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Factorization of weakly continuous differentiable mappings

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Abstract. Given real Banach spaces X and Y, let $C^1_{\text{wbu}}(X, Y)$ be the space, introduced by R.M. Aron and J.B. Prolla, of C^1 mappings from X into Y such that the mappings and their derivatives are weakly uniformly continuous on bounded sets. We show that $f \in C^1_{\text{wbu}}(X, Y)$ if and only if f may be written in the form $f = g \circ S$, where the intermediate space is normed, S is a precompact operator, and g is a Gâteaux differentiable mapping with some additional properties.

Keywords: Fréchet differentiable mapping, Gâteaux differentiable mapping, weakly uniformly continuous mapping on bounded sets, factorization of differentiable mappings.

Mathematical subject classification: Primary: 46E50; Secondary: 46G05.

1 Introduction

Given real Banach spaces X and Y, let $\mathcal{K}(X, Y)$ be the space of compact (linear) operators from X into Y endowed with the supremum norm. Let $C^1_{\text{wbu}}(X, Y)$ be the space of (Fréchet) differentiable mappings $f\colon X \to Y$ such that f and $f'\colon X \to \mathcal{K}(X, Y)$ are weakly uniformly continuous on bounded subsets of X [1, Definition 3.1].

Since f' is compact, it is shown in [4] that there are a normed space Z, a precompact (linear) operator $S\colon X \to Z$, and a Gâteaux differentiable mapping $g\colon Z \to Y$ with some other properties, such that $f = g \circ S$. In the present note, we give conditions on g characterizing the mappings f in the space $C^1_{\text{wbu}}(X, Y)$. This answers a question raised in [6, page 338].

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Throughout, X and Y will denote real Banach spaces, $\mathcal{L}(X, Y)$ will be the space of all (bounded linear) operators from X into Y endowed with the supremum norm. The symbol $\tau_{\rm p}$ will stand for the topology of pointwise convergence on $\mathcal{L}(X, Y)$. By B_X we denote the closed unit ball of X, and X^* is the dual space of X. For $x \in X$ and $\epsilon > 0$, we represent by $B(x, \epsilon)$ the open ball of radius ϵ centered at x. Given $x, y \in X$, we write $I(x, y)$ for the segment with bounds x and y.

Given a mapping $f\colon X \to Y$ and a class $\mathcal{M}$ of subsets of X such that every singleton belongs to $\mathcal{M}$, we say that f is *$\mathcal{M}$-differentiable at $x \in X$* if there exists an operator $f'(x) \in \mathcal{L}(X, Y)$ such that

$$\lim_{\epsilon \to 0} \frac{f(x + \epsilon y) - f(x) - f'(x)(\epsilon y)}{\epsilon} = 0$$

uniformly with respect to y on each member of $\mathcal{M}$ [7, §1.2]. In this case, we shall write $f \in D_{\mathcal{M}}(x, Y)$.

We say that f is *(Fréchet) differentiable at x* if $f \in D_{\mathcal{M}}(x, Y)$ where $\mathcal{M}$ is the class of all bounded subsets of X; f is *Gâteaux differentiable at x* if $f \in D_{\mathcal{M}}(x, Y)$ where $\mathcal{M}$ is the class of all one point subsets of X.

The mapping f is *(Fréchet) differentiable* (respectively, *Gâteaux differentiable*) on X if it is differentiable (respectively, Gâteaux differentiable) at every point $x \in X$.

A mapping $f\colon X \to Y$ is *compact* if it takes bounded subsets of X into relatively compact subsets of Y. We say that $f\colon X \to Y$ is *weakly uniformly continuous on bounded subsets of X* if, for each bounded subset $B \subset X$ and each $\epsilon > 0$, there are $\varphi_1, \ldots, \varphi_k \in X^*$ and $\delta > 0$ such that, if $x, y \in B$ satisfy $|\varphi_i(x - y)| < \delta$ $(i = 1, \ldots, k)$, then $\|f(x) - f(y)\| < \epsilon$. If f is weakly uniformly continuous on bounded subsets, then f is compact [1, Lemma 2.2]. An operator is compact if and only if it is weakly uniformly continuous on bounded subsets [1, Proposition 2.5].

Let $f\colon X \to Y$ be a differentiable mapping. We say that f is *uniformly differentiable on bounded sets* (see [1, Definition 3.5]) if, for every $\epsilon > 0$ and every bounded subset B of X, there exists $\delta > 0$ such that, whenever $x \in B$ and $y \in X$ with $\|y\| < \delta$, we have

$$\left\| f(x + y) - f(x) - f'(x)(y) \right\| < \epsilon \, \|y\| \, .$$

If K is a bounded subset of $\mathcal{L}(X, Y)$, we construct a normed space in the spirit of [3]. As in [4], we define a continuous seminorm on X by

$$\|x\|_K := \sup_{\phi \in K} \|\phi(x)\| \qquad \text{for all } x \in X \, .$$

Then the set

$$V_K := \{x \in X : \ \|x\|_K = 0\}$$

is a closed subspace of X. Let S be the canonical quotient map of X onto the quotient space X/V_K. We define a norm on X/V_K by

$$\|S(x)\| := \|x\|_K \qquad (x \in X).$$

The following result is proved in [4].

Theorem 1. *Let $f: X \to Y$ be a mapping between real Banach spaces. Then the following assertions are equivalent:*

(a) *f is differentiable and $f': X \to \mathcal{K}(X, Y)$ is compact;*

(b) *there exist a normed space Z, a surjective operator $S: X \to Z$, and a mapping $g: Z \to Y$ such that:*

(i) *$f(x) = g(S(x))$ for all $x \in X$;*

(ii) *S is a precompact operator;*

(iii) *$g \in D_{\mathcal{M}}(S(x), Y)$ for every $x \in X$, where*

$$\mathcal{M} := \{S(B) : \ B \text{ is a bounded subset of } X\};$$

(iv) *g' is bounded on $S(nB_X)$ for every positive integer n;*

(v) *for every positive integer n, the set $\{g'(S(x)) \circ S : \ x \in nB_X\}$ is relatively compact in $\mathcal{K}(X, Y)$.*

The following diagram is then commutative.

$$\begin{array}{ccc} X & \xrightarrow{f'} & \mathcal{K}(X, Y) \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle S_Y^*} \\ Z & \xrightarrow[g']{} & \mathcal{L}(Z, Y) \end{array}$$

where $S_Y^*(T)(x) := T(S(x))$ for every $T \in \mathcal{L}(Z, Y)$ and $x \in X$.

Theorem 2. *Let $f: X \to Y$ be a differentiable mapping between real Banach spaces. Then the following assertions are equivalent:*

(a) *$f': X \to \mathcal{K}(X, Y)$ is compact and uniformly continuous on bounded sets;*

(b) *f admits a factorization as in Theorem 1 and, for every bounded set $B \subset X$, the restriction of g' to $S(B)$ is uniformly norm-to-τ_p continuous.*

Proof. (a) $\Rightarrow$ (b). As in the proof of Theorem 1 (see [4]), consider the compact set

$$K := \bigcup_{n\in\mathbb{N}} \frac{\overline{f'(nB_X)}}{n\,\|f'\|_{nB_X}} \bigcup \{0\} \subset \mathcal{K}(X, Y)\,.$$

The intermediate space of the factorization will be $Z := X/V_K$, and the precompact operator $S\colon X \to Z$ is the quotient map.

Let B be a bounded subset of X. We shall prove that g' is uniformly norm-to-τ_{p} continuous on $S(B)$. Let V be a τ_{p}-neighbourhood of zero in $\mathcal{L}(S(X), Y)$. Then there exist $\epsilon > 0$ and nonzero vectors $x_1, \ldots, x_p \in X$ such that

$$V \supseteq \left\{T \in \mathcal{L}(S(X), Y) : \|T(S(x_i))\| < \epsilon\,,\ 1 \le i \le p\right\}.$$

We have to prove that there is $\delta > 0$ such that, if $x, y \in B$ satisfy

$$\|S(x) - S(y)\| < \delta, \quad \text{then} \quad g'(S(x)) - g'(S(y)) \in V.$$

It is enough to see that

$$\begin{aligned} \left\| f'(x)(x_i) - f'(y)(x_i)\right\| &= \left\|[g'(S(x)) - g'(S(y))](S(x_i))\right\| < \epsilon \\ &\qquad (1 \le i \le p)\,. \end{aligned}$$

Let m be an integer such that $B \subset mB_X$. Since f' is uniformly continuous on $(m+1)B_X$, there exists $\delta_1 > 0$ such that

$$\left\| f'(z_1) - f'(z_2)\right\| < \frac{\epsilon}{4 \max\limits_{1\le i\le p} \|x_i\|}$$

whenever $z_1, z_2 \in (m+1)B_X$ with $\|z_1 - z_2\| < \delta_1$. We can assume $\delta_1 < 1$.

Let

$$\delta_2 := \frac{\delta_1}{2 \max\limits_{1\le i\le p} \|x_i\|}\,.$$

Given $x, y \in B$, we have for all $i \in \{1, \ldots, p\}$,

$$\begin{aligned} \left\| f'(x)(x_i) - f'(y)(x_i)\right\| &= \frac{1}{\delta_2} \left\| f'(x)(\delta_2 x_i) - f'(y)(\delta_2 x_i)\right\| \\ &\le \frac{1}{\delta_2} \left\| f'(x)(\delta_2 x_i) + f(x) - f(x + \delta_2 x_i)\right\| \\ &\quad + \frac{1}{\delta_2} \left\| -f'(y)(\delta_2 x_i) - f(y) + f(y + \delta_2 x_i)\right\| \end{aligned}$$

$$+ \frac{1}{\delta_2} \|f(x+\delta_2 x_i) - f(y+\delta_2 x_i)\|$$
$$+ \frac{1}{\delta_2} \|f(x) - f(y)\| \, .$$

By a consequence of the Mean Value Theorem [5, (8.6.2)],

$$\begin{aligned} &\frac{1}{\delta_2} \left\| f'(x)(\delta_2 x_i) + f(x) - f(x+\delta_2 x_i) \right\| \\ &\quad \leq \frac{1}{\delta_2} \sup_{z \in I(x, x+\delta_2 x_i)} \left\| f'(z) - f'(x) \right\| \|\delta_2 x_i\| \\ &\quad = \sup_{z \in I(x, x+\delta_2 x_i)} \left\| f'(z) - f'(x) \right\| \|x_i\| \, . \end{aligned}$$

For $z \in I(x, x+\delta_2 x_i)$, we have

$$\begin{aligned} \|z\| &\leq \|z - x\| + \|x\| \\ &\leq \|\delta_2 x_i\| + m = \frac{\delta_1}{2 \max\limits_{1 \leq i \leq p} \|x_i\|} \|x_i\| + m \\ &< \delta_1 + m < m + 1 \, . \end{aligned}$$

Then

$$\left\| f'(z) - f'(x) \right\| < \frac{\epsilon}{4 \max\limits_{1 \leq i \leq p} \|x_i\|} \, ,$$

which implies

$$\sup_{z \in I(x, x+\delta_2 x_i)} \left\| f'(z) - f'(x) \right\| \leq \frac{\epsilon}{4 \max\limits_{1 \leq i \leq p} \|x_i\|}$$

and then

$$\frac{1}{\delta_2} \left\| f'(x)(\delta_2 x_i) + f(x) - f(x+\delta_2 x_i) \right\| \leq \frac{\epsilon}{4 \max\limits_{1 \leq i \leq p} \|x_i\|} \|x_i\| \leq \frac{\epsilon}{4} \, .$$

Analogously,

$$\frac{1}{\delta_2} \left\| -f'(y)(\delta_2 x_i) - f(y) + f(y+\delta_2 x_i) \right\| \leq \frac{\epsilon}{4} \, .$$

Now, by the Mean Value Theorem [2, Theorem 6.4],

$$\begin{aligned}
\frac{1}{\delta_2}\,\|f(x)-f(y)\| &\le \frac{1}{\delta_2}\sup_{z\in I(x,y)}\left\|f'(z)(x-y)\right\| \\
&\le \frac{1}{\delta_2}\sup_{z\in(m+1)B_X}\left\|f'(z)(x-y)\right\| \\
&= \frac{1}{\delta_2}\sup_{z\in(m+1)B_X}\left\|\frac{f'(z)(x-y)}{(m+1)\|f'\|_{(m+1)B_X}}\,(m+1)\left\|f'\right\|_{(m+1)B_X}\right\| \\
&\le \frac{1}{\delta_2}\,\|S(x)-S(y)\|\,(m+1)\left\|f'\right\|_{(m+1)B_X}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{\delta_2}\,\|f(x+\delta_2 x_i)-f(y+\delta_2 x_i)\| &\le \frac{1}{\delta_2}\sup_{z\in I(x+\delta_2 x_i,\,y+\delta_2 x_i)}\left\|f'(z)(x-y)\right\| \\
&\le \frac{1}{\delta_2}\sup_{z\in(m+1)B_X}\left\|f'(z)(x-y)\right\| \\
&\le \frac{1}{\delta_2}\,\|S(x)-S(y)\|\,(m+1)\left\|f'\right\|_{(m+1)B_X}.
\end{aligned}$$

Let

$$\delta := \frac{\epsilon\delta_2}{4(m+1)\|f'\|_{(m+1)B_X}}\,.$$

Choosing $x, y \in B$ with $\|S(x)-S(y)\| < \delta$, we have

$$\frac{1}{\delta_2}\,\|f(x+\delta_2 x_i)-f(y+\delta_2 x_i)\| < \frac{\epsilon}{4} \quad\text{and}\quad \frac{1}{\delta_2}\,\|f(x)-f(y)\| < \frac{\epsilon}{4}\,.$$

Therefore, for $x, y \in B$ with $\|S(x)-S(y)\| < \delta$, we obtain

$$\left\|f'(x)(x_i)-f'(y)(x_i)\right\| < \epsilon \qquad (1\le i\le p)\,.$$

(b) $\Rightarrow$ (a). By Theorem 1, f' is $\mathcal{K}(X,Y)$-valued and compact, so it remains to prove that f' is uniformly continuous on bounded sets. Choose $\epsilon > 0$. Let $B \subset X$ be a bounded set. Since $S(B_X)$ is precompact, there exist $x_1, \ldots, x_p \in B_X$ such that

$$S(B_X) \subset \bigcup_{i=1}^{p} B\left(S(x_i), \frac{\epsilon}{4M}\right)$$

where $M := \sup_{x\in B}\left\|g'(S(x))\right\|$ is finite by (b),(iv) of Theorem 1. Let

$$V := \left\{T \in \mathcal{L}(S(X), Y) : \|T(S(x_i))\| < \frac{\epsilon}{2}\,,\ 1\le i\le p\right\}$$

be a τ_{p}-neighbourhood of zero in $\mathcal{L}(S(X), Y)$. By (b), there exists $\delta > 0$ such that, whenever $x, y \in B$ satisfy $\|S(x) - S(y)\| < \delta$, we have

$$g'(S(x)) - g'(S(y)) \in V\,,$$

so

$$\left\| f'(x)(x_i) - f'(y)(x_i) \right\| = \left\| [g'(S(x)) - g'(S(y))](S(x_i)) \right\| < \frac{\epsilon}{2} \qquad (1 \le i \le p)\,.$$

Now, let $x, y \in B$ with $\|x - y\| < \delta / \|S\|$. Then

$$\left\| f'(x)(x_i) - f'(y)(x_i) \right\| < \frac{\epsilon}{2} \qquad (1 \le i \le p)\,.$$

Given $z \in B_X$, there exists $k \in \{1, \ldots, p\}$ such that

$$\|S(z) - S(x_k)\| < \frac{\epsilon}{4M}\,.$$

Therefore,

$$\begin{aligned}
&\left\| f'(x)(z) - f'(y)(z) \right\| \\
&\quad \le \left\| f'(x)(z) - f'(x)(x_k) \right\| + \left\| f'(x)(x_k) - f'(y)(x_k) \right\| \\
&\qquad + \left\| f'(y)(x_k) - f'(y)(z) \right\| \\
&\quad < \frac{\epsilon}{2} + \left\| g'(S(x))(S(z) - S(x_k)) \right\| + \left\| g'(S(y))(S(x_k) - S(z)) \right\| \\
&\quad \le \frac{\epsilon}{2} + 2M \, \|S(z) - S(x_k)\| \\
&\quad < \frac{\epsilon}{2} + 2M \, \frac{\epsilon}{4M} = \epsilon\,,
\end{aligned}$$

so

$$\left\| f'(x) - f'(y) \right\| = \sup_{z \in B_X} \left\| f'(x)(z) - f'(y)(z) \right\| \le \epsilon\,,$$

and the proof is finished. □

We shall need the following lemma of independent interest.

Lemma 3. *Let $f\colon X \to Y$ be a differentiable mapping between real Banach spaces. Then f' is uniformly continuous on bounded sets if and only if f is uniformly differentiable on bounded sets.*

Proof. Suppose that f' is uniformly continuous on bounded sets. Let B be a bounded subset of X and let $\epsilon > 0$. The set

$$B' := B + B_X$$

is also bounded. There exists $\delta > 0$ such that, if $x_1, x_2 \in B'$ satisfy $\|x_1 - x_2\| < \delta$, then $\left\| f'(x_1) - f'(x_2) \right\| < \epsilon/2$. We can assume $\delta < 1$. Let $x \in B$, $y \in X$ with $\|y\| < \delta$. Every $z \in I(x, x+y)$ may be written in the form $z = x + t_z y$ with $|t_z| \leq 1$. It follows that $z \in B'$ and $\left\| f'(x) - f'(z) \right\| < \epsilon/2$. Applying [5, (8.6.2)], we have

$$\left\| f(x+y) - f(x) - f'(x)(y) \right\| \leq \sup_{z \in I(x,x+y)} \left\| f'(x) - f'(z) \right\| \|y\| < \epsilon \|y\| ,$$

and f is uniformly differentiable on bounded sets.

Conversely, suppose that f is uniformly differentiable on bounded sets. Let B be a bounded subset of X and let $\epsilon > 0$. There exists $\delta' > 0$ such that, for $x \in B$, $y \in X$ with $\|y\| < \delta'$, we have

$$\left\| f(x+y) - f(x) - f'(x)(y) \right\| < \epsilon \|y\| .$$

Choose

$$0 < \delta < \frac{1}{2} \delta' .$$

Let $x_1, x_2 \in B$ be such that $\|x_1 - x_2\| < \delta$. Let $z \in B_X$. Then

$$\begin{aligned}
\left\| f'(x_1)(z) - f'(x_2)(z) \right\| &= \frac{2}{\delta} \left\| f'(x_1)\left(z\frac{\delta}{2} \right) - f'(x_2)\left(z\frac{\delta}{2} \right) \right\| \\
&\leq \frac{2}{\delta} \left\| f'(x_1)\left(z\frac{\delta}{2} \right) + f(x_1) - f\left(x_1 + z\frac{\delta}{2} \right) \right\| \\
&\quad + \frac{2}{\delta} \left\| f\left(x_1 + z\frac{\delta}{2} \right) - f(x_2) - f'(x_2)\left(x_1 - x_2 + z\frac{\delta}{2} \right) \right\| \\
&\quad + \frac{2}{\delta} \left\| f(x_1) - f(x_2) - f'(x_2)(x_1 - x_2) \right\| \\
&\leq \frac{2}{\delta}\epsilon \left\| z\frac{\delta}{2} \right\| + \frac{2}{\delta}\epsilon \left\| x_1 - x_2 + z\frac{\delta}{2} \right\| + \frac{2}{\delta}\epsilon \|x_1 - x_2\| < 6\epsilon .
\end{aligned}$$

Taking suprema for $z \in B_X$, we obtain

$$\left\| f'(x_1) - f'(x_2) \right\| \leq 6\epsilon ,$$

and f' is uniformly continuous on bounded sets. □

The following result is proved in [4].

Proposition 4. *Given a differentiable mapping $f: X \to Y$ between real Banach spaces so that $f': X \to \mathcal{K}(X, Y)$ is uniformly continuous on bounded sets, the following assertions are equivalent:*

(a) *f is weakly uniformly continuous on bounded sets;*

(b) *f' is weakly uniformly continuous on bounded sets;*

(c) *f' is compact.*

Theorem 5. *Let $f: X \to Y$ be a differentiable mapping between real Banach spaces. The following assertions are equivalent:*

(a) *$f \in C^1_{\mathrm{wbu}}(X, Y)$;*

(b) *$f': X \to \mathcal{K}(X, Y)$ is compact and uniformly continuous on bounded sets;*

(c) *f admits a factorization as in Theorem* 2.

Proof. The equivalence (b) $\Leftrightarrow$ (c) is contained in Theorem 2.

(b) $\Rightarrow$ (a). By Proposition 4, f is weakly uniformly continuous on bounded sets. By Lemma 3, f is uniformly differentiable on bounded sets. By [1, Corollary 3.8], $f \in C^1_{\mathrm{wbu}}(X, Y)$.

(a) $\Rightarrow$ (b). By [1, Corollary 3.8], f is uniformly differentiable on bounded sets, weakly uniformly continuous on bounded sets, and $\mathcal{K}(X, Y)$-valued. By Lemma 3, f' is uniformly continuous on bounded sets. By Proposition 4, f' is compact. □

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