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Coherent orientation of mixed moduli spaces in Morse-Floer theory

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Abstract. We give the coherent orientation for the spaces of intersections of gradient trajectories and holomorphic disks in cotangent bundle. This construction provides the Piunikhin-Salamon-Schwarz isomorphism between Morse homology and Floer homology for Lagrangian intersections in cotangent bundles, with integer coefficients.

Keywords: Lagrangian submanifolds, Floer homology, Morse theory, coherent orientation.

Mathematical subject classification: Primary: 53D40; Secondary: 57R58, 37J05, 53D12.

1 Introduction

Let M be a compact manifold, denote by n its dimension. Suppose $f : M \to \mathbb{R}$ is a Morse function. Denote by $CM_*(f)$ the Morse chain groups, i.e. $\mathbb{Z}_2$-vector spaces generated by critical points of f. Morse homology groups $HM_*(f)$ are the homology groups of $CM_*(f)$ with respect to the boundary operator

$$\partial_M : CM_*(f) \to CM_*(f), \qquad \partial_M(p) := \sum_{q \in Crit(f)} n(p,q)q, \tag{1}$$

where $n(p,q)$ is the number of solutions of

$$\begin{cases} \dfrac{d\gamma}{ds} + \nabla f(\gamma(s)) = 0 \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q. \end{cases}$$

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Denote by $P = T^*M$ a cotangent bundle of M. Let $H : T^*M \times [0, 1] \to \mathbb{R}$ be a compactly supported Hamiltonian. Denote by $L_0 = O_M$ a zero section of T^*M and by $L_1 = \phi_1^H(L_0)$ a Hamiltonian deformation of L_0. Suppose that L_0 and L_1 intersect transversally. Let $CF_*(H)$ denote Floer chain groups generated by intersection points of L_0 and L_1 (or, equivalently, by Hamiltonian paths that begin and end on O_M), also with $\mathbb{Z}_2$ coefficients. Floer homology $HF_*(H)$ is well defined in this situation. It is the homology group of $CF_*(H)$ with respect to the boundary operator,

$$\partial_F : CF_*(H) \to CF_*(H), \qquad \partial_F(x) := \sum_{y \in L_0 \cap L_1} n(x, y)y, \tag{2}$$

where $n(x, y)$ is the number of the solutions of an elliptic system

$$\begin{cases} u : \mathbb{R} \times [0, 1] \to T^*M \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(s, i) \in L_0, \ i \in \{0, 1\} \\ u(-\infty, t) = \phi_t^H\big((\phi_1^H)^{-1}\big)(x) \\ u(+\infty, t) = \phi_t^H\big((\phi_1^H)^{-1}\big)(y), \ x, y \in L_0 \cap L_1. \end{cases}$$

Here J is an almost complex structure (which may smoothly depend on (s, t)) satisfying the following conditions. Let g be a fixed Riemannian metric on M and J_g a fixed almost complex structure such that:

1) J_g is compatible to the canonical symplectic form $\omega = \sum dx_j \wedge dy_j$ on T^*M (meaning that $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is a Riemannian metric).

2) J_g maps vertical tangent vectors to horizontal ones with respect to Levi–Civita connection of g.

3) To every vector $X \in T_q(O_M) \cong T_qM$, J_g assigns a cotangent vector J_gX such that $J_gX(\xi) = g(\xi, J_gX)$ for all $\xi \in T_qM$.

Now suppose that the almost complex structure J is compatible to ω and that $J \equiv J_g$ outside some compact subset of T^*M. We will assume that almost complex structure J satisfies the above conditions through the rest of the paper.

In [15] we constructed the isomorphism between $HM_*(f)$ and $HF_*(H)$, following [19]. The purpose of [15] was to prove that isomorphisms in Floer homology for Lagrangian intersections naturally intertwine with analogous isomorphisms in Morse homology. The isomorphism constructed there was based on counting the objects of mixed type. More precisely, let p be a critical point of Morse function f and $x(t)$ a Hamiltonian path that begins and ends on O_M. Considered the spaces of pairs (γ, u)

$$\gamma : (-\infty, 0] \to M, \quad u : [0, +\infty) \times [0, 1] \to T^*M$$

that satisfy

$$\begin{cases} \dfrac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\rho_R H}(u)\right) = 0 \\ u(0,t), u(s,0), u(s,1) \in O_M \\ \gamma(-\infty) = p, u(+\infty, t) = x(t) \\ \gamma(0) = u\left(0, \dfrac{1}{2}\right). \end{cases} \tag{3}$$

Here $\rho_R : [0, +\infty) \to \mathbb{R}$ is a smooth function such that

$$\rho_R(s) = \begin{cases} 1, & s \geq R+1 \\ 0, & s \leq R \end{cases}$$

and $X_{\rho_R H}$ is a Hamiltonian vector field corresponding to the Hamiltonian function $\rho_R H$. Denote by $\mathcal{M}(p, f; x, H)$ the set of solutions of (3), see Figure 1.

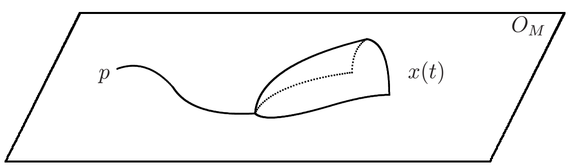

Figure 1: The element of the space $\mathcal{M}(p, f; x, H)$.

Similarly, denote by $\mathcal{M}(x, H; p, f)$ the set of the pairs (u, γ),

$$u : (-\infty, 0] \times [0, 1] \to T^*M, \qquad \gamma : [0, +\infty) \to M$$

that satisfy

$$\begin{cases} \dfrac{d\gamma}{ds} = -\nabla f(\gamma(s)) \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\bar{\rho}_R H}(u)\right) = 0 \\ u(0,t), u(s,0), u(s,1) \in O_M \\ u(-\infty, t) = x(t), \gamma(+\infty) = p \\ \gamma(0) = u\left(0, \frac{1}{2}\right) \end{cases}$$

where $\bar{\rho}_R(s) = \rho_R(-s)$. Let $m_f(p)$ be the Morse index of critical point p and $\mu_H(x)$ the Maslov index of Hamiltonian path x (see [2, 20, 21] for definition of Maslov index, [17] for its application in grading of Floer homology groups and [16] for generalizations). We have the following

Proposition 1 [15]. *For a generic f and H $\mathcal{M}(p, f; x, H)$ is a smooth manifold of dimension*

$$m_f(p) - \left(\mu_H(x) + \frac{n}{2}\right) \quad \textit{and} \quad \mathcal{M}(x, H; p, f)$$

is a smooth manifold of dimension $\mu_H(x) + \frac{n}{2} - m_f(p)$. When $m_f(p) = \left(\mu_H(x) + \frac{n}{2}\right)$, manifolds $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$ are compact, hence finite sets.

For $m_f(p) = \mu_H(x) + \frac{n}{2}$ we denote by $n(p, f; x, H)$ and $n(x, H; p, f)$ the cardinal numbers (mod $\mathbb{Z}_2$) of $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$ respectively. Let $CM_k(f)$ be a $\mathbb{Z}_2$-vector space generated by set of all critical points of f that have Morse index equal to k. Similarly, let $CF_k(H)$ denote a $\mathbb{Z}_2$-vector space generated by the set of Hamiltonian paths with ends in zero section that have Maslov index equal to $k - \frac{n}{2}$. Define homomorphisms

$$\Phi : CF_k(H) \to CM_k(f), \qquad \Psi : CM_k(f) \to CF_k(H) \tag{4}$$

by

$$x \mapsto \sum_{m_f(p)=k} n(x, H; p, f)p, \qquad p \mapsto \sum_{\mu_H(x)+\frac{n}{2}=k} n(p, f; x, H)x$$

on the generators. Homomorphisms Φ and Ψ are also well defined on $HF_k(H)$ and $HM_k(f)$, i.e.

$$\Phi \circ \partial_F = \partial_M \circ \Phi, \qquad \Psi \circ \partial_M = \partial_F \circ \Psi$$

where ∂_M and ∂_F are defined in (1) and (2). Homomorphisms Ψ and Φ are actually isomorphism, it follows from the following

Proposition 2 [15]. *On the homology level it holds:*

$$\Psi \circ \Phi = \mathrm{Id}_{HF}, \qquad \Phi \circ \Psi = \mathrm{Id}_{HM}\,.$$

The proof of the Proposition 2 is based on the analysis of the boundary of one dimensional component of certain mixed type object space (see below or [19, 15, 13, 14]). In this situation no bubbles appear due to the fact that the symplectic manifold in consideration is a cotangent bundle, so there is no holomorphic spheres or disks with boundary in zero section. There is a construction of Piunikhin-Salamon-Schwarz homomorphism for Lagrangian intersections in a general symplectic manifold given in [1]. It is not necessarily an isomorphism in general.

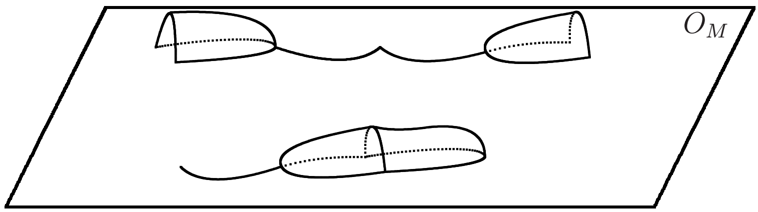

Figure 2: Objects that define $\Psi \circ \Phi$ and $\Phi \circ \Psi$.

Morse and Floer homologies with $\mathbb{Z}$ coefficients are constructed by using the coherent orientation. This construction of coherent orientation was originally done for the case of Floer homology for periodic orbits in [10]. Coherent orientation for Lagrangian Floer homology is discussed in [12] and for Morse theory in [22].

In this paper we carry out the construction of isomorphisms (4) between Morse and Floer homologies, but with $\mathbb{Z}$ coefficients. More explicitly, the main result of the paper is the following

Theorem 3. *For two given coherent orientations that induce Morse and Floer homologies $HM_*(f, \mathbb{Z})$ and $HF_*(H, \mathbb{Z})$ with $\mathbb{Z}$ coefficients, one can associate a sign $\pm$ to every mixed type object, is such way that these signs induce a PSS-type isomorphism between $HM_*(f, \mathbb{Z})$ and $HF_*(H, \mathbb{Z})$.*

In order to prove the Theorem 3, we construct coherent (i.e. compatible with gluing) and canonical orientation of mixed moduli spaces described above. These two orientations induce characteristic signs plus or minus that we associate to every point of zero-dimensional component. This construction via characteristic signs was originally given (for the case of Floer homology for periodic orbits) in [10].

The construction of coherent orientations on moduli spaces involving both holomorphic disks and gradient trajectories is also relevant to some other projects, such as definition of cluster homology [4, 5] and generalization [3] of the original construction by Floer and Hofer [9].

2 Orientation and gluing for Fredholm operators

In [15] we proved that the mixed moduli space $\mathcal{M}(p, f; x, H)$ is a manifold by means of the evaluation map, i.e. we treated it as an intersection of certain stable and unstable manifold. In order to construct the orientation for $\mathcal{M}(p, f; x, H)$ we need to see it as a zero set of certain Fredholm operator, and to define orientation and gluing for the Fredholm operators of that type. (Gluing of Fredholm operators and coherent orientation for moduli spaces of the same type was given in [9], for Floer case and in [22], for Morse case.) We first establish the analytical setting, i.e. we construct Banach manifolds that are domain and target manifold of mentioned Fredholm operator. Let p be a critical point of Morse function f. Denote by $C^\infty(p)$ the set of all smooth maps γ that satisfy

$$\begin{cases} \gamma : (-\infty, 0] \to M \\ \gamma(-\infty) = p. \end{cases} \tag{5}$$

The tangent space space $T_\gamma C^\infty(p)$ at point γ consists of all smooth vector fields ξ with properties

$$\xi : (-\infty, 0] \to TM, \quad \xi(s) \in T_{\gamma(s)}M, \quad \xi(-\infty) = 0.$$

For $r > 1$, let $\|\xi\|_{L^r}$ and $\|\xi\|_{W^{1,r}}$ stand for standard Sobolev norms

$$\|\xi\|_{L^r} = \left(\int_{-\infty}^{0} |\xi|^r \, ds \right)^{\frac{1}{r}}, \qquad \|\xi\|_{W^{1,r}} = \left(\int_{-\infty}^{0} \left(|\xi|^r + |\nabla_s \xi|^r \right) ds \right)^{\frac{1}{r}}.$$

Denote by $W^{1,r}_\gamma(p)$ and $L^r_\gamma(p)$ the completions of $T_\gamma C^\infty(p)$ in $W^{1,r}$- and L^r-Sobolev norms. For a smooth $\gamma \in C^\infty(p)$ and $\xi \in W^{1,r}_\gamma(p)$, the exponential map

$$\xi \mapsto \exp \circ \xi, \quad \xi(s) \mapsto \exp_{\gamma(s)} \xi(s)$$

is defined via the exponential map on the Riemann manifold M. Denote by $\mathcal{D} \subset TM$ the associated injectivity neighborhood of the zero section of the tangent bundle TM and define the set $\mathcal{P}^{1,r}(p)$ as the union

$$\bigcup_{\gamma \in C^\infty(p)} \left\{ \exp \circ \xi \mid \xi \in W^{1,r}_\gamma(p), \ \|\xi\|_{W^{1,r}} \text{ is small, so that } \xi(s) \in \mathcal{D} \right\}.$$

From Sobolev embedding theorem (and the condition $r > 1$) it follows $\mathcal{P}^{1,r}(p) \subset C^0(\mathbb{R}, M)$. The set $\mathcal{P}^{1,r}(p)$ of all continuous γ that satisfy (5) is equipped with a Banach manifold structure (via the exponential map) such that the tangent space at γ is given by

$$T_\gamma \mathcal{P}^{1,r}(p) = W_\gamma^{1,r}(p)$$

(see [22] for more details).

For a Hamiltonian path $x(t)$ with ends in O_M, denote by $C^\infty(x)$ the set of all smooth maps u that satisfy

$$\begin{cases} u : [0, +\infty) \times [0, 1] \to T^*M \\ u(0, t), u(s, 0), u(s, 1) \in O_M \\ u(+\infty, t) = x(t). \end{cases} \tag{6}$$

The tangent space space $T_u C^\infty(x)$ at point u consists of all vector fields ζ such that

$$\begin{cases} \zeta : [0, +\infty) \times [0, 1] \to TT^*M \\ \zeta(s, t) \in T_{u(s,t)} T^*M \\ \zeta(s, 0), \zeta(s, 1), \zeta(0, t) \in TM \cong TO_M \subset TT^*M \\ \zeta(+\infty, t) = 0. \end{cases}$$

Denote by $W_u^{1,r}(x)$ and $L_u^r(x)$ the completions of $T_u C^\infty(x)$ in $W^{1,r}$- and L^r-Sobolev norms:

$$\|\zeta\|_{L^r} = \left(\iint\limits_{[0,+\infty)\times[0,1]} |\zeta|^r \, dsdt \right)^{\frac{1}{r}}$$

$$\|\zeta\|_{W^{1,r}} = \left(\iint\limits_{[0,+\infty)\times[0,1]} \left(|\zeta|^r + |\nabla_s \zeta|^r + |\nabla_t \zeta|^r\right) dsdt \right)^{\frac{1}{r}}.$$

Since u has two-dimensional domain, we will assume that $r > 2$. Sobolev Embedding Theorem will provide the continuity of maps and sections involved. (For $r = 2$ the above spaces become Hilbert spaces, which is more convenient, and these spaces can be of use when one deals with Morse case (see [22]).)

Finally, let $\mathcal{P}^{1,r}(p)$ be the union

$$\bigcup_{u \in C^\infty(x)} \left\{ \exp \circ \zeta \mid \zeta \in W_u^{1,r}(x), \ \|\zeta\|_{W^{1,r}} \text{ is small, so that } \zeta(s) \in \widetilde{\mathcal{D}} \right\}$$

where

$$\left(\exp_u \zeta\right)(s, t) = \exp_{u(s,t)} \zeta(s, t)$$

and $\widetilde{\mathcal{D}} \subset TT^*M$ is the injectivity neighborhood of the zero section in the tangent bundle TT^*M. The set $\mathcal{P}^{1,r}(x)$ of continuous curves u that satisfy (6) is a Banach manifold and the tangent space at u is given by

$$T_u\mathcal{P}^{1,r}(x) = W_u^{1,r}(x)$$

(see [8] for the details).

The topology of $\mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$ (hence the topology of $\mathcal{M}(p, f; x, H) \subset \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$) is induced by the topology of $W_\gamma^{1,r}(p) \times W_u^{1,r}(x)$ by means of the above exponential map.

Let $\widetilde{\mathrm{ev}}$ be the following evaluation map:

$$\begin{aligned} &\widetilde{\mathrm{ev}} : \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x) \to M \times M, \\ &\widetilde{\mathrm{ev}}(\gamma, u) = \left(\gamma(0), u\left(0, \frac{1}{2}\right)\right). \end{aligned}$$

The map $\widetilde{\mathrm{ev}}$ is transversal to the diagonal $\Delta \subset M \times M$ and

$$\mathcal{P}^{1,r}(p, x) := \widetilde{\mathrm{ev}}^{-1}(\Delta)$$

is infinite-dimensional smooth Banach submanifold of $\mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. As before, denote by

$$W_{(\gamma,u)}^{1,r}(p, x) = T_{(\gamma,u)}\mathcal{P}^{1,r}(p, x)$$

the corresponding tangent space. Note that

$$\xi(0) = \zeta\left(0, \frac{1}{2}\right) \qquad \text{for all} \quad (\xi, \zeta) \in T_{(\gamma,u)}\mathcal{P}^{1,r}(p, x).$$

The space $\mathcal{M}(p, f; x, H)$ is the zero set of a restriction of a smooth section

$$\begin{aligned} &F = \widetilde{F}|_{\mathcal{P}^{1,r}(p,x)}, \qquad \widetilde{F} = (F_1, F_2), \\ &F_1(\gamma) = \frac{d\gamma}{ds} + \nabla f(\gamma), \\ &F_2(u) = \bar{\partial}_{\rho_R H, J}\, u = \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{\rho_R H}(u)\right) \end{aligned} \tag{7}$$

of a Banach bundle

$$\mathcal{E}(p, x) \to \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$$

with a fibre $L^r_\gamma(p) \times L^r_u(x)$ over a point $(\gamma, u) \in \mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. The linearization of (7) at the point (γ, u) is

$$
\begin{aligned}
&(DF)_{(\gamma,u)} \colon W^{1,r}_{(\gamma,u)}(p, x) \to L^r_\gamma(p) \times L^r_u(x),\\
&(DF)_{(\gamma,u)} = ((DF_1)_\gamma, (DF_2)_u)\\
&(DF_1)_\gamma \colon W^{1,r}(p) \to L^r(p),\\
&(DF_1)_\gamma \xi = \nabla_{\frac{d\gamma}{ds}} \xi + \nabla_\xi \nabla f(\gamma)\\
&(DF_2)_u \colon W^{1,r}(x) \to L^r(x)\\
&(DF_2)_u \zeta = \nabla_s \zeta + J(u)\nabla_t \zeta + \nabla_\zeta J(u)\partial_t u - \nabla_\zeta J X_{\rho_R H}(u).
\end{aligned}
\tag{8}
$$

Let us consider the trivial case for the moment. If A is an element of $C^0((-\infty, 0], \mathrm{End}(\mathbb{R}^n))$ such that $A(-\infty) \in GL(n, \mathbb{R})$ is symmetric non-degenerate matrix, then the operator $K_A : W^{1,r}((-\infty, 0], \mathbb{R}^n) \to L^r((-\infty, 0], \mathbb{R}^n)$ of the type

$$K_A(\xi)(s) = \dot{\xi}(s) + A(s)\xi(s) \tag{9}$$

is Fredholm (see [22]). In our case, the trivialization of the operator $(DF_1)_\gamma$ is of this type, so it is Fredholm. Its Fredholm index is the dimension of the nonstable manifold of the critical point p:

$$\mathrm{Ind}((DF_1)_\gamma) = m_f(p) \tag{10}$$

(see [22]).

In two-dimensional case, consider the operator $L = L(J, B)$ of the type

$$L(\zeta)(s, t) = \frac{\partial \zeta}{\partial s}(s, t) + J(s, t)\frac{\partial \zeta}{\partial t}(s, t) + B(s, t)\zeta(s, t) \tag{11}$$

for some almost complex structure J satisfying conditions described on the page 254. Suppose that $L^+(x)(t) = J(+\infty, t)\dot{x}(t) + B(+\infty, t)x(t)$ is a self-adjoint isomorphism (which means that it holds $\langle L^+x, y\rangle_{L^2} = \langle x, L^+y\rangle_{L^2}$ for all $x, y \in \widetilde{W}^{1,r}$) with the domain

$$\widetilde{W}^{1,r} := \left\{x \in W^{1,r}([0, 1], \mathbb{R}^{2n}) \mid x(0), x(1) \in \mathbb{R}^n \times \{0\}\right\}$$

and the target set $L^r([0, 1], \mathbb{R}^{2n})$. Then the operator L is Fredholm (see [23]). This condition is fulfilled in the case of the trivialization of the operator $(DF_2)_u$ and

$$\mathrm{Ind}((DF_2)_u) = \frac{n}{2} - \mu_H(x) \tag{12}$$

(see [18]).

Proposition 4. *The operator $(DF)_{(\gamma,u)}$ in* (8) *is Fredholm hence the map* (7) *is a Fredholm map. Its index is equal to*

$$\mathrm{Ind}((DF)_{(\gamma,u)}) = m_f(p) - \mu_H(x) - \frac{n}{2}.$$

Proof. We will omit the subscripts u, γ, (γ, u) in order to abbreviate notations. First we observe that the manifold $\mathcal{P}^{1,r}(p,x)$ is of finite codimension in $\mathcal{P}^{1,r}(p) \times \mathcal{P}^{1,r}(x)$. Indeed, the tangent space $W^{1,r}(p,x)$ of $\mathcal{P}^{1,r}(p,x)$ is the kernel of the differential $D\widetilde{\mathrm{ev}}$ of the evaluation map. It holds:

$$W^{1,r}(p) \times W^{1,r}(x)/\,\mathrm{Ker}(D\widetilde{\mathrm{ev}}) \cong \mathrm{Im}(D\widetilde{\mathrm{ev}}).$$

The image $\mathrm{Im}(D\widetilde{\mathrm{ev}})$ is of finite dimension since the target space of $D\widetilde{\mathrm{ev}}$ is so. It follows that $W^{1,r}(p,x)$ is of finite codimension and from Hahn-Banach theorem that it can be complemented in $W^{1,r}(p) \times W^{1,r}(x)$ by some finite-dimensional space, denote it by X. Since $\mathrm{codim}_{M\times M}(\Delta) = n$ it holds $\dim X = n$.

We consider the auxiliary operator from above

$$\begin{aligned} D\widetilde{F} &: W^{1,r}(p) \times W^{1,r}(x) \to L^r(p) \times L^r(x) \\ D\widetilde{F} &:= (DF_1, DF_2) \end{aligned}$$

defined on the product $W^{1,r}(p) \times W^{1,r}(x)$ in order to compute the index of the operator DF in the terms of the indices of DF_1 and DF_2. Since the operators DF_1 and DF_2 are Fredholm (see Chapter 4.1 in [22] and Appendix A in [18]) and it holds

$$\mathrm{Ker}\, D\widetilde{F} = \mathrm{Ker}\, DF_1 \times \mathrm{Ker}\, DF_2, \quad \mathrm{Coker}\, D\widetilde{F} = \mathrm{Coker}\, DF_1 \times \mathrm{Coker}\, DF_2$$

we conclude that $D\widetilde{F}$ is also Fredholm with the Fredholm index $\mathrm{Ind}(D\widetilde{F}) = \mathrm{Ind}(DF_1) + \mathrm{Ind}(DF_2)$. The operator DF is a restriction of $D\widetilde{F}$ to the space $W^{1,r}(p,x)$. Consider the following (disjoint) decompositions of the spaces $W^{1,r}(p) \times W^{1,r}(x)$ and $L^r(p) \times L^r(x)$:

$$W^{1,r}(p) \times W^{1,r}(x) = X_1 \oplus X_2 \oplus X_3 \oplus X_4, \quad L^r(p) \times L^r(x) = Y_1 \oplus Y_2 \oplus Y$$

where the subspaces X_i, Y_i and Y are defined in the following way:

$$\begin{aligned} &X_3 := W^{1,r}(p,x) \cap \mathrm{Ker}(D\widetilde{F}), && X_1 := W^{1,r}(p,x) \ominus X_3 \\ &X_4 := X \cap \mathrm{Ker}(D\widetilde{F}), && X_2 := X \ominus X_4 \\ &Y_i := D\widetilde{F}(X_i), \text{ for } i = 1, 2, && Y := L^r(p) \times L^r(x) \ominus (Y_1 \oplus Y_2)\,. \end{aligned} \tag{13}$$

The sign $A \ominus B$ stands for a complement of B in A, all the spaces in (13) are well defined due to the Hahn-Banach theorem. All the spaces except X_1 and Y_1 are of finite dimension: X_3 and X_4 are subspaces of Ker $D\widetilde{F}$ – which is of finite dimension; X_2 is the subspace of X – which is of finite dimension; Y_2 is the isomorphic image of X_2; finally Y is of finite dimension since it is isomorphic to the co-kernel of Fredholm operator $D\widetilde{F}$. Set

$$m_2 := \dim(X_2) = \dim(Y_2), \quad m_3 := \dim(X_3)$$
$$m_4 := \dim(X_4), \quad m := \dim(Y).$$

We see that

$$\mathrm{Ker}(D\widetilde{F}) = X_3 \oplus X_4, \quad \mathrm{Coker}(D\widetilde{F}) = Y$$

and, since $DF = D\widetilde{F}|_{X_1 \oplus X_3} : X_1 \oplus X_3 \to Y_1 \oplus Y_2 \oplus Y$,

$$\mathrm{Ker}(DF) = X_3, \quad \mathrm{Coker}(DF) = Y_2 \oplus Y.$$

We conclude that DF is also Fredholm, and moreover, we compute its index:

$$\begin{aligned}
\mathrm{Ind}(DF) &= \dim(\mathrm{Ker}(DF)) - \dim(\mathrm{Coker}(DF)) \\
&= \dim(X_3) - \dim(Y_2 \oplus Y) \\
&= m_3 - (m_2 + m) \\
&= (m_3 + m_4) - m - (m_2 + m_4) \\
&= \dim(\mathrm{Ker}(D\widetilde{F})) - \dim(\mathrm{Coker}(D\widetilde{F})) - \dim(X_2 \oplus X_4) \\
&= \mathrm{Ind}(D\widetilde{F}) - \dim(X) \\
&= \mathrm{Ind}(DF_1) + \mathrm{Ind}(DF_2) - n \\
&\overset{(10),(12)}{=} m_f(p) - \mu_H(x) - \frac{n}{2}.
\end{aligned}$$

□

Let us start with the construction of coherent orientation in the trivial case. In local coordinates the operators F_1 and F_2 in (7) have the forms:

$$\dot{\xi}(s) + A(s)\xi \quad \text{and} \quad \frac{\partial \zeta}{\partial s}(s,t) + J(s,t)\frac{\partial \zeta}{\partial t}(s,t) + B(s,t)\zeta(s,t),$$

so we define two classes of the operators of the special type as follows. Let us denote:

$$W_1^{1,r} := W^{1,r}\left((-\infty, 0], \mathbb{R}^n \times \{0\}\right)$$
$$W_2^{1,r} := \left\{\zeta \in W^{1,r}\left([0,\infty) \times [0,1], \mathbb{R}^{2n}\right), \mid \zeta(s,0), \zeta(s,1), \zeta(0,t) \in \mathbb{R}^n \times \{0\}\right\}$$

$$
\begin{aligned}
L_1^r &:= L^r((-\infty, 0], \mathbb{R}^n \times \{0\}) \\
L_2^r &:= L^r\left([0, \infty) \times [0, 1], \mathbb{R}^{2n}\right) \\
W^{1,r} &:= \left\{(\xi, \zeta) \in W_1^{1,r} \times W_2^{1,r} \mid \xi(0) = \zeta\left(0, \frac{1}{2}\right)\right\} \\
L^r &:= L_1^r \times L_2^r.
\end{aligned}
$$

Let Σ_{triv} be the set of all linear operators

$$(K, L) : W^{1,r} \to L^r$$

such that $K : W_1^{1,r} \to L_1^r$ is of the form (9), so that $K^- = A(-\infty)$ is symmetric non-degenerate matrix and $L : W_2^{1,r} \to L_2^r$ is of the form (11). Here $L^+(x)(t) = J(+\infty, t)\dot{x}(t) + B(+\infty, t)x(t)$ is a self-adjoint isomorphism (meaning that $\langle L^+x, y\rangle_{L^2} = \langle x, L^+y\rangle_{L^2}$ for all $x, y \in \widetilde{W}^{1,r}$) with the domain

$$\widetilde{W}^{1,r} := \left\{x \in W^{1,r}([0, 1], \mathbb{R}^{2n}) \mid x(0), x(1) \in \mathbb{R}^n \times \{0\}\right\} \tag{14}$$

and the target set $L^r([0, 1], \mathbb{R}^{2n})$. We will use notations $L_{(B,J)}$, B_L and J_L for L, B and J as in (11).

In non-trivial case (Section 3) we will consider such trivializations

$$\varphi : u^*T(T^*M) \to [0, +\infty) \times [0, 1] \times \mathbb{R}^{2n} \cong [0, +\infty) \times [0, 1] \times \mathbb{C}^n$$

of $u^*(TT^*M)$ (where $u \in \mathcal{P}^{1,r}(x)$) that satisfy

$$\varphi(H_{u(t)}) = \{t\} \times \mathbb{R}^n, \quad \varphi(V_{u(t)}) = \{t\} \times i\mathbb{R}^n$$

where H_z and V_z are horizontal and vertical splitting

$$u^*T(T^*M) = H_z \oplus V_z$$

with respect to Levi–Civita connection on T^*M of a fixed Riemannian metric g on M. This class is not empty because the set $[0, +\infty) \times [0, 1]$ is contractible (see [17] for details) and we will denote it by $\mathcal{T}$.

Similarly, we consider only trivializations of $w^*(TT^*M)$ (where $w \in \mathcal{P}^{1,r}(p, x)$) that satisfy the same condition.

It is a special structure of a cotangent bundle as an ambient manifold that enables us to choose a canonical class $\mathcal{T}$ of trivializations. In more general situation of relatively spin Lagrangian submanifold, the space of holomorphic disks is also orientable [11]. Although this construction might be generalized to relatively spin case, since we want to construct a PSS isomorphism as in [15] with $\mathbb{Z}$ coefficients, we work in a cotangent bundle in this paper.

It follows from the Proposition 4 that every operator from Σ_{triv} is Fredholm.

Definition 1. *Two elements* $F = (F_1, F_2)$, $G = (G_1, G_2) \in \Sigma_{\text{triv}}$ *are* equivalent *($F \sim G$) if:*

$$F_1^- = G_1^-, \quad F_2^+ = G_2^+.$$

Denote by

$$\widetilde{\Sigma}_{\text{triv}} := \Sigma_{\text{triv}} / \sim .$$

In a standard way, using a linear homotopy as in [9] and [22], one can prove the next Proposition.

Proposition 5. *Let* $[F = (F_1, F_2)]$ *be the equivalence class in* $\widetilde{\Sigma}\text{triv}$*. Then* $[F]$ *is contractible considered as a subset of* Σ_{triv} *with respect to the topology induced by the operator norm.*

Let $\text{Det}[F]$ be the determinant bundle over the class of operator F (for the construction of a determinant bundle see [6, 7] or [9, 22]).

Corollary 6. *The determinant bundle of the family* $[F]$, $\text{Det}[F]$ *is trivial.*

It was shown in [9, 22] how to glue Fredholm operators of the same type, when they are compatible for gluing. More precisely, denote by Σ_{triv}^M the class of Fredholm operators

$$K_A \in \mathcal{L}(W^{1,r}(\mathbb{R}, \mathbb{R}^n), L^r(\mathbb{R}, \mathbb{R}^n))$$

of the type (9) with $K_A^{\pm} = A(\pm\infty)$ self-adjoint isomorphisms. Suppose that two operators $K_{A_i} \in \Sigma_{\text{triv}}^M$ (for $i = 1, 2$) are asymptotically constant (which means $A_i(s) = \text{const}$, for $|s| \geq S, i = 1, 2$) with matching ends, i.e. $K_1^+ = K_2^-$. Define $K_1 \sharp_\rho K_2$ as

$$K_1 \sharp_\rho K_2(\xi)(s) = \dot{\xi}(s) + A_\rho(s)\xi(s)$$

where

$$A_\rho(s) := \begin{cases} A_{K_1}(s + \rho), & s \leq 0 \\ A_{K_2}(s - \rho), & s \geq 0, \end{cases}$$

for ρ large enough.

This gluing construction induces an isomorphism:

$$\text{Det}(K_1) \otimes \text{Det}(K_2) \cong \text{Det}(K_1 \sharp_\rho K_2) \tag{15}$$

(see [22]) such that, when the operators K_1 and K_2 are surjective (so $\text{Det}(K_i) = \text{Ker}(K_i)$, for $i = 1, 2$), this isomorphism coincides with the isomorphism

$\mathrm{Ker}(K_1) \times \mathrm{Ker}(K_2) \cong \mathrm{Ker}(K_1 \sharp K_2)$ that occurs in the construction of gluing trajectories (see [22]).

In the Floer case, denote by Σ^F_{triv} the class of all operators $L \in \mathcal{L}(W^{1,r}([0,1] \times \mathbb{R}, \mathbb{R}^{2n}), L^r([0,1] \times \mathbb{R}, \mathbb{R}^{2n}))$ of the special type (11) such that both $L^{\pm}(x)(t) = J(\pm\infty, t)\dot{x}(t) + B(\pm\infty, t)x(t)$ are self-adjoint isomorphisms (with the domain and the target set as in (14)). There is a similar gluing construction for Fredholm operators $L_1 = L_1(J_1, B_1)$, $L_2 = L_2(J_2, B_2)$ in Σ^F_{triv}. Suppose that they have matching ends $L_1^+ = L_2^-$ and that they asymptotically constant. Define $L_1 \sharp_\rho L_2$ as

$$L_1 \sharp_\rho L_2(\zeta)(s,t) = \frac{\partial \zeta}{\partial s}(s,t) + J_\rho(s,t)\frac{\partial \zeta}{\partial t}(s,t) + B_\rho(s,t)\zeta(s,t)$$

where

$$B_\rho(s,t) := \begin{cases} B_1(s+\rho, t), & s \leq 0 \\ B_2(s-\rho, t), & s \geq 0, \end{cases}$$

for ρ large enough and

$$J_\rho(s,t) := \begin{cases} J_1(s+\rho, t), & s \leq 0 \\ J_2(s-\rho, t), & s \geq 0, \end{cases}$$

for ρ large enough.

As in Morse case, this gluing construction induces an isomorphism:

$$\mathrm{Det}(L_1) \otimes \mathrm{Det}(L_2) \cong \mathrm{Det}(L_1 \sharp_\rho L_2) \tag{16}$$

which is again, when the operators are surjective, a linearization of gluing isomorphism for trajectories (see [9]).

Let us define now gluing of Fredholm operators of mixed type with Fredholm operators from Σ^M_{triv} or Σ^F_{triv}.

Definition 2. *Let $F = (F_1, F_2) \in \Sigma_{\mathrm{triv}}$, and $K \in \Sigma^M_{\mathrm{triv}}$, $L \in \Sigma^F_{\mathrm{triv}}$ asymptotically constant, such that $K^+ = F_1^-$, $F_2^+ = L^-$. Let*

$$\big(K \sharp_\rho F_1\big)(\xi)(s) := \dot{\xi}(s) + A_\rho(s)\xi$$

where

$$A_\rho = \begin{cases} A_K(s+2\rho), & s \leq -\rho \\ A_{F_1}(s), & -\rho \leq s \leq 0 \end{cases}$$

for ρ large enough. Define

$$K \sharp_\rho F := (K \sharp_\rho F_1, F_2) \in \Sigma.$$

Similarly, define

$$\big(F_2 \sharp_\rho L\big)u(s,t) := \frac{\partial u}{\partial s}(s,t) + J_\rho(s,t)\frac{\partial u}{\partial t}(s,t) + B_\rho(s,t)u(s,t)$$

for ρ large enough, where

$$B_\rho(s,t) := \begin{cases} B_{F_2}(s,t), & 0 \le s \le \rho, \\ B_L(s-2\rho,t), & s \ge \rho, \end{cases}$$

$$J_\rho(s,t) := \begin{cases} J_{F_2}(s,t), & 0 \le s \le \rho, \\ J_L(s-2\rho,t), & s \ge \rho \end{cases} \quad \text{and}$$

$$F \sharp_\rho L := \big(F_1, F_2 \sharp_\rho L\big).$$

Proposition 7. *Gluing construction from the Definition 2 induces isomorphisms*

$$\begin{aligned} A &: \mathrm{Det}(K) \otimes \mathrm{Det}(F) \xrightarrow{\cong} \mathrm{Det}(K \sharp_\rho F) \\ B &: \mathrm{Det}(F) \otimes \mathrm{Det}(L) \xrightarrow{\cong} \mathrm{Det}(F \sharp_\rho L). \end{aligned} \tag{17}$$

When K, F, L are surjective, these isomorphisms are actually the gluing isomorphisms for trajectories in linearized case.

Proof. Denote by $W^{1,r}_F$ (resp. L^r_F) the sets of all mappings

$$\eta : \mathbb{R} \times [0,1] \to \mathbb{R}^{2n}, \quad \eta(s,0), \eta(s,1) \in \mathbb{R}^n \times \{0\}$$

of the class $W^{1,r}$, (resp. L^r). Let

$$\psi : \mathbb{R}^k \to L^r, \quad \phi : \mathbb{R}^k \to L^r_F$$

be linear mappings such that the mappings

$$\begin{aligned} &\widehat{F}_\psi : \mathbb{R}^k \times W^{1,r} \to L^r, \quad \widehat{F}_\psi(h,w) = Fw + \psi h \\ &\widehat{L}_\phi : \mathbb{R}^k \times W^{1,r}_F \to L^r_F, \quad \widehat{L}_\phi(h,u) = Lu + \phi h \end{aligned}$$

are surjective. Define

$$\begin{aligned} &F_\psi : \mathbb{R}^k \times W^{1,r} \to \mathbb{R}^k \times L^r, \quad L_\phi : \mathbb{R}^k \times W^{1,r}_F \to \mathbb{R}^K \times L^r_F \\ &F_\psi(a,w) := (0, \widehat{F}_\psi(a,w)), \quad L_\phi(a,u) := (0, \widehat{L}_\phi(a,u)). \end{aligned}$$

It holds

$$\operatorname{Det}(F_\psi) \cong \operatorname{Det}(F) \text{ and } \operatorname{Det}(L_\phi) \cong \operatorname{Det}(L). \tag{18}$$

The above natural isomorphism is obtained by means of the following abstract Lemma.

Lemma 8 [7]. *Given an exact sequence*

$$0 \longrightarrow E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{k-1}} E_k \longrightarrow 0$$

of vector spaces, there exists a canonical isomorphism

$$\phi : \bigotimes_{i \text{ even}} \left(\bigwedge^{\max} E_i \right) \xrightarrow{\cong} \bigotimes_{i \text{ odd}} \left(\bigwedge^{\max} E_i \right).$$

For the proof of the Lemma 8 see also [10, 22].

In our case, consider an exact sequence

$$0 \longrightarrow \operatorname{Ker}(F) \xrightarrow{d_1} \operatorname{Ker}(F_\psi) \xrightarrow{d_2} \mathbb{R}^k \xrightarrow{d_3} \operatorname{Coker}(F) \longrightarrow 0$$

where

$$\begin{aligned} &d_1(k) = (0,k) \\ &d_2(h,k) = h \\ &d_3(h) = \psi(h) \pmod{\operatorname{Im}(F)}. \end{aligned}$$

It follows from Lemma 8 that there exists a canonical isomorphism

$$\bigwedge^{\max} \operatorname{Ker}(F) \otimes \left(\bigwedge^{\max} \mathbb{R}^k \right) \xrightarrow{\cong} \bigwedge^{\max} \operatorname{Ker}(F_\psi) \otimes \bigwedge^{\max} \operatorname{Coker}(F). \tag{19}$$

Note that $\big(\bigwedge^{\max} E\big) \otimes \big(\bigwedge^{\max} E\big)^*$ is canonically isomorphic with $\mathbb{R}$ (the isomorphism is pairing $e \otimes f^* \mapsto f^*(e)$). Multiplying (19) by $\big(\bigwedge^{\max} \mathbb{R}^k\big)^* \otimes \big(\bigwedge^{\max} \operatorname{Coker}(F)\big)^*$

and using the natural identification $A \otimes B \cong B \otimes A$ we get a canonical isomorphism

$$\mathrm{Det}(F) \xrightarrow{\cong} \bigwedge^{\max} \mathrm{Ker}(F_\psi) \otimes \bigwedge^{\max} (\mathbb{R}^k)^*.$$

Since $\mathrm{Coker}(F_\psi) = (H_1 \times \mathbb{R}^k)/H_1 \times \{0\} \cong \mathbb{R}^k$, we conclude

$$\mathrm{Det}(F) \cong \mathrm{Det}(F_\psi).$$

In the same way one obtains the second isomorphism in (18).

For ρ large enough we define glued operator

$$\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi : \mathbb{R}^k \times \mathbb{R}^k \times W^{1,r} \to L^r$$

as follows. We can assume that the supports of the mappings $\psi(a)$ and $\phi(a)$ are contained in $\{|s| \leq R\}$ for some $R > 0$ and all $a \in \mathbb{R}^k$ since the set of surjective Fredholm operators is open. Recall $F = (F_1, F_2)$ and suppose $\psi(a) = (\psi(a)_1, \psi(a)_2)$. For ρ large enough, define

$$\begin{aligned}
&(\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi)(a, b, (\xi, \zeta)) := (\xi_1, \zeta_1)\\
&\xi_1(s) := F_1(\xi)(s) + \psi(a)_1(s)\\
&\zeta_1(s,t) := (F_2 \sharp_\rho L)(\zeta)(s,t) + \psi(a)_2(s,t) + \phi(b)(s - 2\rho, t).
\end{aligned}$$

The above expression makes sense for ρ large enough due to the compactness of supports of ψ and ϕ, although the mappings $\psi(a)$, $(F \sharp_\rho L)w$ and $\phi(b)$ are not *a priori* defined on the same domains.

The proof of the Proposition 7 will follow from the next auxiliary proposition. For $\eta \in W^{1,r}_F$, denote by $\eta_\rho(s,t) := \eta(s + \rho, t)$.

Proposition 9. *There exists lower bound ρ_1 such that, for all $\rho \geq \rho_1$, the operator $\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi$ is surjective. Let Proj_ρ denotes the L^2-orthogonal projection in $\mathbb{R}^k \times \mathbb{R}^k \times W^{1,r}$ to the set $\mathrm{Ker}(\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi)$. The map*

$$\varphi_\rho : \mathrm{Ker}(\widehat{F}_\psi) \times \mathrm{Ker}(\widehat{L}_\phi) \to \mathrm{Ker}(\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi)$$

defined by

$$((a, \varsigma), (b, \eta)) \mapsto \mathrm{Proj}_\rho(a, b, \varsigma + \eta_{-\rho})$$

is an isomorphism.

The proof of the Proposition 9 is a linear version of the Proposition 4 in [14], so we skip it.

Define now

$$\begin{aligned} F_\psi \sharp_\rho L_\phi : \mathbb{R}^k \times \mathbb{R}^k \times W_2^{1,r} &\to \mathbb{R}^k \times \mathbb{R}^k \times L_2^r(a, b, \varsigma) \\ &\mapsto (0, 0, (\widehat{F}_\psi \sharp_\rho \widehat{L}_\phi)(a, b, \varsigma)). \end{aligned}$$

It holds $(F_\psi \sharp_\rho L_\phi) = (F \sharp_\rho L)_{\psi \oplus \phi_{-2\rho}}$, where

$$\psi \oplus \phi_{-2\rho}(a, b)(s, t) = \psi(a)(s, t) + \phi(b)(s - 2\rho, t).$$

From the standard arguments it follows that there is a natural isomorphism

$$\mathrm{Det}(F_\psi \sharp_\rho L_\phi) \cong \mathrm{Det}(F \sharp_\rho L). \tag{20}$$

The isomorphism φ_ρ from the Proposition 9 induces the isomorphism

$$\bigwedge^{\max}(\mathrm{Ker}\, \widehat{F}_\psi) \otimes \bigwedge^{\max}(\mathrm{Ker}\, \widehat{L}_\phi) \cong \bigwedge^{\max}(\mathrm{Ker}\, \widehat{F}_\psi \sharp_\rho \widehat{L}_\phi). \tag{21}$$

Since

$$\left(\bigwedge^{\max} \mathbb{R}^k\right)^* \otimes \left(\bigwedge^{\max} \mathbb{R}^k\right)^* \cong \left(\bigwedge^{\max} (\mathbb{R}^k \times \mathbb{R}^k)\right)^*,$$

then multiplying (21) by $\big(\bigwedge^{\max} \mathbb{R}^k\big)^* \times \big(\bigwedge^{\max} \mathbb{R}^k\big)^*$ and using the fact $\mathrm{Ker}(F_\psi) \cong \mathrm{Ker}(\widehat{F}_\psi)$, we get

$$\mathrm{Det}(F_\psi) \otimes \mathrm{Det}(L_\phi) \cong \mathrm{Det}(F_\psi \sharp_\rho L_\phi).$$

From (20) and the natural isomorphism $\mathrm{Det}(F_\psi) \cong \mathrm{Det}(F)$ we obtain the following isomorphism, induced by the gluing construction:

$$\mathrm{Det}(F) \otimes \mathrm{Det}(L) \cong \mathrm{Det}(F \sharp_\rho L),$$

for ρ large enough. □

The isomorphism A in (17) induces the isomorphism of the determinant bundle of appropriate equivalence classes (which are obviously independent of ρ in the gluing process), i.e.

$$A : \mathrm{Det}[K] \otimes \mathrm{Det}[F] \xrightarrow{\cong} \mathrm{Det}[K \sharp F].$$

The proof is verbatim of arguments in [9]. Let o_K and o_F be two orientations of the families $[K]$ and $[F]$ (i.e. non-zero sections of bundles $\mathrm{Det}[K] \to [K]$ and $\mathrm{Det}[F] \to [F]$). We denote by $o_K \sharp o_F$ the induced orientation of $K \sharp L$, i.e.

$$o_K \sharp o_F := A \circ \big(o_K \otimes o_F\big)$$

and similarly for the second isomorphism B in (17).

Whenever it makes sense, gluing of orientation is an associative operation, i.e. if the three operators K, F and L of special type are compatible for gluing, then it holds

$$(o_K \sharp o_F) \sharp o_L \simeq o_K \sharp (o_F \sharp o_L).$$

Namely, for such K, F and L, one of the next three cases is true:

- $(K, F, L) \in \Sigma^M_{\text{triv}} \times \Sigma^M_{\text{triv}} \times \Sigma^M_{\text{triv}}$
- $(K, F, L) \in \Sigma^M_{\text{triv}} \times \Sigma_{\text{triv}} \times \Sigma^F_{\text{triv}}$
- $(K, F, L) \in \Sigma^F_{\text{triv}} \times \Sigma^F_{\text{triv}} \times \Sigma^F_{\text{triv}}$.

The associativity property can be proven by constructing a smooth line bundle E over $[0, 1]$, such that the boundary fibres E_0 and E_1 are

$$E_0 = \operatorname{Det}((K \sharp_{\rho_1} F) \sharp_{\rho_2} L), \quad E_1 = \operatorname{Det}(K \sharp_{\rho_3} (F \sharp_{\rho_4} L))$$

as well as a natural isomorphism of vector bundles

$$\begin{tikzcd}
{[0,1] \times (\operatorname{Det}(K) \otimes \operatorname{Det}(F) \otimes \operatorname{Det}(L))} \arrow[rr, "\cong"] \arrow[dr] & & E \arrow[dl] \\
& {[0,1]} &
\end{tikzcd}$$

(see [22] for more details).

In the completely analogous way we define the equivalence classes and the gluing of standard operators and operators of mixed type in the case when the latter has the first component of type (9) and the second component of the type (11) (such as the operators that define $\mathcal{M}(x, H; p, f)$). We will use the notation Ξ_{triv} for the set of all linear operators of the form (K, L) defined on the space

$$\Big\{(\zeta, \xi) \in W^{1,r}\big([0 + \infty), \mathbb{R}^n \times \{0\}\big) \times W^{1,r}\big([0, 1] \times (-\infty, 0], \mathbb{R}^{2n}\big)$$
$$\mid \zeta(s, 0), \zeta(s, 1), \zeta(0, t) \in \mathbb{R}^n \times \{0\},\ \zeta\left(0, \frac{1}{2}\right) = \xi(0)\Big\}$$

with the values in

$$L^r\big([0, +\infty), \mathbb{R}^n \times \{0\}\big) \times L^r\big([0, 1] \times (-\infty, 0], \mathbb{R}^{2n}\big)$$

such that K is of the form (9), but with asymptotic assumption at the opposite end and L of the form (11) with asymptotic assumption at $-\infty$. Denote the set of equivalence classes by $\widetilde{\Xi}$.

3 Coherent orientation on the manifold T^*M

In this section we apply the previous construction of coherent orientation to the class of special Fredholm operators but on manifolds. The main difficulty is to choose the suitable trivialization which will enable us to transfer all notions from previous section to the non-linear case.

3.1 Orientation for non-parameterized mixed moduli spaces

Let

$$D := \left((-\infty, 0] \times \left\{\frac{1}{2}\right\}\right) \cup ([0, +\infty) \times [0, 1]). \tag{22}$$

For two symplectic vector bundles $E \to D$ and $F \to D$ over D, denote by $\mathrm{Sp}(E, F)$ a bundle with a fibre $\mathrm{Sp}(E_z, F_z)$ over $z \in D$, consisting of all linear symplectic maps $E_z \to F_z$. In the trivial case $E = F = D \times \mathbb{R}^{2n}$ we have a bundle

$$\begin{array}{ccc} \mathrm{Sp}(n) & \longrightarrow & \mathrm{Sp}(\mathbb{R}^{2n}, \mathbb{R}^{2n}) \\ & & \downarrow \\ & & D \end{array}$$

where $\mathrm{Sp}(n)$ denotes the group of symplectic isomorphisms of $\mathbb{R}^{2n}$ (with respect to standard symplectic form). Denote by $\mathcal{G}^D_{E,F}$ the space of smooth sections of the bundle $\mathrm{Sp}(E, F)$ and by $\mathcal{G}^D_{\mathbb{R}^{2n}}$ the space of smooth sections of $\mathrm{Sp}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$.

Let o and o' be two orientations of some continuous family of Fredholm operators $f : X \to \mathrm{Fred}(E, F)$ (where $\mathrm{Fred}(E, F)$ denotes the set of all Fredholm operators between E and F). We say that o and o' are *compatible* if there exists a mapping

$$o : [0, 1] \times X \to \mathrm{Det}(f)$$

which is a continuous family of nowhere vanishing sections of the bundle $\mathrm{Det}\, f \to X$ such that $o(0, \cdot) = o$, $o(1, \cdot) = o'$. We use the notation $o \simeq o'$. We will use the following auxiliary Lemma.

Lemma 10. *Let $\psi \in \mathcal{G}^D_{\mathbb{R}^{2n}}$ be such that*

$$\psi(-\infty) = \psi(+\infty, t) = \mathrm{Id} \quad \textit{and} \quad F = (F_1, F_2) \in \Sigma_{\mathrm{triv}}.$$

Then $\psi(F) := \psi \circ F \circ \psi^{-1} \in [F]$. If o_F is some orientation of Det F *and $\psi(o_F)$ is the orientation induced by ψ, then these two orientation are compatible, i.e:*

$$\psi(o_F) \simeq o_F.$$

Proof. First we note that $\psi \circ F \circ \psi^{-1}$ is an element of Σ_{triv}: if $w \in W^{1,r}$, then $\psi \circ F \circ \psi^{-1} w \in L^r$. A straightforward check shows that $\psi \circ F \circ \psi^{-1} \in [F]$.

First suppose that $F = (F_1, F_2)$ is of the following type:

$$F_1 = \frac{d}{ds} + \mathrm{Id}, \qquad F_2 = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + \pi\,\cdot\,. \tag{23}$$

Obviously, the orientation $\psi(o)$ only depends on the homotopy class of the mapping ψ in $\mathrm{Sp}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ (and not on the choice of a mapping inside the class). Since the bundle $\mathrm{Sp}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is trivial, it holds:

$$\pi_1(\mathrm{Sp}(\mathbb{R}^{2n}, \mathbb{R}^{2n})) \cong \pi_1(\mathrm{Sp}(n)) \cong \mathbb{Z}.$$

Therefore, it is enough to prove the proposition in the case when ψ is of the form:

$$\psi(s,t) = \begin{bmatrix} e^{2\pi i k\phi(s)} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

where

$$\phi(s) = \begin{cases} 0, & s \le 0 \\ 1, & s \ge 1. \end{cases}$$

In the previous equation we assume that the matrix $\psi(s,t)$ has real coefficients and we use the notation $e^{2\pi i k\phi(s)}$ for a 2×2 block

$$\begin{bmatrix} \cos(2\pi k\phi(s)) & -\sin(2\pi k\phi(s)) \\ \sin(2\pi k\phi(s)) & \cos(2\pi k\phi(s)) \end{bmatrix}.$$

If we set $\phi_r(s,t) := \psi\left(\frac{s}{r}, t\right)$, for $r > 0$, then ϕ_r and ψ are homotopic for all r, so it suffices to prove the assertion for r large enough. We have $\phi_r \circ F \circ {\phi_r}^{-1} = (F_1^r, F_2^r)$, where

$$F_1^r = \phi_r \circ \frac{\partial \phi_r^{-1}}{\partial s}\cdot + \mathrm{Id} + \frac{d}{ds} = F_1 + \Delta_r$$

and

$$F_2^r = \phi_r \circ \frac{\partial \phi_r^{-1}}{\partial s} \cdot + \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} + \pi \cdot = F_2 + \Delta_r$$

so

$$\left(F_1^r, F_2^r\right) = \left(F_1, F_2\right) + \left(\Delta_r, \Delta_r\right).$$

But

$$\Delta_r = \phi_r \circ \frac{\partial \phi_r^{-1}}{\partial s} = \phi_r \circ \frac{\partial}{\partial s} \begin{bmatrix} e^{-2\pi i k \phi\left(\frac{s}{r}\right)} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= -\frac{2\pi i k}{r} \phi' \left(\frac{s}{r}\right) \phi_r \circ \begin{bmatrix} e^{-2\pi i k \phi\left(\frac{s}{r}\right)} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= -\frac{2\pi i k}{r} \phi' \left(\frac{s}{r}\right) \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and its norm converges to zero, when $r \to +\infty$. The operators F_1 and F_2 are isomorphisms. The operators $F = (F_1, F_2)$ and $F^r = (F_1^r, F_2^r)$ are homotopic with fixed end points (through linear homotopy $\tau \mapsto (1-\tau)F + \tau F^r = F + \tau(\Delta_r, \Delta_r)$) and, for r large enough, all the operators during the homotopy are invertible. Hence the orientation $[(1 \otimes 1^*) \otimes (1 \otimes 1^*)]$ of (F_1, F_2) is the same as $\psi(o_F)$ for $\psi(F)$.

The step in the proof that enables us to reduce the proof of Lemma to the special type of operators (23) on gluing of operators is analogous to the corresponding step in [9], so we skip it here. □

For D as in (22), denote by $C^\infty(D, T^*M)$ the space of all pairs $w = (\gamma, u)$ of smooth maps

$$\gamma : (-\infty, 0] \to M, \quad u : [0, +\infty) \times [0, 1] \to T^*M$$

such that:

$$\begin{cases} u(\partial([0, +\infty) \times [0, 1])) \subset O_M \\ \gamma(0) = u\left(0, \frac{1}{2}\right). \end{cases}$$

For given symplectic trivialization $\psi : w^*(TT^*M) \to D \times \mathbb{R}^{2n}$ in $\mathcal{T}$, we define $W^{1,r}(w^*(TT^*M))$ to be the set of all section ς such that $\psi \circ \varsigma \circ \psi^{-1}$ is an element of $W^{1,r}$. We similarly define the set $L^r(w^*(TT^*M))$. For $w \in C^\infty(D, T^*M)$ denote by Σ_w the set of all operators $F = (F_1, F_2) \in \mathcal{L}(W^{1,r}(w^*(TT^*M)), L^r(w^*(TT^*M)))$ such that, for some symplectic trivialization

$$\psi : w^*(TT^*M) \to D \times \mathbb{R}^{2n}$$

the following holds

$$\psi(F) := \psi F \psi^{-1} \in \Sigma_{\mathrm{triv}}.$$

Definition 3. *Let $w_1 = (\gamma_1, u_1)$ and $w_2 = (\gamma_2, u_2)$ be in $C^\infty(D, T^*M)$. Two elements $F = (F_1, F_2) \in \Sigma_{w_1}$ and $G = (G_1, G_2) \in \Sigma_{w_2}$ are* equivalent

$$(w_1, F) \sim (w_2, G)$$

if:

$$\gamma_1(-\infty) = \gamma_2(-\infty), \quad u_1(+\infty, t) = u_2(+\infty, t), \quad F_1^- = G_1^-, \quad F_2^+ = G_2^+.$$

Denote the equivalence class of (w, F) by $[w, F]$ and the set of equivalence classes by $\widetilde{\Sigma}$.

The next step is to define admissible trivializations, i.e. the trivializations that will allow us to orient equivalent operators simultaneously in a unique way. Since our domain D is contractible, our approach is simpler than the one in [9].

Definition 4. *Let $(w_1, F) \sim (w_2, G)$ be as in Definition 3. A pair of symplectic trivializations*

$$\phi_{w_1} : w_1^*(TT^*M) \xrightarrow{\cong} D \times \mathbb{R}^{2n}; \quad \psi_{w_2} : w_2^*(TT^*M) \xrightarrow{\cong} D \times \mathbb{R}^{2n}$$

is called admissible if it holds $\phi_{w_1}(-\infty) = \psi_{w_2}(-\infty)$ and $\phi_{w_1}(+\infty, t) = \psi_{w_2}(+\infty, t)$.

There is always an admissible pair of trivializations. Namely, if $w_i = (\gamma_i, u_i)$, let $w_1 \cdot w_2^{-1}$ be a map defined on $\widetilde{D} := D \times \{1, 2\}/\sim$, where $\sim$ is the identification of two end points

$$\left(-\infty, \frac{1}{2}, 1\right) \sim \left(-\infty, \frac{1}{2}, 2\right), \quad (+\infty, t, 1) \sim (+\infty, t, 2).$$

(Obviously $\widetilde{D}$ has a homotopy type of a circle.) Define

$$w_1 \cdot w_2^{-1} := \begin{cases} \gamma_1\left(s, \frac{1}{2}, 1\right), & s \in [-\infty, -1], \quad \left(s, \frac{1}{2}, 1\right) \in D \times \{1\} \\ u_1(s, t, 1), & s \in [0, +\infty], \quad (s, t, 1) \in D \times \{1\} \\ u_2(-s, t, 2), & s \in [0, +\infty], \quad (s, t, 2) \in D \times \{2\} \\ \gamma_2\left(-s, \frac{1}{2}, 2\right), & s \in [-\infty, 0], \quad \left(s, \frac{1}{2}, 2\right) \in D \times \{2\}. \end{cases}$$

A bundle $(w_1 \cdot w_2^{-1})^*(TT^*M)$ is symplectic vector bundle over the space that has homotopy type of a circle $\mathbb{S}^1$, so it is trivial.

In particular, the operators $\phi_{w_1} F {\phi_{w_1}}^{-1}$ and $\psi_{w_2} G {\psi_{w_2}}^{-1}$ are equivalent in Σ_{triv}. The following Lemma shows that admissible trivializations form a class of trivializations that will enable us to transfer the notion of orientation to the non-linear case.

Lemma 11. *Let $(w_1, F) \sim (w_2, G)$ be as in Definition 3 and (ϕ_{w_1}, ψ_{w_2}), $(\phi'_{w_1}, \psi'_{w_2})$ two pairs of admissible trivializations. Let* $\operatorname{Det} F$ *and* $\operatorname{Det} G$ *be oriented by o_F and o_G. If the pair (ϕ_{w_1}, ψ_{w_2}) induces compatible orientations*

$$\phi_{w_1}(o_F) \simeq \psi_{w_2}(o_G)$$

of trivialized class

$$\left[\phi_{w_1} F \phi_{w_1}^{-1}\right] = \left[\psi_{w_2} G \psi_{w_2}^{-1}\right] = \left[\phi'_{w_1} F \phi'^{-1}_{w_1}\right] = \left[\psi'_{w_2} G \psi'^{-1}_{w_2}\right]$$

then the same is true for the pair $\left(\phi'_{w_1}, \psi'_{w_2}\right)$, i.e it holds

$$\phi'_{w_1}(o_F) \simeq \psi'_{w_2}(o_G). \tag{24}$$

Proof. Consider an element χ of $\mathcal{G}^D_{\mathbb{R}^{2n}}$:

$$\chi := \phi_{w_1} \phi'^{-1}_{w_1} \psi'_{w_2} \psi^{-1}_{w_2} : D \times \mathbb{R}^{2n} \to D \times \mathbb{R}^{2n}. \tag{25}$$

Since by the assumptions

$$\begin{aligned} \phi_{w_1}(-\infty) &= \psi_{w_2}(-\infty), & \phi_{w_1}(+\infty, t) &= \psi_{w_2}(+\infty, t) \\ \phi'_{w_1}(-\infty) &= \psi'_{w_2}(-\infty), & \phi'_{w_1}(+\infty, t) &= \psi'_{w_2}(+\infty, t) \end{aligned}$$

the section χ satisfies the assumptions of the Lemma 10, so it holds

$$\chi(\psi_{w_2}(o_G)) \simeq \psi_{w_2}(o_G). \tag{26}$$

By the assumptions we have

$$\psi_{w_2}(o_G) \simeq \phi_{w_1}(o_F). \tag{27}$$

From (25), (26) and (27) we have

$$\phi_{w_1}\phi'^{-1}_{w_1}\psi'_{w_2}(o_G) \simeq \chi(\psi_{w_2}(o_G)) \simeq \psi_{w_2}(o_G) \simeq \phi_{w_1}(o_F).$$

But it implies

$$\psi'_{w_2}(o_G) \simeq \phi'_{w_1}(o_F).$$ □

We are now able to orient the class of Fredholm operators in non-trivial case.

Definition 5. *Any orientation o_F induces a unique orientation of the equivalence class $[F]$: if $F \sim G$ we define*

$$o_F \simeq o_G$$

if and only if it holds

$$\psi_{w_2}(o_F) \simeq \phi_{w_1}(o_G)$$

for some (hence any) admissible pair of trivializations.

We define pre-gluing of trajectories α and γ, where $\alpha \in C^\infty(\mathbb{R}, M)$, $\gamma \in C^\infty((-\infty, 0], M)$ with matching ends, $\alpha(+\infty) = \gamma(-\infty) = q$ as follows. Let β now denote a smooth increasing cut-off function $0 \leq \beta(s) \leq 1$, such that $\beta(s) = 0$ for $s \leq 0$ and $\beta(s) = 1$ for $s \geq 1$. We have:

$$\alpha\,\sharp^{\,0}_{\rho}\gamma(s) := \begin{cases} \alpha(s+2\rho), & s \leq -\rho-1 \\ \exp_q(\beta(-s-\rho)\xi(s+2\rho)), & -\rho-1 \leq s \leq -\rho \\ q, & -\rho \leq s \leq -\frac{\rho}{2}-1 \\ \exp_q\left(\beta\left(s+\frac{\rho}{2}+1\right)\zeta(s)\right), & -\frac{\rho}{2}-1 \leq s \leq -\frac{\rho}{2} \\ \gamma(s), & -\frac{\rho}{2} \leq s \leq 0 \end{cases} \tag{28}$$

where $\exp_q \xi(s) = \alpha(s)$, for large positive s and $\exp_q \zeta(s) = \gamma(s)$, for large negative s.

Similarly, for $u \in C^\infty([0, +\infty) \times [0, 1], T^*M)$ and $v \in C^\infty(\mathbb{R} \times [0, 1], T^*M)$, $u(s, 0), u(s, 1), u(0, t), v(s, 0), v(s, 1) \in O_M$ such that it holds $u(+\infty, t) = v(-\infty, t) = y(t)$, we have:

$$u\,\sharp^{\,0}_{\rho}v(s,t) := \begin{cases} u(s,t), & 0 \leq s \leq \frac{\rho}{2} \\ \exp_{y(t)}\left(\beta\left(-s+\frac{\rho}{2}+1\right)\xi(s,t)\right), & \frac{\rho}{2} \leq s \leq \frac{\rho}{2}+1 \\ y(t), & \frac{\rho}{2}+1 \leq s \leq \rho \\ \exp_{y(t)}(\beta(s-\rho)\zeta(s-2\rho,t)), & \rho \leq s \leq \rho+1 \\ v(s-2\rho,t), & s \geq \rho+1. \end{cases} \tag{29}$$

Here $u(s,t) = \exp_{y(t)}(\xi(s,t))$ for all t and s large enough and positive, $v(s,t) = \exp_{y(t)}(\zeta(s,t))$ for all t and s large enough and negative.

Using the above gluing maps we construct pre-gluing of mixed trajectories:

Definition 6. *For $\alpha \in C^\infty(\mathbb{R}, M)$, $w = (\gamma, u) \in C^\infty(D, T^*M)$ and $v \in C^\infty(\mathbb{R}\times[0,1], T^*M)$, $v(s,0), v(s,1) \in O_M$ such that*

$$\alpha(+\infty) = \gamma(-\infty), \quad u(+\infty, t) = v(-\infty, t)$$

we define pre-glued mixed objects as:

$$\begin{aligned} \alpha \sharp^0_\rho w &:= (\alpha \sharp^0_\rho \gamma, u) \in C^\infty(D, T^*M), \\ w \sharp^0_\rho v &:= (\gamma, u \sharp^0_\rho v) \in C^\infty(D, T^*M). \end{aligned}$$

Using the gluing in Σ_{triv}, we glue Fredholm operators from Σ_w with standard operators from $\Sigma^M_{\alpha^*(TM)}$ and $\Sigma^F_{v^*(TT^*M)}$. Here $\Sigma^M_{\alpha^*(TM)}$ is the set of all operators K such that for some symplectic trivialization

$$\phi : \alpha^*(TT^*M) \to \mathbb{R}\times\mathbb{R}^{2n} \quad \text{and} \quad \widetilde{\phi} := \phi|_{TM}, \quad \widetilde{\phi} : \alpha^*(TM) \to \mathbb{R}\times\mathbb{R}^n$$

it holds

$$\widetilde{\phi} K \widetilde{\phi}^{-1} \in \Sigma^M_{\mathrm{triv}}$$

and $\Sigma^F_{v^*(TT^*M)}$ is the set of all operators L such that for some symplectic trivialization

$$\varphi : v^*(TT^*M) \to (\mathbb{R}\times[0,1])\times\mathbb{R}^{2n}$$

it holds

$$\varphi L \varphi^{-1} \in \Sigma^F_{\mathrm{triv}}.$$

Definition 7. *Let α, w, v be as in Definition* 6 *and $K \in \Sigma^M_{\alpha^*(TM)}$, $F = (F_1, F_2) \in \Sigma_w$, $L \in \Sigma^F_{v^*(TT^*M)}$. Suppose ϕ_α, ψ_w and φ_v are three trivializations:*

$$\begin{aligned} \phi_\alpha : \alpha^*(TT^*M) \to \mathbb{R}\times\mathbb{R}^{2n}, \quad \psi_w : w^*(TT^*M) \to D\times\mathbb{R}^{2n}, \\ \varphi_v : v^*(TT^*M) \to (\mathbb{R}\times[0,1])\times\mathbb{R}^{2n} \end{aligned}$$

defined as follows. Denote by $x(t) = u(+\infty, t) = v(-\infty, t)$ and let U be some open set containing $x([0,1])$. Smooth trivialization

$$\Gamma : TT^*M\big|_U \xrightarrow{\cong} U \times \mathbb{R}^{2n}$$

induces trivializations ψ_w *and* φ_v *of bundles* $w^*(TT^*M)$ *and* $v^*(TT^*M)$ *such that it holds*

$$\Gamma\big|_{w([R,+\infty]\times[0,1])} = \psi_w\big|_{[R,+\infty]}, \quad \Gamma\big|_{v([-\infty,-R]\times[0,1])} = \varphi_v\big|_{[-\infty,-R]}$$

for R *large enough (such that* $w([R,+\infty]\times[0,1])$, $v([-\infty,-R]\times[0,1]) \subset U$*). Define*

$$\psi_w \sharp_\rho \varphi_v(s,t) := \begin{cases} \psi(s,t), & 0 \le s \le \frac{\rho}{2} \\ \Gamma_{w\sharp^0_\rho v}, & \frac{\rho}{2} \le s \le \rho+1 \\ \varphi(s-2\rho,t), & s \ge \rho+1 \end{cases}$$

for $\rho \ge \max\{2R, R+1\}$ *and similarly in the case of* ϕ_α *and* ψ_w*. Obviously:*

$$\phi_\alpha(+\infty) = \psi_w(-\infty), \quad \psi_w(+\infty,t) = \varphi_v(-\infty,t).$$

Finally, define:

$$K \sharp_\rho F := \left(\phi_\alpha \sharp_\rho \psi_w\right)^{-1} \left(\phi(K) \sharp_\rho \psi(F)\right) \left(\phi_\alpha \sharp_\rho \psi_w\right) \in \Sigma_{\alpha \sharp_\rho w}$$

and, similarly

$$F \sharp_\rho L := \left(\psi_w \sharp_\rho \varphi_v\right)^{-1} \left(\psi(F) \sharp_\rho \varphi(L)\right) \left(\psi_w \sharp_\rho \varphi_v\right) \in \Sigma_{w \sharp_\rho v}.$$

It follows easily, due to homotopy invariance, that the above construction induces a gluing operation of the equivalence classes $[\alpha, K]$, $[v, L]$ and $[w, F]$ which does not depend on the non-canonical elements. The following step is to transfer gluing of orientation to the non-linear case. Note that the construction of gluing of orientation in non-linear case for gradient trajectories in [22] uses different class of admissible trivializations from the ones we use here (Definition 4). Let $\gamma_1, \gamma_2 \in C^\infty(\mathbb{R}, M)$. Pairs of trivializations considered in [22] were

$$\phi_{\gamma_1} : \gamma_1^* TM \xrightarrow{\cong} \mathbb{R}\times\mathbb{R}^n; \quad \psi_{\gamma_2} : \gamma_2^* TM \xrightarrow{\cong} \mathbb{R}\times\mathbb{R}^n$$

such that

$$\begin{aligned} &\phi_{\gamma_1}(-\infty) = \psi_{\gamma_2}(-\infty), \\ &\phi_{\gamma_1}(+\infty)\psi_{\gamma_2}^{-1}(+\infty) = \begin{bmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathrm{Gl}(n,\mathbb{R}). \end{aligned} \tag{30}$$

We will slightly modify the class of admissible trivializations in order to adapt it to our situation. First, we consider only *symplectic* trivializations

$$\psi_\gamma : \gamma^*(TT^*M) \to \mathbb{R} \times \mathbb{R}^{2n}.$$

Then, for the pair

$$\phi_{\gamma_1} : \gamma_1^*(TT^*M) \xrightarrow{\cong} \mathbb{R} \times \mathbb{R}^{2n}; \quad \psi_{\gamma_2} : \gamma_2^*(TT^*M) \xrightarrow{\cong} \mathbb{R} \times \mathbb{R}^{2n}$$

we require

$$\phi_{\gamma_1}(\pm\infty) = \psi_{\gamma_2}(\pm\infty)$$

instead of (30). Such a pair of trivializations we call admissible. The proof of the fact that the pair of admissible trivializations always exist is completely analogous to the proof of the same claim when the basis is D (due to the homotopy equivalence of $\mathbb{R}$ and D). The fact that admissible pairs induce the orientation in non-linear case can be also analogously proved as in the case of "mixed domain" D.

In the case of the spaces of smooth disks in $C^\infty(\mathbb{R} \times [0, 1], T^*M)$ with the boundary in O_M we consider the setting for the gluing of orientation in non-linear case as in [9]. Our situation differs from the one in [9] because the domain of the disk u is contractible here unlike there where it was a cylinder. The class of admissible trivializations is the same in our case of mixed type objects as it was there. More precisely, we call the pair of symplectic trivializations

$$\phi_u : u^*(TT^*M) \xrightarrow{\cong} (\mathbb{R} \times [0, 1]) \times \mathbb{R}^{2n}$$
$$\psi_v : v^*(TT^*M) \xrightarrow{\cong} (\mathbb{R} \times [0, 1]) \times \mathbb{R}^{2n}$$

admissible if

$$\phi_u(\pm\infty, t) = \psi_v(\pm\infty, t) \quad \text{for all } t \in [0, 1].$$

Again, one can prove that admissible trivializations exist and that they induce orientations in non-linear case in the same way as in Lemma 10 and Lemma 11.

We assume now that the gluing of orientations of trajectories from $C^\infty(\mathbb{R}, M)$ in Morse case and disks in $C^\infty(\mathbb{R} \times [0, 1], T^*M)$ with ends in O_M from Floer's (i.e. the classes of corresponding operators of special type) is defined in the similar way as in [9, 22] with the only change of notion of admissible trivializations for $\gamma \in C^\infty(\mathbb{R}, M)$ discussed above. We glue two orientations of a mixed and non-mixed object in following way. Let α, $w = (\gamma, u)$, v, K, F, L, ϕ_α, ψ_w and

φ_v be as in Definition 7. If o_K, o_F and o_L are three orientation from Det$[K]$, Det$[F]$ and Det$[L]$ then we set

$$o_K \sharp o_F := \big(\phi_\alpha \sharp_\rho \psi_w\big)^{-1}\big(\phi_\alpha(o_K) \sharp \psi_w(o_F)\big)$$

and, similarly

$$o_F \sharp o_L := \big(\psi_w \sharp_\rho \varphi_v\big)^{-1}\big(\psi_w(o_F) \sharp \varphi_v(o_L)\big).$$

As before, one can prove that this definition is independent of all choices involved.

Once again, we can repeat the same construction to define orientation compatible for gluing of standard operators and the mixed type operators with the domain

$$D' := \big((-\infty, 0] \times [0, 1]\big) \cup \left([0, +\infty) \times \frac{1}{2}\right)$$

The notations are the following. For $w = (u, \gamma) \in C^\infty(D', T^*M)$, such that $u(s, 0), u(s, 1), u(0, t), \gamma(s) \in O_M$, denote by Ξ_w the set of all operators $F = (F_1, F_2) \in \mathcal{L}(W^{1,r}(w^*(TT^*M)), L^r(w^*(TT^*M)))$ such that, for some symplectic trivialization

$$\psi : w^*(TT^*M) \to D' \times \mathbb{R}^{2n}$$

the following holds

$$\psi(F) := \psi F \psi^{-1} \in \Xi_{\text{triv}}.$$

The set $\mathcal{L}(W^{1,r}(w^*(TT^*M)), L^r(w^*(TT^*M)))$ is defined in the same way as in the case of Σ_w (page 275).

The operation $\sharp$ is associative:

$$\big(o_K \sharp o_F\big)\sharp o_L = o_K \sharp\big(o_F \sharp o_L\big)$$

since it holds in the trivial case. We will denote the orientation for the equivalence classes of operators which is *coherent,* i.e. which commutes with the gluing operation by σ.

Remark 12. There is an isomorphism

$$D \sharp_\rho : \text{Ker}\big(D_w\big) \times \text{Ker}\big(D_v\big) \xrightarrow{\cong} \text{Ker}\big(D_{w \sharp_\rho v}\big)$$

obtained as the differential of the gluing map (see [14]). This isomorphism induces the orientation which is compatible with the glued orientation $o_w \sharp o_v$. Namely the operations $\sharp_\rho$ and $\sharp_\rho^0$ are homotopic in the space $C^\infty(D, T^*M)$. From the very construction of gluing orientations $\sharp$ one sees that this operation is defined using the isomorphism which arises from pre-gluing of trajectories $\sharp_\rho^0$ and Fredholm operators (see [14]). It follows that the glued orientation and the one induced by $\hat{\sharp} = D \sharp_\rho^0$ are compatible.

3.2 Orientation for R-parameterized moduli spaces

In order to prove that Φ and Ψ in (4) are isomorphisms we consider the set

$$\mathcal{M}(R;p,q,f;H) := \left\{ (\gamma_-,u,\gamma_+,R) \;\middle|\; \begin{array}{l} \gamma_- : (-\infty,0] \to M \\ \gamma_+ : [0,+\infty) \to M \\ u : \mathbb{R}\times[0,1] \to T^*M \\ \dfrac{d\gamma_\pm}{ds} = -\nabla f(\gamma_\pm) \\ \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_{\rho_R H}(u)\right) = 0 \\ \gamma_-(-\infty) = p,\ \gamma_+(+\infty) = q \\ u(\partial(\mathbb{R}\times[0,1])) \subset O_M \\ u(\pm\infty,t) = \gamma_\pm(0) \end{array} \right\}. \tag{31}$$

Here ρ_R satisfies

$$\rho_R(s) = \begin{cases} 1, & |s| \le R \\ 0, & |s| \ge R+1. \end{cases}$$

We will also consider the set

$$\begin{aligned} &\mathcal{M}_R(p,q,f;H) := \\ &\big\{(\gamma_-,\gamma_+,u) \mid (\gamma_-,\gamma_+,u) \text{ is a solution of (31) for fixed } R\big\}. \end{aligned} \tag{32}$$

For a linear version, we define the equivalence relation in a special class of operators

$$\begin{aligned} \Theta := \big\{ &F = (F_1,F_2,F_3) \mid \\ &F_i \text{ is of the type (9) for } i = 1,3,\ F_2 \text{ is of the type (11)}\big\} \end{aligned} \tag{33}$$

(we omit the subscript triv to abbreviate the notations) with domain

$$\widetilde{D} = (-\infty,0] \cup (\mathbb{R}\times[0,1]) \cup [0,+\infty)$$

in a following way:

$$\begin{gathered} F = (F_1,F_2,F_3) \sim G = (G_1,G_2,G_3) \quad \text{if and only if} \\ F_1^- = G_1^- \quad \text{and} \quad F_3^+ = G_3^+. \end{gathered}$$

We denote the set of equivalence classes by $\widetilde{\Theta}$. Using linear homotopy one can show that the equivalence class of this type of operators, considered as a subset of operators, is contractible with respect to the operator norm. We define gluing of maps from $C^\infty(\widetilde{D},T^*M)$ with the standard trajectories and gluing of operators of this type with the standard ones (whenever they are compatible for gluing) in an obvious way. A construction of gluing of orientations (both in trivial and non-trivial case) for the triples of operators with the standard operators of special type can be done in a complete analogy as in the case of the pairs.

We treat the tangent space of manifold $\mathcal{M}_R(p, q, f; H)$ as the zero set of a triple of operators (F_1, F_2, F_3) defined above. The case of manifold $\mathcal{M}(R; p, q, f; H)$ is different, it is a zero set of Fredholm map F in two variables: in $R \in \mathbb{R}$ and in $w_R = (\gamma_-, u, \gamma_+)$. Its linearization D_2F in second variable w_R is the Fredholm operator of the type (33). Extending [22] to this situation, we see that there exists a canonical bundle isomorphism

$$\operatorname{Det} D_2F \otimes T_*\mathbb{R} \cong \Lambda^{\max} \operatorname{Ker} DF$$

so the choice of the fixed orientation $\frac{\partial}{\partial R} \equiv 1$ on $T_*\mathbb{R}$ gives rise to natural isomorphism between determinant bundle of the operator D_2F of a special type (33) and the orientation of the tangent bundle of $\mathcal{M}(R; p, q, f; H)$:

$$\operatorname{Det} D_2F \cong \Lambda^{\max} \operatorname{Ker} DF. \tag{34}$$

So the orientation of the operator D_2F induces the orientation of the tangent space $T_{(R,w_R)}\mathcal{M}(R; p, q, f; H)$. Gluing of trajectories, operators and orientation can be constructed in an analogous way as in unparameterized case. Again the two orientation: the first one – induced by the differential $D\sharp$ and the second one – glued, are compatible.

Note that we can glue two operators $G \in \Sigma$ and $H \in \Xi$ to obtain the operator from Θ (unparameterized or parameterized – when $R \to +\infty$, see [15] or [13]). We can also glue the operators from Θ with the operators from Σ^M and the result will be operators from Θ. Gluing of orientation induced by gluing of operators in this case is also associative. Since the details of this construction are analogous to the previous ones, we skip them.

3.3 Coherent orientation

In this section we show that a coherent orientation exists for operators of mixed type and standard operators together. We will divide this construction in several steps.

Morse trajectories. Fix an arbitrary critical point p_0 of f and consider it to be the constant curve. The operator:

$$K_0 = \frac{d}{ds} + A_0 \in \Sigma_{p_0^*TM}$$

is an isomorphism and hence Det $K_0 = \mathbb{R} \otimes \mathbb{R}^*$. Fix the orientation

$$\sigma\big([p_0, K_0]\big) := 1 \otimes 1^* \tag{35}$$

for Det K_0.

We first orient the classes of maps in $C^\infty(\mathbb{R}, M)$ and corresponding operators as in [22], by choosing $[p_0, K_0]$ to be the "anchoring class". More precisely, consider the sets

$$\mathcal{T}^- := \big\{[\gamma, K] \in \Sigma^M \mid \gamma(-\infty) = p_0,\ K^- = K_0\big\}$$

and

$$\mathcal{T}^+ := \big\{[\gamma, K] \in \Sigma^M \mid \gamma(+\infty) = p_0,\ K^+ = K_0\big\}.$$

There is a bijection between $\mathcal{T}^-$ and $\mathcal{T}^+$:

$$\mathcal{T}^- \ni [\gamma, K] \mapsto [\overline{\gamma}, \overline{K}], \quad \text{where} \quad \overline{\gamma}(s) := \gamma(-s),\ \overline{K} = K.$$

We orient the set $\mathcal{T}^- \backslash \{[p_0, K_0]\}$ arbitrarily. Then we orient the set $\mathcal{T}^+ \backslash \{[p_0, K_0]\}$ using the condition

$$\sigma\big([\gamma, F]\big) \sharp\, \sigma\big([\overline{\gamma}, \overline{F}]\big) \simeq \sigma\big([p_0, K_0]\big).$$

For any class $[\gamma, F]$ from $C^\infty(\mathbb{R}, M)$ we find the unique class $[\alpha, F_1] \in \mathcal{T}^-$ and $[\beta, F_2] \in \mathcal{T}^+$ such that

$$\alpha(+\infty) = \gamma(-\infty), \quad \beta(-\infty) = \gamma(+\infty); \quad F_1^+ = F^-, \quad F_2^- = F^+.$$

Define an orientation of $[\gamma, F]$ by the condition

$$\sigma\big([\alpha, F_1]\big) \sharp\, \sigma\big([\gamma, F]\big) \sharp\, \sigma\big([\beta, F_2]\big) \simeq \sigma\big([p_0, K_0]\big)$$

(see Chapter 3.2. in [22] for more details).

Floer trajectories. Now we orient the classes of maps in $C^\infty(\mathbb{R} \times [0,1], T^*M)$ and corresponding operators. Denote by $[u, K]$ the $\sim$ equivalence class of path u and operator K, where $(u_1, K_1) \sim (u_2, K_2)$ if $u_1(\pm\infty, t) = u_2(\pm\infty, t)$ and $K_1^\pm = K_2^\pm$. Define:

$$\mathcal{P}^- = \big\{[w, F] \mid w = (\gamma, u),\ \gamma(-\infty) = p_0,\ F = (F_1, F_2) \in \Sigma_w,\ F_1^- = K_0\big\}.$$

Fix any class $[w_0 = (\gamma_0, u_0), F_0 = (F_{01}, F_{02})]$ from $\mathcal{P}^-$ such that F_0 is an isomorphism (such a class obviously does exist) and orient it by

$$\sigma\big([w_0, F_0]\big) := 1 \otimes 1^*. \tag{36}$$

Let $x_0(t) := u_0(+\infty, t)$. Denote by L_0^- the operator F_{02}^+ where $F_0 = (F_{01}, F_{02})$. The set

$$S^- := \{[u, K] \in \Sigma^F \mid u(-\infty, t) = x_0(t),\ K^- = L_0^-\}$$

contains the class $[x_0, C_0]$ such that $x_0(s, t) = x_0(t)$ and $C_0(s, t) = \frac{\partial}{\partial s} + J_{L_0}(-\infty, t)\frac{\partial}{\partial t} + B_{L_0}(-\infty, t)$ is an isomorphism. Orient class $[x_0, C_0]$ by

$$\sigma\big([x_0, C_0]\big) := 1 \otimes 1^* \tag{37}$$

and the rest of the set S^- arbitrarily. The orientation of classes from the set

$$S^+ := \{[u, K] \in \Sigma^F \mid u(+\infty, t) = x_0(t),\ K^+ = L_0^-\}$$

are determined by the requirement that for $[u_1, K_1] \in S^-$, $[u_2, K_2] \in S^+$ with $K_1^+ = K_2^-$ and $u_1(+\infty, t) = u_2(-\infty, t)$ it holds

$$\sigma\big([u_1, K_1]\big) \sharp \sigma\big([u_2, K_2]\big) \simeq \sigma\big([x_0, C_0]\big).$$

For any class $[u, L]$ in Σ^F, we find unique classes $[u_1, K_1] \in S^-$ and $[u_2, K_2] \in S^+$ such that

$$u_1(+\infty, t) = u(-\infty, t), \quad u(+\infty, t) = u_2(-\infty, t); \quad K_1^+ = L^-, \quad L^+ = K_2^-$$

and define an orientation of $[L]$ by the condition

$$\sigma([u_1, K_1]) \sharp \sigma([u, L]) \sharp \sigma([u_2, K_2]) \simeq \sigma([x_0, C_0])$$

(see Theorem 12 in [9] for more details about the orientation of the operators of the type Σ^F).

PSS trajectories. The next step is to orient the set of classes of mixed objects. Let $w_0 = (\gamma_0, u_0)$ be already mentioned fixed class and $[w = (\gamma, u), F = (F_1, F_2)]$ be such that $F \in \Sigma_w$. There are unique classes $[\alpha, K] \in \Sigma^M$, $[v, L] \in \Sigma^F$ of a special type operators such that

$$\begin{aligned}
&\alpha(-\infty) = \gamma(-\infty), && \alpha(+\infty) = \gamma_0(-\infty) = p_0,\\
&u_0(+\infty, t) = v(-\infty, t), && v(+\infty, t) = u(+\infty, t)\\
&K^- = F_1^-,\ K^+ = F_{01}^- && F_{02}^+ = L^-,\ L^+ = F_2^+.
\end{aligned}$$

Define the orientation of $[w, F]$ in a following way:

$$\sigma\big([w, F]\big) := \sigma\big([\alpha, K]\big) \sharp \sigma\big([w_0, F_0]\big) \sharp \sigma\big([v, L]\big).$$

Due to (35), (37) and (36) this definition is consistent in the relation to anchoring class $[w_0, F_0]$.

If $F' \in \Xi_{w'}$ we orient the class $[w', F']$ similarly. We choose the orientation of some fixed class, for example $[\overline{w_0}, \overline{F}_0]$, where

$$\overline{w_0} = (\bar{u}, \overline{\gamma_0}), \quad \overline{\gamma_0}(s) := \gamma(-s), \quad \overline{u_0}(s,t) := u(-s, 1-t), \quad \overline{F} := (F_2, F_1)$$

to be $1 \otimes 1^*$ and we orient the rest of the set Ξ by requiring:

$$\sigma\big([u, L]\big) \sharp \sigma\big([\overline{w_0}, \overline{F}]\big) \sharp \sigma\big([\gamma, K]\big) \simeq \sigma\big([w', F']\big)$$

where $[u, L]$ and $[\gamma, K]$ are, similarly as before, unique classes that connect, respectively, $\overline{w_0}(-\infty)$ with $w'(-\infty)$ and $w'(+\infty)$ with $\overline{w_0}(+\infty)$ and the corresponding operators.

Glued PSS trajectories. Finally, for arbitrary class $[w_R, F_R] \in \widetilde{\Theta}$ we find classes $[w_1, G_1] \in \widetilde{\Sigma}$ and $[w_2, G_2] \in \widetilde{\Xi}$ such that

$$[w_1, G_1] \sharp [w_2, G_2] = [w_R, F_R].$$

Define $\sigma([w_R, F_R])$ as $\sigma([w_1, G_1]) \sharp \sigma([w_2, G_2])$. These two classes do not have to be unique, but the glued class does not change. Indeed, let $[w_1, G_1]$, $[w_2, G_2]$ and $[w_1', G_1']$, $[w_2', G_2']$ be two pairs of such classes, with $[w_1, G_1](+\infty) = [w_2, G_2](-\infty) = (x, G)$ and $[w_1', G_1'](+\infty) = [w_2', G_2'](-\infty) = (x', G')$. There exists unique class $[u, K] \in \Sigma^F$ that connects (x, G) and (x', G'). Then we have:

$$\begin{aligned}
\sigma\big([w_1, G_1]\big) \sharp \sigma\big([w_2, G_2]\big) &\simeq \sigma\big([w_1', G_1']\big) \sharp \sigma\big([u, K]\big) \sharp \sigma\big([w_2, G_2]\big) \\
&\simeq \sigma\big([w_1', G_1']\big) \sharp \sigma\big([u, K]\big) \sharp \sigma\big([\bar{u}, \overline{K}]\big) \sharp \sigma\big([w_2', G_2']\big) \\
&\simeq \sigma\big([w_1', G_1']\big) \sharp \sigma\big([w_2', G_2']\big).
\end{aligned} \tag{38}$$

Here $[\bar{u}, \overline{K}]$ is unique class such that

$$\bar{u}(\pm\infty) = u(\mp\infty), \quad (\overline{K})^{\pm} = K^{\mp}.$$

The last equality in (38) is true due to the above construction of coherent orientation of Σ^F. Indeed, if $[u, K] \in \Sigma^F$ let $[v, L], [w, F] \in \Sigma^F$ be the unique classes such that

$$\begin{aligned}
&(u, K)(-\infty) = (v, L)(+\infty), \ (u, K)(+\infty) = (w, F)(-\infty), \\
&(v, L)(-\infty) = (v_0, C_0)(-\infty), \ (w, F)(+\infty) = (v_0, C_0)(+\infty),
\end{aligned}$$

i.e. $(v, L) \in S^-$, $(w, F) \in S^+$. From

$$\sigma\left(([v,L]\,\sharp\,[u,K]\,\sharp\,[w,F])\,\sharp\,(\overline{[v,L]\,\sharp\,[u,K]\,\sharp\,[w,F]})\right) =$$
$$\sigma\left(([v,L]\,\sharp\,[u,K]\,\sharp\,[w,F])\,\sharp\,([\overline{w},\overline{F}]\,\sharp\,[\bar{u},\overline{K}]\,\sharp\,[\bar{v},\overline{L}])\right) \simeq 1\otimes 1^*$$
$$\sigma\big([v,L]\,\sharp\,([\bar{v},\overline{L}])\big) \simeq 1\otimes 1^* \quad \sigma\big([w,F]\,\sharp\,([\overline{w},\overline{F}])\big) \simeq 1\otimes 1^*$$

it follows

$$\sigma\big([\bar{u},\overline{K}]\big)\,\sharp\,\sigma\big([u,K]\big) \simeq 1\otimes 1^*.$$

Thus the definition is correct.

Since the gluing of orientations is an associative operation, the coherent orientation is well defined by the above description.

Denote by Λ the set of equivalence classes of maps and operators of all mentioned types: trajectories, disks, mixed objects (of all three types) and R-parameterized mixed objects and by C_Λ the set of all coherent orientations on Λ. Consider the action of a group

$$\Gamma := \left\{ f \in \{-1,1\}^\Lambda \mid f([w,F]\sharp[u,L]) = f([w,F])f([u,L]) \right\}$$

under a pointwise multiplication. We assume here that the classes $[w, F]$ and $[u, L]$ are of any type admissible for gluing.

Proposition 13. *The group Γ acts freely and transitively on C_Λ by*

$$(f\cdot\sigma)([w,F]) = f([w,F])\sigma([w,F]).$$

Proof. Let σ_1 and σ_2 be two coherent orientations. Define f such that $\sigma_1 = f\cdot\sigma_2$. We want to check that f is indeed an element in Γ. For $[u, K]$ and $[v, L]$ compatible for gluing we have

$$\begin{aligned}
\sigma_1([u,K])\,\sharp\,\sigma_1([v,L]) &\simeq \sigma_1([u,K]\,\sharp\,[v,L])\\
&\simeq f([u,K]\,\sharp\,[v,L])\sigma_2([u,K]\,\sharp\,[v,L])\\
&\simeq f([u,K]\,\sharp\,[v,L])\sigma_2([u,K])\,\sharp\,\sigma_2([v,L])\\
&\simeq f([u,K]\,\sharp\,[v,L])f([u,K])f([v,L])\sigma_1([u,K])\,\sharp\,\sigma_1([v,L])
\end{aligned}$$

which exactly means that

$$f([u,K]\,\sharp\,[v,L]) = f([u,K])f([v,L]). \qquad \square$$

4 Canonical orientation and construction of isomorphism using characteristic signs

In this chapter we construct canonical orientation and compare it to the coherent one, constructed in the previous chapter, in order to associate a sign $+$ or $-$ to each isolated trajectory involved in our construction of isomorphism (4).

The canonical orientation for mixed moduli space is given only for its zero-dimensional components. Unlike in [22] and [9] we have no $\mathbb{R}$-action in definition of $\mathcal{M}(p, f; x, H)$ and $\mathcal{M}(x, H; p, f)$. Thus an element from the kernel of the operator $F = (F_1, F_2)$, a mixed object w, is an isolated trajectory if the linearization of F, $D_w F$ has trivial kernel. The determinant bundle, $\mathrm{Det}[w]$, is trivial, and we can orient zero-dimensional components canonically, by $1 \otimes 1^*$.

We also need to orient canonically the isolated non mixed objects, gradient trajectories and (perturbed) holomorphic disks. In this case there is an $\mathbb{R}$-action. Denote by $\mathcal{M}(p, q, f)$ the set of solutions of:

$$\begin{cases} \dfrac{d\gamma}{ds} + \nabla f(\gamma) = 0 \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q, \end{cases}$$

and by $\mathcal{M}(x, y, H)$ the set of solutions of:

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(s, i) \in L_0, \ i \in \{0, 1\} \\ u(-\infty, t) = x(t) \\ u(+\infty, t) = y(t). \end{cases}$$

Let $\widehat{\mathcal{M}}(p, q, f)$ and $\widehat{\mathcal{M}}(x, y, H)$ denote these sets modulo $\mathbb{R}-$ action $\gamma(\cdot) \mapsto \gamma(\cdot + \tau)$, and $u(\cdot, \cdot) \mapsto u(\cdot + \tau, \cdot)$. Hence $\gamma \in \mathrm{Ker}\, K$ and $u \in \mathrm{Ker}\, L$ are isolated if $\mathrm{Ker}\, D_\gamma K$ and $\mathrm{Ker}\, D_u L$ are one-dimensional. We have the "flow orientation" determined by

$$0 \neq \frac{d\gamma}{ds} \in \mathrm{Ker}\, D_\gamma K$$

for $\mathcal{M}(p, q, f)$ and

$$0 \neq \frac{\partial u}{\partial s} \in \mathrm{Ker}\, D_u L$$

for $\mathcal{M}(x, y, H)$. We denote these canonical orientations by $[1 \otimes 1_w^*]$, $[\gamma_s]$ and $[u_s]$.

Now define numbers $\tau(\gamma)$, $\tau(u)$ and $\tau(w)$ in $\{-1, 1\}$ to satisfy:

$$\sigma([w]) = \tau(w)[1 \otimes 1_w^*], \quad \sigma([\gamma]) = \tau(\gamma)[\gamma_s], \quad \sigma([u]) = \tau(u)[u_s].$$

Using these signs we define homomorphisms

$$\Psi : CM_k(f) \to CF_k(H), \qquad \Phi : CF_k(H) \to CM_k(f)$$

by

$$p \mapsto \sum_{w \in \mathcal{M}(p,f;x,H)} \tau(w)x, \qquad x \mapsto \sum_{w \in \mathcal{M}(x,H;p,f)} \tau(w)p$$

on the generators. Homomorphisms Ψ and Φ induce homomorphisms on homologies if

$$\Psi \circ \partial_M = \partial_F \circ \Psi, \quad \Phi \circ \partial_F = \partial_M \circ \Phi.$$

The first equality is equivalent to

$$\begin{gathered}\sum_{\gamma_1 \in \widehat{\mathcal{M}}(p,r,f)} \left(\sum_{w_1 \in \mathcal{M}(r,f;x,H)} \tau(w_1)\tau(\gamma_1) \right) = \\ \sum_{w_2 \in \mathcal{M}(p,f;y,H)} \left(\sum_{u_2 \in \widehat{\mathcal{M}}(y,x,H)} \tau(u_2)\tau(w_2) \right).\end{gathered} \tag{39}$$

The proof of (39) will follow from the identity

$$\begin{gathered}\partial \mathcal{M}(p,f;x,H) = \\ \bigcup \widehat{\mathcal{M}}(p,r,f) \times \mathcal{M}(r,f;x,H) \cup \bigcup \mathcal{M}(p,f;y,H) \times \widehat{\mathcal{M}}(y,x,H),\end{gathered} \tag{40}$$

the fact that the number of the boundary of one-dimensional manifold is even and the next Theorem. (The proof of the second identity is analogous.) The proof of the identity (40) follows from Gromov compactness and gluing arguments (see [13, 14]).

Theorem 14. *Let* $m_f(p) = \left(\mu_H(x) + \frac{n}{2}\right) + 1$. *Assume that* $\mathcal{M}(p,f;x,H)$ *has only one connected non-compact component, i.e.* $\mathcal{M}(p,f;x,H) \approx (-1,1)$. *The boundary of component* $\mathcal{M}(p,f;x,H)$ *can be one of the next three possibilities:*

1. $(\gamma_1, w_1) \in \widehat{\mathcal{M}}(p,r,f) \times \mathcal{M}(r,f;x,H)$ *and* $(\gamma_1, w_2) \in \widehat{\mathcal{M}}(p,r',f) \times \mathcal{M}(r',f;x,H)$;
2. $(w_1, u_1) \in \mathcal{M}(p,f;y,H) \times \widehat{\mathcal{M}}(y,x,H)$ *and* $(w_2, u_2) \in \mathcal{M}(p,f;y',H) \times \widehat{\mathcal{M}}(y',x,H)$;

3. $(\gamma_1, w_1) \in \widehat{\mathcal{M}}(p, r, f) \times \mathcal{M}(r, f; x, H)$ *and* $(w_2, u_2) \in \mathcal{M}(p, f; y, H) \times \widehat{\mathcal{M}}(y, x, H)$. *For each of these possibilities it holds:*

   1. $\tau(\gamma_1)\tau(w_1) = -\tau(\gamma_1)\tau(w_2)$;
   2. $\tau(w_1)\tau(u_1) = -\tau(w_2)\tau(u_2)$;
   3. $\tau(\gamma_1)\tau(w_1) = \tau(w_2)\tau(u_2)$.

Proof. Case 1. It holds

$$
\begin{aligned}
[\gamma_{1s}]\sharp[1 \otimes 1_{w_1}] &\overset{(i)}{\simeq} \Big(\tau(\gamma_1)\sigma([\gamma_1])\Big)\sharp\Big(\tau(w_1)\sigma([w_1])\Big) \\
&\overset{(ii)}{\simeq} \tau(\gamma_1)\tau(w_1)\Big(\sigma([\gamma_1])\sharp\sigma([w_1])\Big) \\
&\overset{(iii)}{\simeq} \tau(\gamma_1)\tau(w_1)\Big(\sigma([\gamma_1\sharp w_1])\Big) \\
&\overset{(iv)}{\simeq} \tau(\gamma_1)\tau(w_1)\Big(\sigma([\gamma_1\sharp w_2])\Big) \\
&\overset{(v)}{\simeq} \tau(\gamma_1)\tau(w_1)\Big(\sigma([\gamma_1])\sharp\sigma([w_2])\Big) \\
&\overset{(vi)}{\simeq} \tau(\gamma_1)\tau(w_1)\Big(\tau(\gamma_1)[\gamma_{2s}]\sharp\tau(w_2)[1 \otimes 1_{w_2}]\Big) \\
&\overset{(vii)}{\simeq} \tau(\gamma_1)\tau(w_1)\tau(\gamma_1)\tau(w_2)\Big([\gamma_{2s}]\sharp[1 \otimes 1_{w_2}]\Big).
\end{aligned}
\tag{41}
$$

Equalities (i) and (vi) are just the definitions of the characteristic numbers τ; (ii) and (vii) follow from the definition of gluing of orientation. Equalities (iii) and (v) follow from the fact that σ is coherent; finally equality (iv) is true because $\gamma_1\sharp^0 w_1 \sim \gamma_2\sharp^0 w_2$. So the assertion for the first case will follow from

$$
\big[\gamma_{1s}\big]\sharp\big[1 \otimes 1_{w_1}\big] \simeq -\big[\gamma_{2s}\big]\sharp\big[1 \otimes 1_{w_2}\big]. \tag{42}
$$

To prove (42) consider the set

$$
\mathcal{M}(p, f; x, H) \approx (-1, 1) \tag{43}
$$

and suppose that its orientation is given by $\dfrac{d}{d\tau} \in T_\tau(-1, 1)$. With the identification (43), gluing is the map

$$
\sharp : \{\gamma_1\} \times \{w_1\} \times \big(\rho_0, +\infty\big) \to \big(-1, -1 + \varepsilon\big).
$$

All the isolated points of form $\{w_1\}$ are canonically and uniformly oriented, hence gluing can also be considered as the map

$$\sharp : \{\gamma_1\} \times (\rho_0, +\infty) \to (-1, -1+\varepsilon).$$

Since γ_1 is in $\widehat{\mathcal{M}}(p, r, f)$ and $\mathcal{M}(p, r, f) = \widehat{\mathcal{M}}(p, r, f) \times \mathbb{R}$, $\gamma_1 \in \widehat{\mathcal{M}}(p, r, f)$ corresponds to $\gamma_1 \times \mathbb{R} \in \mathcal{M}(p, r, f)$. We identify $\mathbb{R} \approx (\rho_0, +\infty)$ by the orientation preserving map, so we have an identification

$$(\rho_0, +\infty) \approx \{\gamma_1\} \times (\rho_0, +\infty) \approx \{\gamma_1\} \times \mathbb{R}$$

such that the following correspondence holds

$$\operatorname{Det} D_{\gamma_1} K \ni [\gamma_{1s}] \leftrightarrow \frac{d}{d\rho} \in T_\rho(\rho_0, +\infty).$$

Hence gluing $\sharp$ is a mapping

$$\sharp : (\rho_0, +\infty) \to (-1, -1+\varepsilon)$$

that does not preserve the orientation since the point $+\infty$ corresponds to the point -1. Analogously, for the other pair γ_1 and w_2, the gluing (denote it by $\sharp'$) can be considered as the map

$$\sharp' : (\rho_0, +\infty) \to (1-\varepsilon, 1),$$

which preserves orientation because the point $+\infty$ corresponds to point 1 now. So (42) follows.

Case 2 is completely analogous to the **Case 1**.

Case 3. It follows from the same arguments as in (41) that

$$[\gamma_{1s}]\sharp[1 \otimes 1_{w_1}] \simeq \tau(\gamma_1)\tau(w_1)\tau(w_2)\tau(u_2)\Big([1 \otimes 1_{w_2}]\sharp[u_{2s}]\Big),$$

so the only thing we have to verify is

$$[\gamma_{1s}]\sharp[1 \otimes 1_{w_1}] \simeq [1 \otimes 1_{w_2}]\sharp[u_{2s}]. \tag{44}$$

We do have the same identifications as in **Case 1**, and the mappings $\sharp$ and $\sharp'$ as

$$\sharp : (\rho_0, +\infty) \to (-1, -1+\varepsilon), \qquad \sharp' : (\rho_0, +\infty) \to (1-\varepsilon, 1).$$

But for gluing $\sharp$, γ_1 is the first trajectory in the definition (28), and for gluing $\sharp'$, u_2 is the second in (29), so ρ appears with the opposite sign there. Thus there is one more reverse of orientation than in **Case 1**, so (44) holds. $\square$

The fact that homomorphisms Φ and Ψ are isomorphisms with $\mathbb{Z}_2$ coefficients follows from the analysis of the boundary of $\mathcal{M}(R; p, q, f; H)$ defined by (31) (see [15] and [13]). In order to show that Φ and Ψ defined in this way (with $\mathbb{Z}$ coefficients) are also isomorphisms, we need to choose the canonical orientations for zero dimensional component of $\mathcal{M}_R(p, q, f; H)$ and $\mathcal{M}(R; p, q, f; H)$. We orient all these zero dimensional components canonically by $1 \otimes 1^*$.

Remark 15. For $m_f(p) = m_f(q) - 1$, $\mathcal{M}(R; p, q, f; H)$ is zero-dimensional manifold and its tangent space is zero of certain Fredholm operator, denote it by DF. Then $DF = (D_1 F, D_2 F)$, where $D_1 = D_R$ and $D_2 = D_{w_R}$ are the derivatives in R and w_R respectively. Since $m_f(p) = m_f(q) - 1$, the operator DF_2 is injective and has one-dimensional cokernel. After stabilization by the R direction it becomes an isomorphism so that its determinant bundle has a canonical orientation.

As before, denote by $\tau(\cdot)$ the characteristic sign which relates coherent and canonical orientation.

Remark 16. For $m_f(p) = m_f(q)$, $\mathcal{M}(R; p, q, f; H)$ is one-dimensional manifold and its tangent space is zero of Fredholm operator $DF = (D_1 F, D_2 F)$, as in Remark 15. The value R is regular with respect to

$$\pi : \mathcal{M}(R; p, q, f; H) \to [R_0, +\infty)$$

if and only if $D_2 F$ is onto. Recall the isomorphism (34) between Det $D_2 F$ and $\Lambda^{\max}$ Ker DF. It identifies the canonical orientation $1 \otimes 1^*$ on $[D_2 F]$ with that orientation of Ker $DF = T_{(R,w)}\mathcal{M}(R; p, q, f; H)$ which is mapped (by π) onto the canonical orientation d/dR of $[R_0, +\infty)$.

In [15] we showed that, for $m_f(p) = m_f(q)$, the boundary of $\mathcal{M}(R; p, q, f; H)$ is identified with

$$\partial \mathcal{M}(R; p, q, f; H) = \partial_1 \cup \partial_2 \cup \partial_3 \cup \partial_4, \tag{45}$$

where (see Figure 3)

$$\begin{aligned}
\partial_1 &= \mathcal{M}_{R_0}(p, q, f; H), \\
\partial_2 &= \bigcup_{m_f(r) = m_f(p) - 1} \widehat{\mathcal{M}}(p, r, f) \times \mathcal{M}(R; r, q, f; H), \\
\partial_3 &= \bigcup_{m_f(r) = m_f(q) + 1} \mathcal{M}(R; p, r, f; H) \times \widehat{\mathcal{M}}(r, q, f), \\
\partial_4 &= \bigcup_{\mu_H(x) = m_f(p)} \mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f).
\end{aligned}$$

Consider homomorphisms

$$T : CM_k(f) \to CM_k(f), \qquad p \mapsto \sum_{m_f(q)=k} n_{R_0}(p, q, f; H)q$$

where

$$n_{R_0}(p, q, f; H) := \sum_{w_{R_0} \in \mathcal{M}_{R_0}(p,q,f;H)} \tau(w_{R_0})$$

and

$$K : CM_k(f) \to CM_{k+1}(f), \qquad p \mapsto \sum_{m_f(q)=k+1} n(R; p, q, f; H)q,$$

where

$$n(R; p, q, f; H) := \sum_{(R,w_R) \in \mathcal{M}(R;p,q,f;H)} \tau((R, w_R)).$$

In order to prove that $\Phi \circ \Psi = \mathrm{Id}$ we have to check that:

$$\begin{aligned} &\sum_{(w_1,w_2)\in\partial_4} \tau(w_1)\tau(w_2) - \sum_{w_{R_0}\in\partial_1} \tau(w_{R_0}) = \\ &\sum_{(u,\gamma_2)\in\partial_3} \tau(u)\tau(\gamma_2) + \sum_{(\gamma_1,v)\in\partial_2} \tau(\gamma_1)\tau(v). \end{aligned} \tag{46}$$

From (46) it follows that

$$\Phi \circ \Psi - T = \partial_M \circ K + K \circ \partial_M,$$

i.e. $\Phi \circ \Psi$ is chain homotopic to T. By deforming the two dimensional piece in the mixed object that defines T (i.e. an element of $\mathcal{M}_{R_0}(p, q, f; H)$) one can prove that the mapping T induces the identity on the homology. This proves the Proposition 2 (see [15]).

The connected non-compact component of $\mathcal{M}(R; p, q, f; H)$ is identified with one the next four intervals:

1. $[0, 1]$ – both ends are in $\partial_1 \cup \partial_4$;

2. $[0, +\infty)$ – one end is in ∂_1 and the other in $\partial_2 \cup \partial_3$;

3. $(-\infty, 1]$ – one end is in $\partial_2 \cup \partial_3$ and the other in ∂_4;

4. $(-\infty, +\infty)$ – both ends are in $\partial_2 \cup \partial_3$.

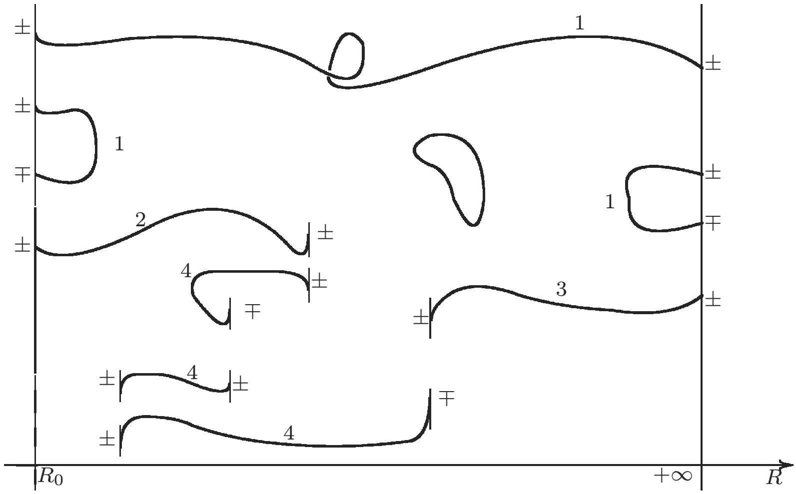

Figure 3: One-dimensional manifold $\mathcal{M}(R;\, p, q, f;\, H)$.

We will discuss four cases separately (see also Figure 3 for illustration of every particular case). For the sake of simplicity we always denote the ends of the connected component of $\mathcal{M}(R;\, p, q, f;\, H)$ (of any type) by w_1 and w_2.

Case 1. If both ends, w_1 and w_2, are in ∂_1, then it follows from (34) that the coherent orientations of isolated trajectories in $\mathcal{M}_{R_0}(p, q, f;\, H)$, induce the coherent orientation of one-dimensional component $\mathcal{M}(R;\, p, q, f;\, H)$ and from Remark 16 that we can identify the canonical orientation with $\pi_*^{-1}\left(\frac{d}{dR}\right)$. So we conclude:

$$\tau\big(w_1\big) = -\tau\big(w_2\big).$$

If the ends w_1 and w_2 belong to different sets ∂_1 and ∂_4, then $w_2 = (u_1, u_2) \in \mathcal{M}(p, f;\, x;\, H) \times \mathcal{M}(x, H;\, q, f)$. Denote by $\big(R_1, w_{R_1}\big)$ the element $u_1 \sharp u_2$ of a zero-dimensional component of $\mathcal{M}(R;\, p, q, f;\, H)$, for R_1 large enough. From the construction of coherent orientation in Chapter 3.3 we have

$$\sigma\big(u_1\big) \sharp \sigma\big(u_2\big) \simeq \sigma\left(\big(R_1, w_{R_1}\big)\right) \simeq \sigma\left(w_{R_1}\right)$$

for R_1 regular with respect to π. It holds

$$\begin{aligned} \tau\big(u_1\big)\tau\big(u_2\big)\big[1 \otimes 1^*_{u_1}\big] \sharp \big[1 \otimes 1^*_{u_2}\big] &\simeq \sigma\big(u_1\big) \sharp \sigma\big(u_2\big) \simeq \sigma\left(w_{R_1}\right) \\ &\simeq \tau\left(w_{R_1}\right)\left[1 \otimes 1^*_{w_{R_1}}\right] \simeq \tau\left(w_{R_1}\right)\big[1 \otimes 1^*_{u_1}\big] \sharp \big[1 \otimes 1^*_{u_2}\big], \end{aligned}$$

so we have $\tau(w_{R_1}) = \tau(u_1)\tau(u_2)$. Since from the Remark 16 it follows $\tau(w_{R_1}) = \tau(w_1)$, we conclude

$$\tau\big(w_1\big) = \tau\big(u_1\big)\tau\big(u_2\big).$$

If both ends, $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$, are in ∂_4, in the same way as in the previous case, using the construction of coherent orientation and the Remark 16, we conclude

$$\tau\big(u_1\big)\tau\big(v_1\big) = -\tau\big(u_2\big)\tau\big(v_2\big).$$

Case 2. Reasoning as in the Case 1, we conclude

$$\sigma\big(\mathcal{M}(R; p, q, f; H)\big) \simeq \tau\big(w_1\big)\sigma_R, \tag{47}$$

where σ_R denotes canonical orientation of $\mathcal{M}(R; p, q, f; H)$ identified with $\pi_*^{-1}(\frac{d}{dR})$. If the other end, w_2, belongs to ∂_2, denote by u_2 and v_2 the parts of broken trajectory w_2, $w_2 = (u_2, v_2)$. Reasoning in the same way as in the proof of the Theorem 14, we see that canonical orientation of u_2 and reversed canonical orientation of v_2 (because of the Remark 15) induce the same orientation σ_R, so:

$$\begin{aligned} -\sigma\big(\mathcal{M}(R; p, q, f, g; H, J)\big) &\simeq -\sigma\big(u_2\big) \sharp\, \sigma\big(v_2\big) \\ &\simeq \tau\big(u_2\big)\tau\big(v_2\big)\big[u_{2s}\big] \sharp \big[-1 \otimes 1^*_{v_2}\big] \simeq \tau\big(u_2\big)\tau\big(v_2\big)\sigma_R. \end{aligned} \tag{48}$$

From (47) and (48) we conclude

$$\tau\big(w_1\big) = -\tau\big(u_2\big)\tau\big(v_2\big).$$

On the other hand, if $w_2 \in \partial_3$, $w_2 = (u_2, v_2)$, the trajectory v_2 is now the one which gives the canonical orientation of w_2 (unlike before), and it is the second ingredient in gluing process, so it holds

$$\tau\big(w_1\big) = -\tau\big(u_2\big)\tau\big(v_2\big).$$

Case 3. Here both ends are broken, but $w_1 = (u_1, v_1)$ is the element of $\partial_2 \cup \partial_3$ and $w_2 = (u_2, v_2)$ of ∂_4. As in **Case 1** we have $\tau(w_R) = \tau(u_2)\tau(v_2)$ for R large enough. Now in the same way as in **Case 2** we see that it holds:

$$\tau\big(u_1\big)\tau\big(v_1\big) = \tau\big(u_2\big)\tau\big(v_2\big) \quad \text{if } w_2 \in \partial_2 \cup \partial_3.$$

Case 4. Reasoning in the similar way as in previous discussion we obtain:

$$\tau\big(u_1\big)\tau\big(v_1\big) = -\tau\big(u_2\big)\tau\big(v_2\big).$$

Hence we conclude that in (46), in both left and right side, we count only the ends that are the ends of either $[0, +\infty)$ or $(-\infty, +\infty]$, exactly with signs as in (46).

The first half of the Proposition 2, the identity $\Psi \circ \Phi = \mathrm{Id}_{HF}$, also holds in homology with $\mathbb{Z}$ coefficients. To prove this, we introduce the following auxiliary spaces. For two Hamiltonian paths x and y with the ends in O_M, define:

$$\mathcal{M}(\varepsilon, x, y, H; f) := \left\{ (u_-, u_+, \gamma) \;\middle|\; \begin{array}{l} u_- : (-\infty, 0] \times [0,1] \to T^*M \\ u_+ : [0, +\infty) \times [0,1] \to T^*M \\ \gamma : [-\varepsilon, \varepsilon] \to M \\ \dfrac{d\gamma}{dt} = -\nabla f(\gamma) \\ \dfrac{\partial u_\pm}{\partial s} + J\left(\dfrac{\partial u_\pm}{\partial t} - X_{\rho_R H}(u_\pm)\right) = 0 \\ u_-\big(\partial((-\infty, 0] \times [0,1])\big) \subset O_M \\ u_+\big(\partial([0, +\infty) \times [0,1])\big) \subset O_M \\ u_\pm\left(0, \dfrac{1}{2}\right) = \gamma(\pm\varepsilon) \\ u_-(-\infty, t) = x(t) \\ u_+(+\infty, t) = y(t) \end{array} \right\}$$

where ρ_R now is a smooth function such that

$$\rho_R(t) = \begin{cases} 1, & |t| \le R \\ 0, & |t| \ge R + 1 \end{cases}$$

for fixed $R > 0$. Define also

$$\mathcal{M}_\varepsilon(x, y, H; f) := \big\{ (u_-, u_+, \gamma, \varepsilon) \mid (u_-, u_+, \gamma) \in \mathcal{M}(\varepsilon; x, y, H; f) \big\}.$$

We define a special class of Fredholm operators associated to the above spaces and the corresponding equivalent class in the same way as in the case of the classes from Θ. We construct a coherent orientation for these classes of operators similarly as in the Section 3.3, by gluing an orientation of a class from $\widetilde{\Xi}$ with an orientation of a class from $\widetilde{\Sigma}$. The canonical orientation is again $1 \otimes 1^*$ for every zero dimensional component. As before, we define a sign τ as a number ± 1 such that $\sigma(w) = \tau(w)[1 \otimes 1^*]$.

Now define

$$P : CH_k(H) \to CH_k(H), \qquad x \mapsto \sum_{\mu_H(y)=k} n_\varepsilon(x, y, H; f)y$$

where

$$n_\varepsilon(x, y, H; f)y := \sum_{w_\varepsilon \in \mathcal{M}_\varepsilon(x,y,H;f)} \tau(w_\varepsilon)$$

and

$$L : CH_k(H) \to CH_{k+1}(H), \qquad x \mapsto \sum_{\mu_H(y)=k+1} n(\varepsilon; x, y, H; f)y,$$

where

$$n(\varepsilon; x, y, H; f)y := \sum_{(\varepsilon, w_\varepsilon)\in\mathcal{M}(\varepsilon,x,y,H;f)} \tau(\varepsilon, w_\varepsilon).$$

The symbol w_ε stands for the triple (u_-, u_+, γ). From an analysis of the boundary of one-dimensional manifold $\mathcal{M}(\varepsilon, x, y, H; f)$ similar to one given for the case $\Phi \circ \Psi$, we conclude

$$\Psi \circ \Phi - P = L \circ \partial_M + \partial_M \circ L,$$

i.e. the map $\Psi \circ \Phi$ is chain homotopic to L, which is again the identity map in the homology.

5 Conclusion

We proved that, given the coherent orientation for all trajectory spaces involved (mixed and non-mixed, parameterized and non-parameterized objects) there exists the isomorphism between Morse and Floer homology with $\mathbb{Z}$ coefficients. We also showed that this coherent orientation exists. The question is whether there exists such coherent orientation (and, consequently, the isomorphism) in the case when two coherent orientations for Morse and Floer homology are *given* as in [22] and [9] respectively.

Let σ^M and σ^F be the given coherent orientations for operators in Morse and Floer homologies respectively. Let p_0, K_0, $\mathcal{P}^-$, w_0, F_0, x_0, v_0 and C_0 be as in Chapter 3.3, and σ the coherent orientation given there. Let

$$\sigma_1 := \sigma|_{\Sigma^M} \quad \sigma_2 := \sigma|_{\Sigma^F}.$$

Since σ_1 and σ^M (σ_2 and σ^F respectively) are two coherent orientation of Σ^M (respectively Σ^F), we can choose $f^M \in \Gamma^M$ $f^F \in \Gamma^F$ such that

$$f^M \sigma^M = \sigma_1, \qquad f^F \sigma^F = \sigma_2$$

where C^M_Λ, C^F_Λ are the groups of transformations for coherent orientation in Morse and Floer homology. We can construct the extension $f \in \Gamma$ that satisfies

$$f|_{C^M_\Lambda} = f^M, \qquad f|_{C^F_\Lambda} = f^F.$$

It is determined uniquely by value of f at w_0, since the sets Σ^M and Σ^F (and the corresponding coherent orientations) are disjoint. So choose $f([w_0, F_0]) := 1$ and extend f to C_Λ by the requirements

$$f|_{C^M_\Lambda} = f^M, \qquad f|_{C^F_\Lambda} = f^F, \qquad f([w, F]\sharp[u, L]) = f([w, F])f([u, L]),$$

for all $[w, F]$, $[u, L]$ compatible for gluing. By inspection one shows that $f \in \Gamma$. Define the coherent orientation by

$$\sigma' := f\sigma.$$

Due to the construction it follows that σ' is coherent and that it coincides with σ^M and σ^F on C^M_Λ and C^F_Λ. So we proved the following Theorem, and hence the Theorem 3.

Theorem 17. *For two given coherent orientation of Σ^M and Σ^F that induce Morse and Floer homologies $HM_*(f, \mathbb{Z})$ and $HF_*(H, \mathbb{Z})$ with $\mathbb{Z}$ coefficients, there exists a coherent orientation of Λ (the set of all involved classes of operators) that coincides with two given coherent orientations on corresponding classes, inducing the isomorphism between $HM_*(f, \mathbb{Z})$ and $HF_*(H, \mathbb{Z})$.* □

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