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The Aubry set for periodic Lagrangians on the circle

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Abstract. For periodic convex Lagrangians on $\mathbb{S}^1$, we show that, generically, in the sense of Mañé, there exists a dense open set of cohomology classes such that the Aubry set of these Lagrangians is a hyperbolic periodic orbit. This allows us to prove Mañé's conjecture on $\mathbb{S}^1$.

Keywords: minimizing measures, Aubry set, beta function.

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1 Introduction

Let M be a closed manifold, a C^∞ function $L : TM \times \mathbb{S}^1 \to \mathbb{R}$ will be a periodic Lagrangian (with period 1) if it satisfies the convexity completeness and superlinearity condition. A periodic Lagrangian defines a flow $\varphi_t : TM \times \mathbb{S}^1 \hookleftarrow$, $t \in \mathbb{R}$. The goal of this work is to study the Aubry set for periodic Lagrangians. The Aubry set is a specific invariant set associated with L. We restricted our study to the case $M = \mathbb{S}^1$. Even though it is a particular case, it contains interesting situations, for instance, J. Moser in [9] showed that any exact twist map on the cilinder $\mathbb{S}^1 \times \mathbb{R}$ can be regarded as the time one map $\varphi_t : T\mathbb{S}^1 \times \{0\} \hookleftarrow$ of the flow of a periodic Lagrangian on $\mathbb{S}^1$.

Our main result is the following one:

Theorem A. *For a generic Lagrangian $L : T\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}, (x, v, t) \to L(x, v, t)$ there exists a dense open set $U \subset H^1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R}$ such that $\forall c \in U$, the Aubry set of $(x, v, t) \to L(x, v, t) - cv$ is a hyperbolic periodic orbit.*

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2 Preliminaries

Let M be a closed Riemannian manifold, a Lagrangian L on M will be a C^∞ function $L : TM \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions

i) Periodicity: $L(\theta, t+1) = L(\theta, t)$ for all $(\theta, t) \in TM \times \mathbb{R}$,

ii) Convexity: For all $x \in M$, $v \in T_xM$ and $t \in \mathbb{R}$, the Hessian matrix calculated in any linear coordinate system on T_xM is positive definite,

iii) Superlinearity: It holds that $\lim_{||v||\to\infty} \frac{L(x,v,t)}{||v||} = +\infty$ uniformly on $x \in M,\ t \in \mathbb{R}$,

iv) Completeness: The Euler-Lagrange equation of L, defines a complete flow φ_t on $TM \times \mathbb{R}$.

Let us recall the main concepts introduced by Mather in [7]. Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra on $TM \times \mathbb{S}^1$ that have compact support and are invariant under the Euler-Lagrange flow φ_t. Let $H_1(M, \mathbb{R})$ be the first real homology group of M. Given a closed one-form ω on M and $\rho \in H_1(M, \mathbb{R})$, let $\langle [\omega], \rho \rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its homology is defined as the unique $\rho(\mu) \in H_1(M, \mathbb{R})$ such that

$$\langle [\omega], \rho(\mu) \rangle = \int_{TM\times\mathbb{S}^1} \omega d\mu, \tag{1}$$

for any closed 1-form on M. The integral on the right-hand side is with respect to μ, with ω considered as a function $\omega : TM \times \mathbb{S}^1 \to \mathbb{R}$, independent of the second variable.

The action of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) := \int_{TM\times\mathbb{S}^1} L d\mu. \tag{2}$$

We say that $\mu \in \mathcal{M}(L)$ is a minimizing measure of L if

$$A_L(\mu) := \min \left\{ A_L(\nu) |\ \nu \in \mathcal{M}(L), \rho(\nu) = \rho(\mu) \right\}. \tag{3}$$

Mather in ([7]) proved that for every $h \in H_1(M, \mathbb{R})$, there exist minimizing measures $\mu \in \mathcal{M}(L)$ with $\rho(\mu) = h$ and has denoted by $\mathcal{M}_h(L)$ the subset of $\mathcal{M}(L)$ consisting of these measures.

Given a closed 1-form ω on M, note that the function $L - \omega$ satisfies the conditions i)–iv) and has the same solutions as L. We define $\mathcal{M}^\omega(L)$ as the set of $\mu \in \mathcal{M}(L)$ such that

$$A_{L-\omega}(\mu) := \min \left\{ A_{L-\omega}(\nu) |\ \nu \in \mathcal{M}(L) \right\}, \tag{4}$$

note that $\mathcal{M}^\omega(L) = \mathcal{M}^0(L - \omega)$.

We define the Mather's beta function $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) := \min\left\{A_L(\mu) \mid \mu \in \mathcal{M}(L),\ \rho(\mu) = h\right\}, \tag{5}$$

the β-function is convex and superlinear. The Mather's alpha function $\alpha = \beta^* : H^1(M, \mathbb{R}) \to \mathbb{R}$ is the convex dual of β, in particular

$$\alpha(\omega) := -\min\left\{A_{L-\omega}(\mu) \mid \mu \in \mathcal{M}(L)\right\}, \tag{6}$$

thus for $[\omega] \in H^1(M, \mathbb{R})$, we have

$$\mathcal{M}^\omega(L) = \left\{\mu \in \mathcal{M}(L) \mid A_{L-\omega}(\mu) = -\alpha([\omega])\right\}.$$

Given $[\omega] \in H^1(M, \mathbb{R})$, let Λ^ω be the closure of the union of the supports of all measures in $\mathcal{M}^\omega(L)$. This is called the *Mather set*. We also consider the *Aubry set* $\mathcal{A}_\omega(L)$, see [1] for its definition. It is known that these sets only depend on $[\omega]$, and these are compact invariant sets for the Euler-Lagrange flow and $\Lambda^\omega \subset \mathcal{A}_\omega(L)$.

Let h be a homology class. A cohomology class ω is said to be a subderivative for β at h if $\langle[\omega], h\rangle = \beta(h) + \alpha(\omega)$. The subderivatives for β at h form a face F_h of the epigraph of the alpha function. If $\omega \in F_h$, then $\mathcal{M}_h(L) \subset \mathcal{M}^\omega(L)$ (see [7]).

In [6], Proposition 6, it has been proved that for cohomology classes $[\omega_0]$, $[\omega_1] \in H^1(M, \mathbb{R})$ in the interior of a face F_h, then $\mathcal{A}_{\omega_0} = \mathcal{A}_{\omega_1}$. Since the Mather set $\Lambda^{[\omega]}$ is the closure of the union of the support of all invariant measures contained in $\mathcal{A}_{[\omega]}$, for $[\omega_0], [\omega_1] \in \text{int}(F_h)$ implies $\Lambda^{[\omega_0]} = \Lambda^{[\omega_1]}$. We denote by $\mathcal{A}(F_h)$ and $\Lambda(F_h)$ this common set to all the cohomologies in the interior of F_h. We have the following result:

Lemma 2.1. *Let L be a periodic Lagrangian on $\mathbb{S}^1$. If h is a rational homology then $\Lambda(F_h) = \mathcal{A}(F_h)$.*

We say that $h \in H_1(M, \mathbb{R})$ is a vertex of β if there exists an open set of subderivatives for β at h. Note that when $\dim H_1(M, \mathbb{R}) = 1$, β is differentiable at $h \in H_1(M, \mathbb{R})$ if and only if h is not a vertex of β, in particular the faces of alpha correspond to vertices of beta.

It was proved in [8] (see also [10]); when $M = \mathbb{S}^1$ and L is a periodic Lagrangian, that

a) β is differentiable at every $h \in H_1(\mathbb{S}^1, \mathbb{R})$ irrational;

b) If $h \in H_1(\mathbb{S}^1, \mathbb{R})$ is a vertex of β, then h is rational and the ergodic measures in $\mathcal{M}_h(L)$ are supported in periodic orbits.

Denote by $C^\infty(M \times \mathbb{S}^1, \mathbb{R})$ the set of C^∞ real functions on $M \times \mathbb{S}^1$ with the topology of the uniform convergence. This is a complete metric space. We say that a property is *generic* in the sense of Mañé [5], if for each periodic Lagrangian L, there exists a residual subset $\mathcal{O} \subset C^\infty(M \times \mathbb{S}^1, \mathbb{R})$ such that, the given property holds for every Lagrangian of the form $L + \phi$, with $\phi \in \mathcal{O}$. Recall that a set is residual if it is a countable intersection of open and dense sets. Since $C^\infty(M \times \mathbb{S}^1, \mathbb{R})$ is complete, then any residual set is dense.

3 Proofs

Let's begin with the following results:

Proof of Lemma 2.1. We consider $H_1(\mathbb{S}^1, \mathbb{R}) \equiv \mathbb{R} \equiv H^1(\mathbb{S}^1, \mathbb{R})$, and take $[\omega] \in F_h$ with h a rational homology. From Mather's Graph theorem, the map $\pi^{-1} : \pi(\mathcal{A}_{[\omega]}) \to \mathcal{A}_{[\omega]}$ is Lipschitz. Therefore we can induce a flow ψ_t on $\pi(\mathcal{A}_{[\omega]})$, and by considering the Poincaré return map, we obtain an homeomorphism that preserves order. Thus we can use the theory of homeomorphisms of the circle. In our case the Poincaré return map has a rotation number h rational, and we then see that $\mathcal{A}_{[\omega]}$ is a union of periodic orbits.

Since $\Lambda^{[\omega]} \subset \mathcal{A}_{[\omega]}$, the first part implies that $\Lambda^{[\omega]}$ is a union of periodic orbits. Assume that $\mathcal{A}_{[\omega]} \setminus \Lambda^{[\omega]} \neq \emptyset$, and take $\theta \in \mathcal{A}_{[\omega]} \setminus \Lambda^{[\omega]}$. Let γ_θ be the periodic orbit through θ. Now consider open neihborhoods V, V_θ in TM of $\Lambda^{[\omega]}$ and γ_θ respectively. Since $\Lambda^{[\omega]}$ and γ_θ are compact sets, we can assume that $V \cap V_\theta = \emptyset$. Since the measure μ_{γ_θ} supported on γ_θ is minimizing, we must have $\text{supp}(\gamma_\theta) \subset \Lambda^{[\omega]}$, which is a contradiction, and $\mathcal{A}_{[\omega]} = \Lambda^{[\omega]}$.

Now, by taking $[\omega] \in \text{int}(F_h)$, we then obtain $\Lambda(F_h) = \mathcal{A}(F_h)$. □

Lemma 3.1. *Let L be a periodic Lagrangian on $\mathbb{S}^1$. The set $S(L)$ of subderivatives of β at rational homology classes is dense in $H^1(\mathbb{S}^1, \mathbb{R})$.*

Proof. First observe that since the α-function is convex and superlinear, for every $[\omega] \in H^1(\mathbb{S}^1, \mathbb{R})$ there is $h \in H_1(\mathbb{S}^1, \mathbb{R})$ such that $\langle [\omega], h \rangle = \beta(h) + \alpha(\omega)$.

Assume that there exists an open set $G \subset H^1(\mathbb{S}^1, \mathbb{R})$ such that $G \cap S(L) = \emptyset$. We may assume G to be convex. Then the set

$$H := \left\{ h \in H_1(\mathbb{S}^1, \mathbb{R}) \mid \exists \omega \in G, \langle [\omega], h \rangle = \beta(h) + \alpha(\omega) \right\}$$

is also convex and by the above observation $H \neq \emptyset$.

Now, since any convex set with two different points contains a rational, then $H = \{h_0\}$ for some $h_0 \in H_1(\mathbb{S}^1, \mathbb{R})$. Therefore, h_0 is a vertex (the elements of G are subderivatives of β at h_0), and from b) of section 2, h_0 is rational. Thus $G \cap S(L) \neq \emptyset$, which proves the result. □

Proof of Theorem A. The proof closely follows the ideas of Massart in [6].

Given a periodic Lagrangian L on $\mathbb{S}^1$, from [3] there exists a residual set $\mathcal{H} \subset C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$, such that $\beta_{L+\phi}$ is not differentiable at any rational point, for all $\phi \in \mathcal{H}$.

We consider a fixed homology $r \in \mathbb{Q} \subset H_1(\mathbb{S}^1, \mathbb{R})$. In [5], th. D, Mañé proves that there exists a residual set $\mathcal{O}_r \subset C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$ such that for all $\phi \in \mathcal{O}_r$, $\mathcal{M}_r(L+\phi)$ has a unique minimizing measure and this measure is uniquely ergodic. Then for all $\phi \in \mathcal{H} \cap O_r$ there exists a unique measure $\mu_{\phi,r} \in \mathcal{M}_r(L+\phi)$. Since $\phi \in \mathcal{H}$, then r is vertex of $\beta_{L+\phi}$. Therefore, from Lemma 2.1, $\mu_{\phi,r}$ is supported on periodic orbits. Moreover $\mu_{\phi,r}$ is uniquely ergodic. Therefore $\operatorname{supp}(\mu_{\phi,r})$ is a single periodic orbit, and from [2] we can assume that is hyperbolic.

Taking $\mathcal{O} := \mathcal{H} \cap (\cap_{r\in\mathbb{Q}} \mathcal{O}_r) \subset C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$ it is a residual subset of $C^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R})$. We then get for all $\phi \in \mathcal{O}$ and for all $r \in \mathbb{Q}$, there exists a uniquely ergodic measure in $\mathcal{M}_r(L+\phi)$ supported on a single hyperbolic periodic orbit.

For $\phi \in \mathcal{O}$, we consider the set $S'(L+\phi)$ of cohomology classes subderivative to $\beta_{L+\phi}$ at rational homology and in the interior of a face (since $\phi \in \mathcal{H}$, the rational homology classes are vertices). We then get that $S'(L+\phi)$ is open and by Lemma 3.1 is dense.

Now, for all $[\omega] \in S'(L+\phi)$, $\Lambda^{[\omega]}(L+\phi)$ consists of a single hyperbolic periodic orbit, and from Lemma 2.1 $\Lambda^{[\omega]}(L+\phi) = \mathcal{A}_{[\omega]}(L+\phi)$. In conclusion, for all

$$\phi \in \mathcal{O},\ [\omega] \in S'(L+\phi),\ \Lambda^{[\omega]}(L+\phi) = \mathcal{A}_{[\omega]}(L+\phi)$$

is a hyperbolic periodic orbit. □

Mañé ([5], [2]) showed that for a generic Lagrangian, there exists a unique minimizing measure and it is conjectured in [5] *(Mañé): For a generic Lagrangian L on a closed manifold M, there exists a dense open set $U \subset H^1(M, \mathbb{R})$ such that $\forall \omega \in U$, Λ^ω consists of a single periodic orbit, or fixed point.*

Note that for periodic Lagrangians, there do not exist fixed points. On the other hand, the Aubry set contains the Mather set, and thus as an application of Theorem A, we have:

Corollary 3.2. *The Mañé's conjecture is true for periodic Lagrangians on* $\mathbb{S}^1$.

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