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On Cauchy and Martinelli–Bochner Integral Formulae in Hermitean Clifford Analysis

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Abstract. Euclidean Clifford analysis is a higher dimensional function theory, refining harmonic analysis, centred around the concept of monogenic functions, i.e. null solutions of a first order vector valued rotation invariant differential operator, called the Dirac operator. More recently, Hermitean Clifford analysis has emerged as a new and successful branch of Clifford analysis, offering yet a refinement of the Euclidean case; it focusses on the simultaneous null solutions of two Hermitean Dirac operators, invariant under the action of the unitary group. In this paper, a Cauchy integral formula is established by means of a matrix approach, allowing the recovering of the traditional Martinelli–Bochner formula for holomorphic functions of several complex variables as a special case.

Keywords: Cauchy integral formula, Martinelli–Bochner integral formula, Hermitean Clifford analysis.

Mathematical subject classification: 30G35, 32A26, 46F10.

1 Introduction

The Cauchy integral formula, a key result for the theory of holomorphic functions in the complex plane, may be generalized to the case of several complex variables in two ways: either one takes a holomorphic kernel and an integral over the distinguished boundary $\partial_0 \widetilde{D} = \prod_{j=1}^{n} \partial \widetilde{D}_j$ of a polydisk $\widetilde{D} = \prod_{j=1}^{n} \widetilde{D}_j$ in $\mathbb{C}^n$,

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leading to the formula

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} \, d\xi_1 \wedge \cdots \wedge d\xi_n \,, \qquad z_j \in \overset{\circ}{\tilde{D}}_j \tag{1.1}$$

or one takes an integral over the (piecewise) smooth boundary ∂D of a bounded domain D in $\mathbb{C}^n$ in combination with the Martinelli–Bochner kernel, see e.g. [14], which is not holomorphic anymore but still harmonic, resulting into

$$f(z) \;=\; \int_{\partial D} f(\xi) \, U(\xi, z) \,, \quad z \in \overset{\circ}{D} \tag{1.2}$$

with

$$U(\xi, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\xi_j^c - z_j^c}{|\xi - z|^{2n}}$$
$$d\xi_1^c \wedge \cdots \wedge d\xi_{j-1}^c \wedge d\xi_{j+1}^c \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

where $\cdot^c$ denotes the complex conjugate. The history of formula (1.2), obtained independently and through different methods by Martinelli and by Bochner, has been described in detail in [13]. It reduces to the traditional Cauchy integral formula when $n = 1$; for $n > 1$, it is related to the double layer potential, while at the same time, it establishes a connection between harmonic and holomorphic functions.

A third alternative for a generalization of the Cauchy integral formula is offered by so–called Clifford analysis, where functions defined in Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and taking values in a Clifford algebra are considered. This multidimensional function theory focusses on so–called monogenic functions, i.e. null solutions of the elliptic Dirac operator $\partial_{\underline{X}}$ factorizing the Laplace operator: $\partial_{\underline{X}}^2 = -\Delta_{2n}$, and may thus be seen as both a generalization of the theory of holomorphic functions in the complex plane and as a refinement of classical harmonic analysis. As the Dirac operator is rotation invariant, or more precisely: invariant under the action of the special orthogonal group, the name orthogonal Clifford analysis is used nowadays to refer to this setting. Standard references in this respect are [6, 9, 12, 11]. In this framework the Cauchy kernel appearing in the Clifford–Cauchy formula is monogenic, up to a pointwise singularity, while the integral remains being taken over the complete boundary:

$$f(\underline{X}) = \int_{\partial D} E(\underline{\Xi} - \underline{X}) \, d\sigma_{\underline{\Xi}} \, f(\underline{\Xi}) \,, \quad \underline{X} \in \overset{\circ}{D}$$

with

$$E(\underline{\Xi} - \underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{\Xi}} - \overline{\underline{X}}}{\left|\underline{\Xi} - \underline{X}\right|^{2n}}$$

a_{2n} being the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $\bar{\cdot}$ denoting the Clifford conjugation and $d\sigma_{\underline{\Xi}}$ being a Clifford algebra valued differential form of order $(2n-1)$ (see Section 3 for its precise definition). This Clifford–Cauchy integral formula is a corner stone in the function theoretic development of orthogonal Clifford analysis.

In a series of recent papers, so–called Hermitean Clifford analysis has emerged as yet a refinement of the orthogonal case; it focusses on the simultaneous null solutions of the complex Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ which decompose the Laplace operator in the sense that $4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}}) = \Delta_{2n}$ and which are invariant under the action of the special unitary group. The study of complex Dirac operators was initiated in [16, 15, 17]; a systematic development of the associated function theory, including the invariance properties with respect to the underlying Lie groups and Lie algebras, is still in full progress, see e.g. [7, 3, 4, 10, 1, 2].

Naturally a Cauchy integral formula for Hermitean monogenic functions taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ is essential in the further development of this function theory, but has not yet been obtained in a satisfactory way. A first result in this direction was obtained in [18], however for functions which are null solutions of only one of the Hermitean Dirac operators and moreover presenting a "fake" – as termed by the authors – Cauchy kernel, failing to be monogenic.

In this paper a Cauchy integral formula for Hermitean monogenic functions is established. However, from the start of our quest, it was clear that the formula aimed at could not have a traditional form as in (1.1) or in (1.2). Indeed, it is a known fact (see [4]) that in the special case where the functions considered do not take their values in the whole Clifford algebra $\mathbb{C}_{2n}$, but in the n-homogeneous part $\mathbb{S}_n$ of the complex spinor space $\mathbb{S} = \mathbb{C}_{2n}I \cong \mathbb{C}_n I$, I being a self-adjoint primitive idempotent, Hermitean monogenicity turns out to be equivalent with holomorphy in the complex variables $(z_1, \ldots, z_n)$.

It turned out that a matrix approach is the key to obtain the desired result, see Theorem 4.4. Moreover and as could be expected, the obtained Hermitean Cauchy integral formula (4.11) reduces to the traditional Martinelli–Bochner formula (1.2) in the special case of $\mathbb{S}_n$ valued functions (Section 5). This also means that the theory of Hermitean monogenic functions not only refines orthogonal Clifford analysis (and thus harmonic analysis as well), but also has strong connections with the theory of functions of several complex variables, even encompassing some of its results.

2 Preliminaries

We first consider the real Clifford algebra $\mathbb{R}_{0,m}$, constructed over the vector space $\mathbb{R}^{0,m}$ endowed with a non–degenerate quadratic form of signature $(0, m)$, and generated by the orthonormal basis $(e_1, \ldots, e_m)$. The non–commutative, so–called geometric, multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_j e_k + e_k e_j = -2\delta_{jk} \quad , \quad j, k = 1, \ldots, m \tag{2.1}$$

As a basis for $\mathbb{R}_{0,m}$ we consider for any set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, while for the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so–called k–vector part of a $(k = 0, 1, \ldots, m)$. The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(X_1, \ldots, X_m)$ with the Clifford vector $\underline{X}$ given by

$$\underline{X} = \sum_{j=1}^{m} e_j \, X_j$$

Note that the square of $\underline{X}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{X}^2 = - < \underline{X}, \underline{X} > = -|\underline{X}|^2$. The Fischer dual of $\underline{X}$ is the vector valued first order differential operator

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j \, \partial_{X_j}$$

called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^{0,m}$ and taking values in $\mathbb{R}_{0,m}$ is called (left) monogenic in Ω if $\partial_{\underline{X}}[f] = 0$ in Ω. As the Dirac operator factorizes the Laplacian: $\Delta_m = -\partial_{\underline{X}}^2$, monogenicity can be regarded as a refinement of harmonicity. We refer to this setting as the orthogonal case, since the fundamental group leaving the Dirac operator $\partial_{\underline{X}}$ invariant is the special orthogonal group $\mathrm{SO}(m; \mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is also called rotation invariant.

When allowing for complex constants and moreover taking the dimension to be even: $m = 2n$, the set of generators $(e_1, \ldots, e_{2n})$, still satisfying the multiplication rules (2.1), produces the complex Clifford algebra $\mathbb{C}_{2n}$, being the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\, \mathbb{R}_{0,2n}$. Any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may thus be written as $\lambda = a+ib$, $a, b \in \mathbb{R}_{0,2n}$, an observation leading to the definition of the Hermitean conjugation $\lambda^\dagger = (a+ib)^\dagger = \overline{a} - i\overline{b}$, where the bar notation stands for the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitean conjugation also leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$.

The above will be the framework for so–called Hermitean Clifford analysis, yet a refinement of orthogonal Clifford analysis. An elegant way for introducing this setting consists in considering a so–called complex structure, i.e. a specific $\mathrm{SO}(2n; \mathbb{R})$–element J for which $J^2 = -\mathbf{1}$ (see [3, 4]). Here, J is chosen to act upon the generators $e_1, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \qquad \text{and} \qquad J[e_{n+j}] = e_j, \qquad j = 1, \ldots, n$$

With J one may associate two projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ which will produce the main protagonists of the Hermitean setting by acting upon the corresponding objects in the orthogonal framework. First of all, the so–called Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for the complex Clifford algebra $\mathbb{C}_{2n}$ are obtained through the action of $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements e_j:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\, e_{n+j}), \quad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\, e_{n+j}), \quad j = 1, \ldots, n \end{aligned}$$

These Witt basis elements satisfy the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0 \;\;,\;\; j, k = 1, \ldots, n$$

and the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk} \;\;,\;\; j, k = 1, \ldots, n$$

Next we identify a vector $\underline{X} = (X_1, \ldots, X_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{0,2n}$ with the Clifford vector $\underline{X} = \sum_{j=1}^n (e_j\, x_j + e_{n+j}\, y_j)$ and we denote by $\underline{X}|$

the action of the complex structure J on $\underline{X}$, i.e.

$$\underline{X}| \;=\; J[\underline{X}] \;=\; \sum_{j=1}^{n} \left(e_j\, y_j - e_{n+j}\, x_j\right)$$

Observe that the Clifford vectors $\underline{X}$ and $\underline{X}|$ anti–commute, since the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product. The actions of the projection operators on the Clifford vector $\underline{X}$ then produce the Hermitean Clifford variables $\underline{Z}$ and its Hermitean conjugate $\underline{Z}^\dagger$:

$$\begin{aligned}
\underline{Z} \;&=\; \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] \;=\; \frac{1}{2}(\underline{X} + i\,\underline{X}|)\\
\underline{Z}^\dagger \;&=\; -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] \;=\; -\frac{1}{2}(\underline{X} - i\,\underline{X}|)
\end{aligned}$$

which may also be rewritten in terms of the Witt basis elements as

$$\underline{Z} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j\, z_j \quad \text{and} \quad \underline{Z}^\dagger \;=\; (\underline{Z})^\dagger \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\, z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. Finally, the Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$ are derived from the orthogonal Dirac operator $\partial_{\underline{X}}$:

$$\begin{aligned}
\partial_{\underline{Z}^\dagger} \;&=\; \frac{1}{4}(\mathbf{1} + iJ)[\partial_{\underline{X}}] \;=\; \frac{1}{4}(\partial_{\underline{X}} + i\,\partial_{\underline{X}|})\\
\partial_{\underline{Z}} \;&=\; -\frac{1}{4}(\mathbf{1} - iJ)[\partial_{\underline{X}}] \;=\; -\frac{1}{4}(\partial_{\underline{X}} - i\,\partial_{\underline{X}|})
\end{aligned}$$

where we have introduced the so–called twisted Dirac operator

$$\partial_{\underline{X}|} \;=\; J[\partial_{\underline{X}}] \;=\; \sum_{j=1}^{n} \left(e_j\, \partial_{y_j} - e_{n+j}\, \partial_{x_j}\right)$$

As was the case with $\partial_{\underline{X}}$, a notion of monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well. Again passing to the Witt basis, the Hermitean Dirac operators are expressed as

$$\partial_{\underline{Z}} \;=\; \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\, \partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}^\dagger} \;=\; (\partial_{\underline{Z}})^\dagger = \sum_{j=1}^{n} \mathfrak{f}_j\, \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j–planes, $j = 1, \ldots, n$.

Then a continuously differentiable function g on an open region Ω of $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a (left) Hermitean monogenic (or h–monogenic) function in Ω if and only if it simultaneously is $\partial_{\underline{X}}$– and $\partial_{\underline{X}|}$–monogenic in Ω, i.e. it satisfies in Ω the system

$$\partial_{\underline{X}}\, g = 0 = \partial_{\underline{X}|}\, g$$

or equivalently, the system

$$\partial_{\underline{Z}}\, g = 0 = \partial_{\underline{Z}^\dagger}\, g$$

It remains to recall the group invariance underlying this system. To this end we consider the group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$, given by

$$\widetilde{\mathrm{U}}(n) = \left\{ s \in \mathrm{Spin}(2n) \mid \exists\, \theta \geq 0 : \overline{s} I = \exp(-i\theta) I \right\}$$

its definition involving the self-adjoint primitive idempotent

$$I = I_1 \ldots I_n \tag{2.2}$$

with

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}\big(1 - i e_j e_{n+j}\big), \qquad j = 1, \ldots, n$$

It has been proved, see [7], that this group constitutes a realisation in the Clifford algebra of the unitary group $\mathrm{U}(n)$, and moreover, that its associated action leaves the Hermitean Dirac operators invariant. Less precisely, one thus says that these operators are invariant under the action of the unitary group, and so is the notion of h–monogenicity.

For further use, observe that the Hermitean vector variables and Dirac operators are isotropic, since the Witt basis elements are, i.e.

$$(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4\big(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}}\big)$$

while also

$$\underline{Z}\,\underline{Z}^\dagger + \underline{Z}^\dagger\underline{Z} = |\underline{Z}|^2 = |\underline{Z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}|\,|^2$$

3 Clifford–Cauchy and Clifford–Stokes theorems

In this section we will denote, as above, by Ω some open region in $\mathbb{R}^{2n}$, and we consider a $2n$–dimensional compact differentiable and oriented manifold $\Gamma \subset \Omega$ with C^∞ smooth boundary $\partial\Gamma$.

Further, we denote by $\widetilde{d\sigma}_{\underline{X}}$ the vector valued oriented surface element on $\partial\Gamma$ given by the differential form

$$\widetilde{d\sigma}_{\underline{X}} = \sum_{j=1}^{n} e_j \, (-1)^{j-1} \, \widetilde{\widetilde{dx_j}} + \sum_{j=1}^{n} e_{n+j} \, (-1)^{n+j-1} \, \widetilde{\widetilde{dy_j}}$$

of order $(2n-1)$, and by $\widetilde{d\sigma}_{\underline{X}|}$ its twisted analogue, i.e.

$$\widetilde{d\sigma}_{\underline{X}|} = J[\widetilde{d\sigma}_{\underline{X}}]$$

given by the differential form

$$\widetilde{d\sigma}_{\underline{X}|} = \sum_{j=1}^{n} e_j \, (-1)^{n+j-1} \, \widetilde{\widetilde{dy_j}} - \sum_{j=1}^{n} e_{n+j} \, (-1)^{j-1} \, \widetilde{\widetilde{dx_j}}$$

Here

$$\begin{aligned}
\widetilde{\widetilde{dx_j}} &= dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \\
\widetilde{\widetilde{dy_j}} &= dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n
\end{aligned}$$

reflecting the original consecutive ordering of the variables $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The corresponding oriented volume elements on Γ then read

$$\begin{aligned}
\widetilde{dV}(\underline{X}) &= dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \\
\widetilde{dV}(\underline{X}|) &= dy_1 \wedge \cdots \wedge dy_n \wedge (-dx_1) \wedge \cdots \wedge (-dx_n)
\end{aligned}$$

for which it is easily checked that

$$\widetilde{dV}(\underline{X}) = \widetilde{dV}(\underline{X}|) \tag{3.1}$$

In orthogonal Clifford analysis the theorem of Stokes may be formulated as follows (see [6]).

Theorem 3.1 (Clifford–Stokes theorem). *Let f and g be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} f(\underline{X}) \, \widetilde{d\sigma}_{\underline{X}} \, g(\underline{X}) = \int_{\Gamma} \big((f\partial_{\underline{X}})\, g + f\,(\partial_{\underline{X}} g)\big) \, \widetilde{dV}(\underline{X})$$

As an immediate consequence one obtains the basic theorem of Cauchy (see also [6]).

Theorem 3.2 (Clifford–Cauchy theorem). *Let the function g be $\partial_{\underline{X}}$–monogenic in Ω and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} \widetilde{d\sigma}_{\underline{X}}\, g(\underline{X}) \;=\; 0$$

Clearly, both theorems may be restated for $\underline{X}|$, $\widetilde{d\sigma}_{\underline{X}|}$ and $\partial_{\underline{X}|}$, leading to the formulations below, where (3.1) has been taken into account.

Corollary 3.1. *Let f and g be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\Gamma \subset \Omega$ be a $2n$-dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} f(\underline{X})\, \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}) \;=\; \int_{\Gamma} \big((f\partial_{\underline{X}|})\, g + f\,(\partial_{\underline{X}|} g)\big)\, \widetilde{dV}(\underline{X})$$

When moreover g is $\partial_{\underline{X}|}$–monogenic in Ω, then

$$\int_{\partial\Gamma} \widetilde{d\sigma}_{\underline{X}|}\, g(\underline{X}) \;=\; 0$$

We now introduce the Hermitean counterparts of the pair of oriented surface elements $(\widetilde{d\sigma}_{\underline{X}}, \widetilde{d\sigma}_{\underline{X}|})$. To this end, we should note that we may also write

$$\begin{aligned} \widetilde{d\sigma}_{\underline{X}} &= (-1)^{\frac{n(n-1)}{2}}\, d\sigma_{\underline{X}} \\ \widetilde{d\sigma}_{\underline{X}|} &= (-1)^{\frac{n(n-1)}{2}}\, d\sigma_{\underline{X}|} \end{aligned}$$

with the alternative pair of surface elements $(d\sigma_{\underline{X}}, d\sigma_{\underline{X}|})$ only involving a re-ordering of the variables according to n complex planes, i.e.

$$\begin{aligned} d\sigma_{\underline{X}} &= \sum_{j=1}^{n} \Big(e_j\ (dx_1 \wedge dy_1) \wedge \cdots \wedge ([dx_j] \wedge dy_j) \wedge \cdots \wedge (dx_n \wedge dy_n) \Big) \\ &\quad + \sum_{j=1}^{n} \Big(-e_{n+j}\ (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_j \wedge [dy_j]) \wedge \cdots \wedge (dx_n \wedge dy_n) \Big) \end{aligned}$$

and

$$\begin{aligned} d\sigma_{\underline{X}|} &= \sum_{j=1}^{n} \Big(-e_{n+j}\ (dx_1 \wedge dy_1) \wedge \cdots \wedge ([dx_j] \wedge dy_j) \wedge \cdots \wedge (dx_n \wedge dy_n) \Big) \\ &\quad + \sum_{j=1}^{n} \Big(-e_j\ (dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_j \wedge [dy_j]) \wedge \cdots \wedge (dx_n \wedge dy_n) \Big) \end{aligned}$$

where $[\cdot]$ denotes omitting that particular differential. It is then easily seen that

$$\begin{aligned}\widetilde{d\sigma}_{\underline{X}} - i\,\widetilde{d\sigma}_{\underline{X}|} &= (-1)^{\frac{n(n-1)}{2}}\left(d\sigma_{\underline{X}} - i\,d\sigma_{\underline{X}|}\right)\\ &= (-1)^{\frac{n(n-1)}{2}}(-4)\left(\frac{i}{2}\right)^n \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\,\widehat{dz_j}\end{aligned}$$

while

$$\begin{aligned}\widetilde{d\sigma}_{\underline{X}} + i\,\widetilde{d\sigma}_{\underline{X}|} &= (-1)^{\frac{n(n-1)}{2}}\left(d\sigma_{\underline{X}} + i\,d\sigma_{\underline{X}|}\right)\\ &= (-1)^{\frac{n(n-1)}{2}}(-4)\left(\frac{i}{2}\right)^n \sum_{j=1}^{n} \mathfrak{f}_j\,\widehat{dz_j^c}\end{aligned}$$

with

$$\begin{aligned}\widehat{dz_j} &= (dz_1 \wedge dz_1^c) \wedge \cdots \wedge ([dz_j] \wedge dz_j^c) \wedge \cdots \wedge (dz_n \wedge dz_n^c)\\ \widehat{dz_j^c} &= (dz_1 \wedge dz_1^c) \wedge \cdots \wedge (dz_j \wedge [dz_j^c]) \wedge \cdots \wedge (dz_n \wedge dz_n^c)\end{aligned}$$

This observation leads to the definition of the Hermitean oriented surface elements

$$\begin{aligned}d\sigma_{\underline{Z}} &= \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\,\widehat{dz_j}\\ d\sigma_{\underline{Z}^\dagger} &= \sum_{j=1}^{n} \mathfrak{f}_j\,\widehat{dz_j^c}\end{aligned}$$

for which it holds that

$$\begin{aligned}d\sigma_{\underline{Z}} &= -\frac{1}{4}(-2i)^n\left(d\sigma_{\underline{X}} - i\,d\sigma_{\underline{X}|}\right)\\ d\sigma_{\underline{Z}^\dagger} &= -\frac{1}{4}(-2i)^n\left(d\sigma_{\underline{X}} + i\,d\sigma_{\underline{X}|}\right)\end{aligned}$$

or equivalently

$$d\sigma_{\underline{Z}} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n\left(\widetilde{d\sigma}_{\underline{X}} - i\,\widetilde{d\sigma}_{\underline{X}|}\right) \tag{3.2}$$

$$d\sigma_{\underline{Z}^\dagger} = -\frac{1}{4}(-1)^{\frac{n(n+1)}{2}}(2i)^n\left(\widetilde{d\sigma}_{\underline{X}} + i\,\widetilde{d\sigma}_{\underline{X}|}\right) \tag{3.3}$$

Note that we in fact have applied the same technique as in Section 2, by means of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ acting on $d\sigma_{\underline{X}}$, up to a deliberately chosen constant.

We also consider the associated volume element $dW(\underline{Z}, \underline{Z}^\dagger)$ defined as

$$dW(\underline{Z}, \underline{Z}^\dagger) \quad = \quad \left(dz_1 \wedge dz_1^c\right) \wedge \left(dz_2 \wedge dz_2^c\right) \wedge \cdots \wedge \left(dz_n \wedge dz_n^c\right)$$

reflecting integration over the respective complex z_j–planes, $j = 1, \ldots, n$. One has that

$$\widetilde{dV}(\underline{X}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n dW(\underline{Z}, \underline{Z}^\dagger) \tag{3.4}$$

A first result is then easily obtained.

Theorem 3.3 (Hermitean Clifford–Stokes theorems). *Let f and g be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} f\, d\sigma_{\underline{Z}}\, g \quad = \quad \int_\Gamma \left[(f\, \partial_{\underline{Z}}\ )\, g + f\, (\partial_{\underline{Z}}\ \ g)\right] dW(\underline{Z}, \underline{Z}^\dagger)$$

$$\int_{\partial\Gamma} f\, (-d\sigma_{\underline{Z}^\dagger})\, g \quad = \quad \int_\Gamma \left[(f\, \partial_{\underline{Z}^\dagger})\, g + f\, (\partial_{\underline{Z}^\dagger}\, g)\right] dW(\underline{Z}, \underline{Z}^\dagger)$$

Proof. Start from the orthogonal Clifford–Stokes theorem(s) and invoke the expressions (3.2)–(3.3), as well as the relation (3.4) between the orthogonal volume element $\widetilde{dV}(\underline{X})$ and the Hermitean volume element $dW(\underline{Z}, \underline{Z}^\dagger)$. □

Theorem 3.4 (Hermitean Clifford–Cauchy theorems). *Let the function g be* h*-monogenic in Ω and let $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$, then*

$$\int_{\partial\Gamma} d\sigma_{\underline{Z}}\, g \quad = \quad 0$$

$$\int_{\partial\Gamma} d\sigma_{\underline{Z}^\dagger}\, g \quad = \quad 0$$

Proof. Start from the orthogonal Clifford–Cauchy theorem(s) and invoke the expressions (3.2)–(3.3), or alternatively, take $f = 1$ and g an h–monogenic function in the above Hermitean Clifford–Stokes theorems. □

4 Cauchy integral formulae

The fundamental solutions of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$, i.e. the orthogonal Cauchy kernels, are respectively given by

$$E\ (\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}}}{|\underline{X}|^{2n}} \tag{4.1}$$

$$E|(\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}}|}{|\underline{X}|^{2n}} \tag{4.2}$$

where a_{2n} denotes the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. Explicitly, this means

$$\partial_{\underline{X}}\ E\ (\underline{X}) = \delta(\underline{X}) \tag{4.3}$$

$$\partial_{\underline{X}|} E|(\underline{X}) = \delta(\underline{X}|) = \delta(\underline{X}) \tag{4.4}$$

By a lengthy calculation, we also find

Lemma 4.1.

$$\begin{aligned}\partial_{\underline{X}} E|(\underline{X}) &= -\frac{i}{n}(2\beta - n)\delta(\underline{X}) + 2n\frac{1}{a_{2n}}\ \mathrm{Fp}\ \frac{\underline{X}\underline{X}|}{|\underline{X}|^{2n+2}} \\ &\quad - 2i(2\beta - n)\frac{1}{a_{2n}}\ \mathrm{Fp}\ \frac{1}{|\underline{X}|^{2n}}\end{aligned} \tag{4.5}$$

$$\begin{aligned}\partial_{\underline{X}|} E(\underline{X}) &= \frac{i}{n}(2\beta - n)\delta(\underline{X}) + 2n\frac{1}{a_{2n}}\ \mathrm{Fp}\ \frac{\underline{X}|\underline{X}}{|\underline{X}|^{2n+2}} \\ &\quad + 2i(2\beta - n)\frac{1}{a_{2n}}\ \mathrm{Fp}\ \frac{1}{|\underline{X}|^{2n}}\end{aligned} \tag{4.6}$$

where β is the so-called spin Euler operator given by $\frac{1}{2}\sum_{j=1}^{n}(1 - i e_j e_{n+j})$ (see e.g. [7]*) and Fp stands for the traditional "finite part" distribution.*

Similarly as above, we now introduce the Hermitean counterparts to the pair of fundamental solutions $(E, E|)$, by putting

$$\mathcal{E} = -(E + i\ E|)$$

$$\mathcal{E}^{\dagger} = (E - i\ E|)$$

Explicitly this yields

$$\mathcal{E}\;(\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}}{|\underline{Z}|^{2n}} \tag{4.7}$$

$$\mathcal{E}^{\dagger}(\underline{Z}) = \frac{2}{a_{2n}}\frac{\underline{Z}^{\dagger}}{|\underline{Z}|^{2n}} \tag{4.8}$$

Note however that $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ are not the fundamental solutions to the respective Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^{\dagger}}$! Indeed, invoking (4.3)–(4.6), we obtain

Lemma 4.2.

$$\begin{aligned}\partial_{\underline{Z}}\;\mathcal{E}(\underline{Z}) &= \frac{1}{n}\beta\delta(\underline{Z},\underline{Z}^{\dagger}) + \frac{2}{a_{2n}}\beta\,\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2}{a_{2n}}n\,\mathrm{Fp}\,\frac{\underline{Z}^{\dagger}\underline{Z}}{r^{2n+2}}\\ \partial_{\underline{Z}^{\dagger}}\mathcal{E}(\underline{Z}) &= 0\end{aligned}$$

and

$$\begin{aligned}\partial_{\underline{Z}}\;\mathcal{E}^{\dagger}(\underline{Z}) &= 0\\ \partial_{\underline{Z}^{\dagger}}\mathcal{E}^{\dagger}(\underline{Z}) &= \frac{1}{n}(n-\beta)\delta(\underline{Z},\underline{Z}^{\dagger}) + \frac{2}{a_{2n}}(n-\beta)\,\mathrm{Fp}\,\frac{1}{r^{2n}} - \frac{2}{a_{2n}}n\,\mathrm{Fp}\,\frac{\underline{Z}\,\underline{Z}^{\dagger}}{r^{2n+2}}\end{aligned}$$

A first attempt at constructing a Hermitean Cauchy integral formula has been undertaken in [18], however presenting the "fake" – as termed by the authors themselves – Cauchy kernel $\frac{1}{2}\mathcal{E}^{\dagger} = \frac{1}{2}(E - iE|)$, which obviously fails to be h-monogenic.

Nevertheless it is clear that, in order to establish the desired formula, the functions $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ will need to be involved. Indeed, surprisingly, combining the above calculations, we are lead to the following result (see also [15]).

Theorem 4.1. *Introducing the particular circulant* (2×2) *matrices*

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^{\dagger})} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^{\dagger}} \\ \partial_{\underline{Z}^{\dagger}} & \partial_{\underline{Z}} \end{pmatrix}, \quad \boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^{\dagger} \\ \mathcal{E}^{\dagger} & \mathcal{E} \end{pmatrix}, \quad \textit{and} \quad \boldsymbol{\delta} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

one obtains that

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^{\dagger})}\boldsymbol{\mathcal{E}}(\underline{Z}) = \boldsymbol{\delta}(\underline{Z})$$

This means that $\boldsymbol{\mathcal{E}}$ may be considered as a fundamental solution of $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^{\dagger})}$, the latter concept being reinterpreted in a matrical context. It is precisely this simple observation which has lead us to the idea of a matrix approach to arrive at a Cauchy integral formula in the Hermitean setting. Also note, as another remarkable fact, that the Dirac matrix $\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}$ in some sense factorizes the Laplacian, since

$$4\,\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\left(\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\right)^\dagger = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix}$$

Thus, in the same setting of circulant (2×2) matrices we associate, with continuously differentiable functions g_1 and g_2 defined in Ω and taking values in $\mathbb{C}_{2n}$, the matrix function

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix} \tag{4.9}$$

Definition 4.1. *We call $\boldsymbol{G}_2^1$ (left) $\boldsymbol{H}$–monogenic if and only if it satisfies the system*

$$\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1 = \boldsymbol{O} \tag{4.10}$$

where $\boldsymbol{O}$ denotes the matrix with zero entries.

The above system (4.10) for $\boldsymbol{H}$–monogenicity explicitly reads

$$\begin{cases} \partial_{\underline{Z}}\,[g_1] + \partial_{\underline{Z}^\dagger}[g_2] = 0 \\ \partial_{\underline{Z}^\dagger}[g_1] + \partial_{\underline{Z}}\,[g_2] = 0 \end{cases}$$

Choosing in particular $g_1 = g$ and $g_2 = g^\dagger$, it is clear that, in general, the $\boldsymbol{H}$–monogenicity of the corresponding matrix function

$$\boldsymbol{G} = \begin{pmatrix} g & g^\dagger \\ g^\dagger & g \end{pmatrix}$$

will not imply the h–monogenicity of the function g and vice versa. As a simple example consider the matrix $\boldsymbol{\mathcal{E}}$ for which we have found above that it is $\boldsymbol{H}$–monogenic in $\mathbb{R}^{2n} \setminus \{\underline{0}\}$, while clearly the function $\mathcal{E}$ is not h–monogenic. An exception to this general remark clearly occurs in the special case of scalar (i.e. complex) valued functions, where h–monogenicity (of g) and $\boldsymbol{H}$–monogenicity (of $\boldsymbol{G}$) are found to be equivalent notions.

Another special yet very important case occurs when considering the matrix function

$$\boldsymbol{G}_0 = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

Since its $\boldsymbol{H}$–monogenicity is easily seen to be equivalent with the h–monogenicity of the function g, this specific matrix will form the key for the construction of a Hermitean Cauchy integral formula. A first step in this direction is the reformulation of the Hermitean Clifford–Stokes theorems, established in the previous section, in a matrical form. To this end, we still introduce the matrix

$$\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{Z}} & -d\sigma_{\underline{Z}^\dagger} \\ -d\sigma_{\underline{Z}^\dagger} & d\sigma_{\underline{Z}} \end{pmatrix}$$

which will play the rôle of the differential form. We then have the following result.

Theorem 4.2. *Let f_1, f_2, g_1 and g_2 be arbitrary functions in $C^1(\Omega; \mathbb{C}_{2n})$ and consider the corresponding matrix functions of the form* (4.9)*; let as above $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$. It then holds that*

$$\int_{\partial\Gamma} \boldsymbol{F}_2^1\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1 = \int_\Gamma \left[(\boldsymbol{F}_2^1\, \boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)})\, \boldsymbol{G}_2^1 + \boldsymbol{F}_2^1\, (\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1) \right] dW(\underline{Z}, \underline{Z}^\dagger)$$

Proof. Follows by taking deliberate combinations of the Hermitean Clifford–Stokes formulae found in Theorem 3.3. □

From now on Γ^+ will stand for $\overset{\circ}{\Gamma}$ and Γ^- for $\Omega \setminus \Gamma$; furthermore we reserve the notations $\underline{Y}$ and $\underline{Y}|$ for Clifford vectors associated to points in $\Gamma^\pm$. Their Hermitean counterparts are denoted by

$$\begin{aligned} \underline{V} &= \frac{1}{2}(\mathbf{1} + iJ)[\underline{Y}] = \frac{1}{2}(\underline{Y} + i\,\underline{Y}|) \\ \underline{V}^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[\underline{Y}] = -\frac{1}{2}(\underline{Y} - i\,\underline{Y}|) \end{aligned}$$

The following Hermitean Cauchy-Pompeiu formula is then established.

Theorem 4.3 (Hermitean Cauchy-Pompeiu formula). *Let g_1 and g_2 be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let $\boldsymbol{G}_2^1$ be the corresponding matrix function of the form* (4.9)*; let as above $\Gamma \subset \Omega$ be a $2n$–dimensional compact differentiable and oriented manifold with C^∞ smooth boundary $\partial\Gamma$. It then holds that*

$$\begin{aligned} &\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) - \int_\Gamma \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X}) \right] dW(\underline{Z}, \underline{Z}^\dagger) \\ &= \begin{cases} \boldsymbol{O}\,, & \text{if } \underline{Y} \in \Gamma^- \\ (-1)^{\frac{n(n+1)}{2}} (2i)^n\, \boldsymbol{G}_2^1(\underline{Y})\,, & \text{if } \underline{Y} \in \Gamma^+ \end{cases} \end{aligned}$$

Proof. First, let $\underline{Y} = \underline{V} - \underline{V}^\dagger \in \Gamma^-$. In this case we have that, considered as functions of $\underline{X} = \underline{Z} - \underline{Z}^\dagger$,

$$\mathcal{E}(\underline{Z} - \underline{V}) = \frac{2}{a_{2n}} \frac{\underline{Z} - \underline{V}}{|\underline{Z} - \underline{V}|^{2n}} = \frac{2}{a_{2n}} \frac{\underline{Z} - \underline{V}}{|\underline{X} - \underline{Y}|^{2n}}$$

and

$$\mathcal{E}^\dagger(\underline{Z} - \underline{V}) = \frac{2}{a_{2n}} \frac{\underline{Z}^\dagger - \underline{V}^\dagger}{|\underline{Z} - \underline{V}|^{2n}} = \frac{2}{a_{2n}} \frac{\underline{Z}^\dagger - \underline{V}^\dagger}{|\underline{X} - \underline{Y}|^{2n}}$$

are continuously differentiable in Γ^+, so that the Hermitean Clifford–Stokes Theorem 4.2 can be applied, yielding the desired statement, since we have that in Ω

$$\boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} = \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} = \boldsymbol{O}$$

Next, let $\underline{Y} = \underline{V} - \underline{V}^\dagger \in \Gamma^+$, and take $R > 0$ such that $B(\underline{X}; R) \subset \Gamma$. Invoking the previous case, we may then write

$$\int_{\partial(\Gamma \setminus B(\underline{X}; R))} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z}, \underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})$$
$$- \int_{\Gamma \setminus B(\underline{X}; R)} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger) = \boldsymbol{O}$$

Taking limits for $R \to 0$ the second term at the left-hand side yields

$$\lim_{R \to 0} \int_{\Gamma \setminus B(\underline{X}; R)} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger)$$
$$= \int_{\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \left[\boldsymbol{\mathcal{D}}_{(\underline{Z}, \underline{Z}^\dagger)} \boldsymbol{G}_2^1(\underline{X})\right] dW(\underline{Z}, \underline{Z}^\dagger)$$

since the integrand only contains functions which are integrable on Γ. Furthermore we may write

$$\int_{\partial(\Gamma \setminus B(\underline{X}; R))} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z}, \underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})$$
$$= \int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z}, \underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) - \int_{\partial B(\underline{X}; R)} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z}, \underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})$$

In order to calculate the last integral in the above expression, we apply once more the Hermitean Clifford–Stokes Theorem 4.2:

$$\int_{\partial B(\underline{X};R)} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X})$$

$$= \frac{2}{a_{2n}R^{2n}}\Big[\int_{B(\underline{X};R)} [\boldsymbol{G}_{\underline{Z}-\underline{V}}\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}]\, \boldsymbol{G}_2^1\, dW(\underline{Z},\underline{Z}^\dagger)$$

$$+ \int_{B(\underline{X};R)} \boldsymbol{G}_{\underline{Z}-\underline{V}}\, [\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\boldsymbol{G}_2^1]\, dW(\underline{Z},\underline{Z}^\dagger)\Big]$$

where

$$\boldsymbol{G}_{\underline{Z}-\underline{V}} = \begin{pmatrix} \underline{Z}-\underline{V} & \underline{Z}^\dagger-\underline{V}^\dagger \\ \underline{Z}^\dagger-\underline{V}^\dagger & \underline{Z}-\underline{V} \end{pmatrix}$$

By direct calculation of the first term we obtain

$$\lim_{R\to 0} \frac{2}{a_{2n}R^{2n}} \int_{B(\underline{X};R)} \big[\boldsymbol{G}_{\underline{Z}-\underline{V}}\boldsymbol{\mathcal{D}}_{(\underline{Z},\underline{Z}^\dagger)}\big]\, \boldsymbol{G}_2^1(\underline{X})\, dW(\underline{Z},\underline{Z}^\dagger)$$

$$= (-2i)^n\, (-1)^{\frac{n(n-1)}{2}}\, \boldsymbol{G}_2^1(\underline{Y})$$

while the second term may be shown to converge to $\boldsymbol{O}$ for $R \to 0$. □

This theorem then leads to the following Hermitean Cauchy integral formulae for $\boldsymbol{H}$–monogenic matrix functions $\boldsymbol{G}_2^1$ and h–monogenic functions g, respectively.

Theorem 4.4 (Hermitean Cauchy integral formula I). *If the matrix function $\boldsymbol{G}_2^1$ is $\boldsymbol{H}$–monogenic in Ω then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_2^1(\underline{X}) = \begin{cases} \boldsymbol{O}\,, & \text{if } \underline{Y}\in\Gamma^- \\ (-1)^{\frac{n(n+1)}{2}}(2i)^n\, \boldsymbol{G}_2^1(\underline{Y})\,, & \text{if } \underline{Y}\in\Gamma^+ \end{cases}$$

Proof. Apply Theorem 4.3 while taking into account the $\boldsymbol{H}$–monogenicity of the matrix function $\boldsymbol{G}_2^1$. □

Theorem 4.5 (Hermitean Cauchy integral formula II). *If the function g is* h*–monogenic in Ω then*

$$\int_{\partial\Gamma} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_0(\underline{X}) = \begin{cases} \boldsymbol{O}\,, & \text{if } \underline{Y}\in\Gamma^- \\ (-1)^{\frac{n(n+1)}{2}}(2i)^n\, \boldsymbol{G}_0(\underline{Y})\,, & \text{if } \underline{Y}\in\Gamma^+ \end{cases} \tag{4.11}$$

The previous theorem may be considered as a Hermitean Cauchy integral formula for the h–monogenic function g; therefore the matrix function $\boldsymbol{\mathcal{E}}$ appearing in this formula is called the Hermitean Cauchy kernel.

Remark 4.1. The Cauchy integral is a well–known integral operator applying to functions defined on $\partial\Gamma$. It has been thoroughly studied in the framework of orthogonal Clifford analysis; in particular it has lead to the definition of Clifford–Hardy spaces and of a multidimensional Clifford vector valued Hilbert transform, when considering its non–tangential boundary limits in L_2–sense in the interior or exterior of the domain of interest (see [11, 8]). It is clear that, by means of the matrical Hermitean Cauchy kernel defined in this section, also a Hermitean Cauchy integral may be defined; the study of its boundary limits, leading to Hermitean Clifford–Hardy spaces and to a Hermitean Hilbert transform, is the subject of the paper [5].

5 The Martinelli–Bochner formula revisited

In this section we will restrict ourselves to so–called spinor valued functions, i.e. functions on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, taking values in the complex spinor space

$$\mathbb{S} = \mathbb{C}_{2n}\, I \cong \mathbb{C}_n\, I$$

where I is the primitive idempotent introduced in Section 2, (2.2). In [4], it has been shown that $\mathbb{S}$, considered as a $\widetilde{\mathrm{U}}(n)$–module, decomposes as

$$\mathbb{S} = \bigoplus_{j=1}^{n} \mathbb{S}_j = \bigoplus_{j=1}^{n} \left(\mathbb{C}\Lambda_n^\dagger\right)^{(j)} I \tag{5.1}$$

into the $\widetilde{\mathrm{U}}(n)$–invariant and irreducible subspaces

$$\mathbb{S}_j = \left(\mathbb{C}\Lambda_n^\dagger\right)^{(j)} I, \qquad j = 0, \ldots, n$$

consisting of j-vectors from $\mathbb{C}\Lambda_n^\dagger$ multiplied by the idempotent I, where $\mathbb{C}\Lambda_n^\dagger$ is the Grassmann algebra generated by the Witt basis elements $\{\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger\}$. Therefore, the spaces $\mathbb{S}_j$ are also called the "homogeneous parts" of spinor space. Clearly, the system for h–monogenic spinor valued functions will then split into n independent subsystems for functions with values in $\mathbb{S}_j$, $j = 0, \ldots, n$, the individual subsystems not being mutually equivalent (see [4]).

In particular, for $j = n$, the homogeneous n–space of spinor space is generated by the basis element $\mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger I$, so that $\mathbb{S}_n$ valued functions take the form

$$g(z_1, \ldots, z_n) = g_n(z_1, \ldots, z_n)\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \tag{5.2}$$

where g_n is a smooth complex valued function on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. It is easily seen that h–monogenicity for such a function g is equivalent to holomorphy of the corresponding scalar function g_n in the complex variables $(z_1, \ldots, z_n)$. Thus, considering functions g of the form (5.2) establishes a connection between Hermitean Clifford analysis and the theory of holomorphic functions of several complex variables, whence it is interesting to investigate the true nature of the Hermitean Cauchy integral formula obtained in the previous section for this type of functions.

To this end, we will explicitly calculate the left-hand side of formula (4.11), taking g to be of the form (5.2). We obtain

$$
\begin{aligned}
\boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_0(\underline{X}) &= \begin{pmatrix} \sum_{k=1}^{n} \left(\widehat{dz_k \mathfrak{f}_k^\dagger}\right) g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I & * \\ -\sum_{k=1}^{n} \left(\widehat{dz_k^c \mathfrak{f}_k}\right) g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I & * \end{pmatrix} \\
&= \begin{pmatrix} 0 & * \\ -d\sigma_{\underline{Z}^\dagger}\, g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I & * \end{pmatrix}
\end{aligned}
$$

where the second column has not been written, since it only duplicates the first one (in reversed order) seen the circulant structure of the involved matrices, and the matrix entry $[\cdot_{11}]$ reduces to zero on account of the anti–commutation and the isotropy of the Witt basis element $\mathfrak{f}_k^\dagger$. Further calculation yields

$$
\boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\, \boldsymbol{d\Sigma}_{(\underline{Z},\underline{Z}^\dagger)}\, \boldsymbol{G}_0(\underline{X}) = \begin{pmatrix} -\mathcal{E}^\dagger(\underline{Z} - \underline{V})\, d\sigma_{\underline{Z}^\dagger}\, g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I & * \\ -\mathcal{E}\ \ (\underline{Z} - \underline{V})\, d\sigma_{\underline{Z}^\dagger}\, g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I & * \end{pmatrix}
$$

where, putting $\rho = |\underline{Z} - \underline{V}|$,

$$
\mathcal{E}^\dagger(\underline{Z} - \underline{V})\, d\sigma_{\underline{Z}^\dagger} = \frac{2}{a_{2n}\rho^{2n}} \left(\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \big(z_j^c - v_j^c\big) \right) \left(\sum_{k=1}^{n} \mathfrak{f}_k \widehat{dz_k^c} \right)
$$

whence

$$
\begin{aligned}
&\mathcal{E}^\dagger(\underline{Z} - \underline{V})\, d\sigma_{\underline{Z}^\dagger}\, g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \\
&= \frac{2}{a_{2n}\rho^{2n}} \sum_{j=1}^{n} (z_j^c - v_j^c) \widehat{dz_j^c}\, g_n\big(z_1, \ldots, z_n\big)\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I
\end{aligned}
$$

where we have invoked the duality identities for the Witt basis elements and once more their isotropy. Similarly

$$\mathcal{E}(\underline{Z}-\underline{V})\, d\sigma_{\underline{Z}^\dagger} = \frac{2}{a_{2n}\rho^{2n}} \left(\sum_{j=1}^{n} \mathfrak{f}_j \left(z_j^c - v_j^c \right) \right) \left(\sum_{k=1}^{n} \mathfrak{f}_k \widehat{dz_k^c} \right)$$

whence

$$\begin{aligned} &\mathcal{E}(\underline{Z}-\underline{V})\, d\sigma_{\underline{Z}^\dagger}\, g_n\big(z_1,\ldots,z_n\big)\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \\ &= \frac{2}{a_{2n}\rho^{2n}} \sum_{j\neq k} \big(z_j^c - v_j^c\big) \widehat{dz_k^c}\, g_n\big(z_1,\ldots,z_n\big)\, \mathfrak{f}_j \mathfrak{f}_k\, \mathfrak{f}_1^\dagger \mathfrak{f}_2^\dagger \ldots \mathfrak{f}_n^\dagger\, I \end{aligned}$$

Thus, the Hermitean Cauchy integral formula (4.11) for $\underline{Y} \in \Gamma^+$ yields two statements. The first one reads

$$(-1)^{\frac{n(n+1)}{2}} g_n(\underline{Y}) = -\int_{\partial\Gamma} \frac{(n-1)!}{(2\pi i)^n} \frac{1}{\rho^{2n}} \sum_{j=1}^{n} \big(z_j^c - v_j^c\big) \widehat{dz_j^c}\, g_n(\underline{X})$$

which exactly coincides with the Martinelli–Bochner formula (1.2), when taking into account the appropriate reordering of the involved differential forms. Here we have used $a_{2n} = \frac{2\pi^n}{(n-1)!}$. The second statement is

$$0 = -\int_{\partial\Gamma} \frac{2}{a_{2n}} \frac{1}{\rho^{2n}} \sum_{j\neq k} \big(z_j - v_j\big) \widehat{dz_k^c}\, g_n(\underline{X})\, \mathfrak{f}_j \mathfrak{f}_k\, \mathfrak{f}_1^\dagger \ldots \mathfrak{f}_n^\dagger\, I$$

which, by means of some more Witt basis calculations, decomposes into

$$\int_{\partial\Gamma} \frac{z_j - v_j}{\rho^{2n}} \widehat{dz_k^c}\, g_n(\underline{X}) = \int_{\partial\Gamma} \frac{z_k - v_k}{\rho^{2n}} \widehat{dz_j^c}\, g_n(\underline{X}), \qquad j,k = 1,\ldots,n,\ j \neq k$$

a result which can be proved directly using standard techniques.

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