1-\alpha)g\} = [\alpha f + (1-\alpha) g]_{\mathcal{R}} = [\beta f + (1-\beta) g]_{\mathcal{R}} = \{\beta f + (1-\beta) g\}$, which implies $f = g$. □

As a last application of Lemma 2.5, we present now a characterization of Banach spaces in which the exposed faces of the unit ball are pairwise disjoint.

Theorem 2.10. *Let X be a Banach space. The following conditions are equivalent:*

(1) *The exposed faces of B_X are pairwise disjoint.*

(2) *$[f]_{\mathcal{R}} = x^{-1}(1) \cap \mathsf{B}_{X^*}$ for every norm-attaining $f \in \mathsf{S}_{X^*}$ and every $x \in f^{-1}(1) \cap \mathsf{B}_X$.*

(3) *$[f]_{\mathcal{R}}$ is a ω^*-exposed face of B_{X^*} for every norm-attaining $f \in \mathsf{S}_{X^*}$.*

(4) *If $x, y \in \mathsf{S}_X$ and $[x, y] \subset \mathsf{S}_X$, then both x and y are smooth points of the unit ball of* $\operatorname{span}\{x, y\}$.

Proof. In the first place, assume that the exposed faces of B_X are pairwise disjoint. Let us fix an arbitrary norm-attaining $f \in \mathsf{S}_{X^*}$ and an element $x \in f^{-1}(1) \cap \mathsf{B}_X$. We already know from Lemma 2.5 that $[f]_{\mathcal{R}} \subseteq x^{-1}(1) \cap \mathsf{B}_{X^*}$. Now, if $g \in x^{-1}(1) \cap \mathsf{B}_{X^*}$, then $f^{-1}(1) \cap \mathsf{B}_X$ and $g^{-1}(1) \cap \mathsf{B}_X$ have non-empty intersection, thus $g \in [f]_{\mathcal{R}}$ by hypothesis. As a consequence, $[f]_{\mathcal{R}}$ is a ω^*-exposed face of B_{X^*}. In the second place, suppose that (3) holds. Let $x \neq y \in \mathsf{S}_X$ with $[x, y] \subset \mathsf{S}_X$. It suffices to show that x is a smooth point of the unit ball of $Y = \operatorname{span}\{x, y\}$, since the same argument can be applied to y. So, suppose to the contrary that x is not a smooth point of B_Y. Then, by [3, Theorem 2.1], we have that x is an exposed point of B_Y. Thus, let $f \in \mathsf{S}_{Y^*}$ be the functional that supports B_Y only at x. Let $g \in \mathsf{S}_{Y^*}$ be such that $g([x,y]) = \{1\}$. Let f_0 and g_0 be the Hahn-Banach extensions of f and g with norm 1, respectively. By hypothesis, we have that $g_0 \in x^{-1}(1) \cap \mathsf{B}_{X^*} = [f_0]_{\mathcal{R}}$, which means the contradiction that $f_0^{-1}(1) \cap \mathsf{B}_X = g_0^{-1}(1) \cap \mathsf{B}_X$. In the third and last place, assume that (3) holds. Let $f, g \in \mathsf{S}_{X^*}$ be two norm-attaining functionals such that $f^{-1}(1) \cap \mathsf{B}_X$ and $g^{-1}(1) \cap \mathsf{B}_X$ are different and have non-empty intersection. Take any element x in that intersection, and suppose without loss of generality the existence of an element $y \in f^{-1}(1) \cap \mathsf{B}_X \setminus g^{-1}(1) \cap \mathsf{B}_X$. Now, $[x,y] \subset \mathsf{S}_X$ so x is a smooth point of $Y = \operatorname{span}\{x,y\}$. Next, $f|_Y(x) = f(x) = 1 = g(x) = g|_Y(x)$, therefore $f|_Y = g|_Y$. Then, $g(y) = g|_Y(y) = f|_Y(y) = f(y) = 1$ which is a contradiction. □

Corollary 2.11. *Let X be a Banach space. If the exposed faces of B_X are pairwise disjoint, then $[f]_{\mathcal{R}}$ is a ω^*-exposed face of B_{X^*} for every $f \in \mathsf{S}_{X^*}$ such that $f^{-1}(1) \cap \mathsf{B}_X$ is a maximal face of B_X. The converse does not hold.*

Proof. To show that the converse does not hold, we look at any separable Banach space X whose unit ball exposed faces are not pairwise disjoint (like for instance, c_0 or ℓ_1). Observe that, since X is separable, then $(\mathsf{S}_{X^*}, \omega^*)$ is a separable metric space, therefore, for every $f \in \mathsf{S}_{X^*}$ such that $f^{-1}(1) \cap \mathsf{B}_X$ is a maximal face of B_X, we have that $[f]_{\mathcal{R}}$ can be expressed as a countable intersection of ω^*-exposed faces, so it is a ω^*-exposed face (see [3, Lemma 2.4] for more details.) □

Theorem 2.12. *Let X be a Banach space. If X is smooth or the faces of B_X are pairwise disjoint, then the exposed faces of S_X are also pairwise disjoint. The converse does not hold.*

Proof. To show that the converse does not hold, it is sufficient to consider the 2-dimensional real Banach space whose unit ball is the intersection of the following two bodies:

(1) The union of the square $|x|, \left|y - \frac{1}{2}\right| \leq 1$ and the two circles $(x-1)^2 + \left(y - \frac{1}{2}\right)^2 \leq 1$ and $(x+1)^2 + \left(y - \frac{1}{2}\right)^2 \leq 1$.

(2) The union of the square $|x|, \left|y + \frac{1}{2}\right| \leq 1$ and the two circles $(x-1)^2 + \left(y^2 + \frac{1}{2}\right) \leq 1$ and $(x+1)^2 + \left(y + \frac{1}{2}\right)^2 \leq 1$.

□

To finalize the manuscript, we will show now that, opposite to above, the rotund Banach spaces are exactly those spaces whose unit ball faces are pairwise disjoint.

Theorem 2.13. *Let X be a Banach space. The following conditions are equivalent:*

(1) *The faces of B_X are pairwise disjoint.*

(2) *X is rotund.*

Proof. If X is rotund, then the only faces are the extreme points. Conversely, suppose that X is not rotund. Then, there exists a face C with $\operatorname{diam}(C) > 0$. In accordance to [4, Chapter 2], C has a proper subface D, with contradicts the hypothesis. □

References

[1] A. Aizpuru and F.J. García-Pacheco, *Some questions about rotundity and renormings in Banach spaces*. J. Austral. Math. Soc., **79** (2005), 131–140.

[2] A. Aizpuru and F.J. García-Pacheco, *Rotundity in transitive and separable Banach spaces*. Quaest. Math., **30** (2007), 1–12.

[3] A. Aizpuru and F.J. García-Pacheco, *A short note about exposed points in real Banach spaces*. Acta Math. Sci. Ser. B Engl. Ed., **28B**(4) (2008), 797–800.

[4] F.J. García-Pacheco, *Four non-linear problems on normed spaces*. VDM Verlag, (2008).

[5] J.P. Moreno, *On the ω^*-Mazur intersection property and Fréchet differentiability norms on dense open sets*. Bull. Sci. Math., **122**(2) (1998), 93–105.

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