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Norm optimization problem for linear operators in classical Banach spaces

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Abstract. The main result of the paper shows that, for $1 < p < \infty$ and $1 \leq q < \infty$, a linear operator $T\colon \ell_p \to \ell_q$ attains its norm if, and only if, there exists a not weakly null maximizing sequence for T (counterexamples can be easily constructed when $p = 1$). For $1 < p \neq q < \infty$, as a consequence of the previous result we show that any not weakly null maximizing sequence for a norm attaining operator $T : \ell_p \to \ell_q$ has a norm-convergent subsequence (and this result is sharp in the sense that it is not valid if $p = q$). We also investigate lineability of the sets of norm-attaining and non-norm attaining operators.

Keywords: norm attaining operators, lineability.

Mathematical subject classification: 47B37, 47B99, 47B05.

1 Introduction

Let E and F be two Banach spaces. We denote by $\mathcal{L}(E, F)$ the space of all bounded linear operators from E into F. A linear operator $T\colon E \to F$ is said to attain its norm if there exists a $v \in E$, $\|v\|_E = 1$, such that

$$\|T(v)\|_F = \|T\|_{\mathcal{L}(E,F)} := \sup_{e \in \mathbb{S}_E} \|T(e)\|_F,$$

where $\mathbb{S}_E$ denotes the unit sphere of E, i.e., $\mathbb{S}_E = \{e \in E : \|e\|_E = 1\}$. We will denote by $\mathcal{NA}(E, F)$ the subset of all norm attaining bounded linear operators from E into F.

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The question whether a given linear operator attains its norm is doubtless one of the most important lines of investigation from the applied functional analysis point of view. Often, the solvability of certain (continuous or discrete) differential equations is intrinsically related to the norm attaining property of a determined linear operator acting between appropriate Banach spaces.

When the target space is the real line, i.e., $F = \mathbb{R}$, a deep and, by now, well known result due to James, see [9], asserts that $\mathcal{NA}(E, \mathbb{R}) = \mathcal{L}(E, \mathbb{R})$ if and only if E is reflexive. Another classical result in this theory, Bishop-Phelps' Theorem, [6], states that $\mathcal{NA}(E, \mathbb{R})$ is always norm-dense in $\mathcal{L}(E, \mathbb{R})$.

From now on the Banach spaces will be considered over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

The question whether $\mathcal{NA}(E, F)$ is dense in $\mathcal{L}(E, F)$ for an arbitrary Banach space F becomes much more involved. A remarkable result due to Lindenstrauss assures that if F is reflexive then indeed $\mathcal{NA}(E, F)$ is dense in $\mathcal{L}(E, F)$. This result was further generalized by Bourgain in [7], who showed that the Radon-Nikodym property on F suffices for $\mathcal{NA}(E, F)$ to be dense in $\mathcal{L}(E, F)$. On the converse, Gowers in [8] showed there exists a Banach space $\mathfrak{G}$, such that $\mathcal{NA}(\ell_p, \mathfrak{G})$ is not dense in $\mathcal{L}(\ell_p, \mathfrak{G})$, for $1 < p < \infty$. The case $p = 1$ was settled by Acosta in [1].

The first goal of this note is to provide a simple yet useful characterization of norm attaining operators acting on ℓ_p type spaces. Hereafter T will always denote a bounded linear operator from ℓ_p into ℓ_q. For a not weakly null maximizing sequence for T we mean a sequence $u^n \in \ell_p$, with,

$$\|u^n\|_{\ell_p} = 1, \ \|T(u^n)\|_{\ell_q} \to \|T\|$$

and u^n does not converge weakly to zero.

Initially, let us recall that Pitt's Theorem, [14], states that any bounded linear operator $T\colon \ell_p \to \ell_q$, with $1 \leq q < p$ is compact. Therefore, $\mathcal{NA}(\ell_p; \ell_q) = \mathcal{L}(\ell_p; \ell_q)$ provided that $q < p$. On the other hand, for $1 \leq p \leq q < \infty$ it is well-known that $\mathcal{NA}(\ell_p; \ell_q) \neq \mathcal{L}(\ell_p; \ell_q)$ (see [10, Proposition 4.2]). Yet in the lights of Pitt's Theorem, for $p > q$, if $T \neq 0$ and $(x^n)_{n=1}^{\infty}$ is a maximizing sequence for T, then $(x^n)_{n=1}^{\infty}$ is not weakly null. Clearly this result is no longer valid for $p \leq q$ (the inclusion provides an example). Our first and main result shows that when $p \leq q$, the existence of a not weakly null maximizing sequence for T occurs precisely when T is norm attaining.

Theorem 1. *Let $1 < p < \infty$, $1 \leq q < \infty$ and $T\colon \ell_p \to \ell_q$ be a bounded linear operator. Then T attains its norm if, and only if, there exists a not weakly null maximizing sequence for T.*

Theorem 1 is sharp in the sense that it is no longer true for $p = 1$. In fact, if $1 \leq p \leq q < \infty$ the operator $T \in \mathcal{L}(\ell_p; \ell_q)$ given by

$$T(x) = \left(\frac{n x_n}{n+1} \right)_{n=1}^{\infty} \tag{1}$$

does not attain its norm. The canonical basis $(e_j)_{j=1}^{\infty}$ is a maximizing sequence for T which is not weakly null when $p = 1$. The authors thank R. Aron for this observation.

The strategy for proving Theorem 1 relies on an asymptotic analysis involving weak convergence in ℓ_p-type spaces, similar to [13]. From the proof of Theorem 1, as long as $p \neq q$, we can actually infer pre-compactness of any not weakly null maximizing sequence for a linear operator $T \in \mathcal{L}(\ell_p; \ell_q)$. This is the content of our next result.

Theorem 2. *Let $T : \ell_p \to \ell_q$ be a not identically zero norm attaining operator and $1 < p < \infty$, $1 \leq q < \infty$, $p \neq q$. Then any not weakly null maximizing sequence for T has a norm convergent subsequence.*

Initially, let us point out that Theorem 2 is sharp in the sense it does not hold true if $p = q$. Indeed, the sequence

$$\begin{aligned} u^n &:= \left(\frac{1}{\sqrt[p]{2}}, 0, 0, \cdots, \frac{1}{\sqrt[p]{2}}, 0, \cdots \right) \\ &= \frac{1}{\sqrt[p]{2}} e_1 + \frac{1}{\sqrt[p]{2}} e_n \end{aligned}$$

is a not weakly null maximizing sequence for the identity map, $\mathrm{Id} : \ell_p \to \ell_p$, but has no norm convergent subsequence.

Theorem 1 is particulary useful in discrete problems involving some sort of symmetry or special invariances. For instance, in practical applications, one is often able to find a hyperplane $\Pi := \left\{ f(x) \leq \epsilon \right\}$ with $\epsilon > 0$ such that

$$\|T(\xi)\|_{\ell_q} < \|T\| \quad \text{for all} \quad \xi \in \Pi \cap B_1.$$

Thus a maximizing sequence can be found within $B_1 \cap \{ f(x) > \epsilon \}$. In particular such a maximizing sequence is not weakly null.

The simplest norm-invariant operation for sequences is permutation. In the sequel, we state a definition and afterwards a consequence of Theorem 1 related to permutation of sequences.

Definition 3. *Given a real sequence* $\alpha = \{\alpha_j\}_{j=1}^{\infty} \in c_0$*, the non-increasing permutation of* α*, denoted as* $\sigma(\alpha) = \{\beta_j\}_{j=1}^{\infty}$ *is given by*

$$\beta_1 := \max_{j\in\mathbb{N}}\{|\alpha_j|\},$$

$$\beta_2 := \max_{j\in\mathbb{N}}\left(\{|\alpha_j|\} \setminus \{\beta_1\}\right), \cdots,$$

$$\beta_k := \max_{j\in\mathbb{N}}\left(\{|\alpha_j|\} \setminus \{\beta_1, \beta_2, \cdots, \beta_{k-1}\}\right), \cdots,$$

where $\beta_k = 0$ *if* $\left(\{|\alpha_j|\} \setminus \{\beta_1, \beta_2, \cdots, \beta_{k-1}\}\right) = \phi$*. A linear operator* $T : \ell_p \to \ell_q$ *is said to be monotone with respect to non-increasing permutation if* $\|T(\sigma(x))\|_{\ell_q} \geq \|T(x)\|_{\ell_q}$ *for every* $x \in \ell_p$*.*

From now on, the notation $\langle , \rangle$ will represent the canonical duality between ℓ_p and $\ell_{p'}$ where $1/p + 1/p' = 1$, i.e., if $x = (x_j)_{j=1}^{\infty} \in \ell_p$ and $y = (y_j)_{j=1}^{\infty} \in \ell_{p'}$ then

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j.$$

We also use the symbols e_j to denote the canonical jth unit vectors in each ℓ_p.

A typical, but not the only, way of verifying that a given operator T is monotone with respect to non-increasing permutation is by checking that

$$\langle Te_1, e_j\rangle \geq \langle Te_2, e_j\rangle \geq \cdots \geq \langle Te_k, e_j\rangle \geq \cdots \geq 0, \quad \forall j = 1, 2, \cdots$$

Concerning monotone with respect to non-increasing permutation operators, we have the following general result.

Theorem 4. *Let* $1 < p < \infty$, $1 \leq q < \infty$ *and* $T : \ell_p \to \ell_q$ *be monotone with respect to non-increasing permutation. Assume for some* $\epsilon > 0$, $T\big|_{\ell_{p+\epsilon}} \hookrightarrow \ell_q$ *continuously. Then* T *attains its norm.*

As another simple yet interesting application of Theorem 1 (and Bessaga-Pelczyński selection principle) we obtain, up to subsequences, a structural behavior of any maximizing sequence for an operator

$$T \in \mathcal{L}(\ell_p; \ell_q) \setminus \mathcal{NA}(\ell_p; \ell_q).$$

We recall that if $\{e_i\}$ and $\{f_i\}$ are two basic sequences in Banach spaces, then we say $\{e_i\}$ is equivalent to $\{f_i\}$ if for any sequence of scalars $\{\lambda_i\}$, $\sum_i \lambda_i e_i$ converges if and only if $\sum_i \lambda_i f_i$ converges.

Proposition 5. *Let $1 < p < \infty$, $1 \le q < \infty$ and $T : \ell_p \to \ell_q$ be a bounded linear operator. Assume T does not attain its norm. Then, any maximizing sequence u^n for T has a subsequence, u^{n_k}, such that u^{n_k} is isometrically equivalent to the canonical basis of ℓ_p, whose image is equivalent to the canonical basis of ℓ_q.*

In a parallel direction, we also carry out the investigation of lineability properties related to the sets $\mathcal{NA}(E; \ell_q)$ as well as $\mathcal{L}(E; \ell_q) \setminus \mathcal{NA}(E; \ell_q)$. Recall that in an infinite-dimensional vector space E, a set $A \subset E$ is said to be lineable if $A \cup \{0\}$ contains an infinite-dimensional subspace. The term "lineable" seems to have been coined by V. Gurariy and has been broadly explored in different contexts (see, for example [2, 4, 5] and references therein).

Let E and F be Banach spaces. For a fixed vector $x_0 \in \mathbb{S}_E$, let us denote $\mathcal{NA}^{x_0}(E; F)$ the set of all linear operators in $\mathcal{L}(E; F)$ that attain their norms at x_0, that is,

$$\mathcal{NA}^{x_0}(E; F) := \left\{ T \in \mathcal{NA}(E; F) : \|Tx_0\|_F = \|T\|_{\mathcal{L}(E;F)} \right\}.$$

The linear structure of the sets $\mathcal{NA}(E; \mathbb{K})$ and $\mathcal{L}(E; \mathbb{K}) \smallsetminus \mathcal{NA}(E; \mathbb{K})$ was subject of several recent works (see, e.g., [1, 2] and references therein) and, of course, the geometry of E plays a decisive role in this study. If we replace $\mathbb{K}$ by an infinite-dimensional Banach space F, as it will be shown, it seems that the geometry of F will be decisive, rather than the particular properties of E. We believe that the study of lineability properties related to $\mathcal{NA}(E; F)$, where F is a hereditarily indecomposable space may be an interesting subject for further investigation.

It is worth mentioning that the presence of an infinite-dimensional Banach space F in the place of the scalar field $\mathbb{K}$ allows to investigate the lineability of sets of norm-attaining operators at a fixed point x_0.

In general, $\mathcal{NA}^{x_0}(E; F)$ is a quite more restrictive subset of $\mathcal{NA}(E; F)$. Nevertheless we have managed to show that if F contains an isometric copy of ℓ_q, then $\mathcal{NA}^{x_0}(E; F)$ is lineable in $\mathcal{L}(E; F)$. In particular $\mathcal{NA}^{x_0}(\ell_p; \ell_q)$ is lineable in $\mathcal{L}(\ell_p; \ell_q)$. This is the content of our next result.

Proposition 6. *Let E and F be Banach spaces so that F contains an isometric copy of ℓ_q for some $1 \le q < \infty$, and let $x_0 \in \mathbb{S}_E$. Then $\mathcal{NA}^{x_0}(E; F)$ is lineable in $\mathcal{L}(E; F)$.*

An adaptation of the argument used to prove Proposition 6 allows us to conclude that $\mathcal{L}(\ell_p; \ell_q) \smallsetminus \mathcal{NA}(\ell_p; \ell_q)$ is also lineable in $\mathcal{L}(\ell_p; \ell_q)$ when $1 \le p \le q < \infty$. In fact we prove a more general result:

Proposition 7. *Let E and F be Banach spaces so that F contains an isometric copy of ℓ_q for some $1 \leq q < \infty$. If $\mathcal{L}(E;\ell_q) \neq \mathcal{NA}(E;\ell_q)$, then $\mathcal{L}(E;F) \setminus \mathcal{NA}(E;F)$ is lineable in $\mathcal{L}(E;F)$.*

Corollary 8. *If $1 \leq p \leq q < \infty$ then $\mathcal{L}(\ell_p;\ell_q) \setminus \mathcal{NA}(\ell_p;\ell_q)$ is lineable in $\mathcal{L}(\ell_p;\ell_q)$.*

The arguments used throughout the article are fairly clear and simple in nature. We do believe they provide insights for further generalizations to more abstract settings.

2 Characterization of operators in $\mathcal{NA}(\ell_p;\ell_q)$

Proof of Theorem 1. Clearly, if T attains its norm, there exists a not weakly null maximizing sequence for T. We shall just address the converse. As mentioned in the introduction, when $p > q$, *any* bounded linear operator attains its norm. The really interesting situation for us is therefore when $1 < p \leq q$.

Let u^n be a not weakly null maximizing sequence for T. Passing to a subsequence if necessary, we may assume,

$$u^n \rightharpoonup u \neq 0,$$

where the symbol $\rightharpoonup$ stands for the weak convergence in ℓ_p. Our first observation is that there exists a subsequence (v^n) of (u^n) so that

$$\left(|\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle|^{q-1}\right)_{i=1}^{\infty} \rightharpoonup 0 \tag{2}$$

in $\ell_{\frac{q}{q-1}}$. Indeed, since

$$\left(|\langle T(u^n), e_i\rangle - \langle T(u), e_i\rangle|^{q-1}\right)_{i=1}^{\infty}$$

is a bounded sequence in $\ell_{\frac{q}{q-1}}$, there is a subsequence (v^n) of (u^n) so that

$$\left(|\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle|^{q-1}\right)_{i=1}^{\infty} \rightharpoonup f,$$

for some $f \in \ell_{\frac{q}{q-1}}$. Since $v^n \rightharpoonup u$ we have that $T(v^n) \rightharpoonup T(u)$; so, for each $i = 1, 2, \cdots$

$$\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle \to 0,$$

thus f must be the zero vector.

Let $r > 1$ be a real number and let us consider the auxiliary function $\varphi_r : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by

$$\varphi_r(X) = \frac{\Big| \, |X|^r - |X-1|^r - 1 \, \Big|}{|X-1|^{r-1}}.$$

It is simple to verify that

$$\lim_{|X|\to\infty} \varphi_r(X) = r.$$

Hence, given $\varepsilon > 0$, we can find a constant C_ε such that

$$\Big| \, |X|^r - |X-1|^r - 1 \, \Big| \le C_\varepsilon |X-1|^{r-1}$$

whenever $|X-1| > \varepsilon$. On the other hand if $|X-1| \le \varepsilon$, we have

$$\Big| \, |X|^r - |X-1|^r - 1 \, \Big| \le \varepsilon^r + \tilde{\delta}(\varepsilon),$$

where

$$\tilde{\delta}(\varepsilon) := \sup_{|t-1|\le\varepsilon} | \, |t|^r - 1 \, |.$$

Adding up the above two inequalities we obtain

$$\Big| \, |X|^r - |X-1|^r - 1 \, \Big| \le C_\varepsilon |X-1|^{r-1} + \delta(\varepsilon), \tag{3}$$

for every $X \in \mathbb{R}$, where $\delta(\varepsilon) = \varepsilon^r + \tilde{\delta}(\varepsilon)$.

The idea now is to apply estimate (3) to each coordinate of $T(v^n)$ in ℓ_q. More precisely, we apply inequality (3) to $r = q$ and

$$X_i := \frac{\langle T(v^n), e_i \rangle}{\langle T(u), e_i \rangle},$$

whenever $\langle T(u), e_i \rangle$ is nonzero. In any case, when we add these inequalities up, we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \Big| |\langle T(v^n), e_i \rangle|^q - |\langle T(v^n), e_i \rangle - \langle T(u), e_i \rangle|^q - |\langle T(u), e_i \rangle|^q \Big| \\ \le C_\varepsilon \Delta_n + \delta(\varepsilon) \|T(u)\|_{\ell_q}^q, \end{aligned} \tag{4}$$

where

$$\Delta_n = \sum_{i=1}^{\infty} |\langle T(u), e_i \rangle| \cdot |\langle T(v^n), e_i \rangle - \langle T(u), e_i \rangle|^{q-1}.$$

Since $T(u) \in \ell_q = \left[\ell_{\frac{q}{q-1}}\right]^*$, we have

$$\Delta_n = \left\langle T(u), \left(|\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle|^{q-1}\right)_{i=1}^{\infty} \right\rangle$$

and using (2) it follows that

$$\lim_{n\to\infty} \Delta_n = \langle T(u), 0\rangle = 0. \tag{5}$$

Letting $n \to \infty$ in (4) we get, for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} \left| \sum_{i=1}^{\infty} \left(|\langle T(v^n), e_i\rangle|^q - |\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle|^q - |\langle T(u), e_i\rangle|^q\right) \right| \leq \delta(\varepsilon)\|T(u)\|_{\ell_q}^q,$$

Letting $\varepsilon \searrow 0$ we conclude that the lim sup is in fact the limit and is equal to zero:

$$\lim_{n\to\infty} \left| \sum_{i=1}^{\infty} \left(|\langle T(v^n), e_i\rangle|^q - |\langle T(v^n), e_i\rangle - \langle T(u), e_i\rangle|^q - |\langle T(u), e_i\rangle|^q\right) \right| = 0.$$

In particular,

$$\|T(v^n)\|_{\ell_q}^q = \left(\|Tu\|_{\ell_q}^q + \|T(v^n - u)\|_{\ell_q}^q + \mathrm{o}(1)\right), \tag{6}$$

where $\mathrm{o}(1) \to 0$ as $n \to \infty$.

A similar computation, using $r = p$ on inequality (3) and applying it on the points

$$Y_i := \frac{\langle v^n, e_i\rangle}{\langle u, e_i\rangle}$$

can be performed, as long as $\langle u, e_i\rangle$ is nonzero. Reasoning as before, we reach

$$\|w^n - u\|_{\ell_p}^p = 1 - \|u\|_{\ell_p}^p + \mathrm{o}(1) \tag{7}$$

for some subsequence (w^n) of (v^n). Combining (6) and (7) with the well known inequality

$$(\alpha + \beta)^\theta \leq \alpha^\theta + \beta^\theta,$$

for $\alpha, \beta \geq 0$ and $0 \leq \theta \leq 1$, we can write

$$\begin{aligned} \|T(w^n)\|_{\ell_q}^p &\leq \left(\|T(u)\|_{\ell_q}^q + \|T(w^n - u)\|_{\ell_q}^q + \mathrm{o}(1)\right)^{p/q} \\ &\leq \|T(u)\|_{\ell_q}^p + \|T(w^n - u)\|_{\ell_q}^p + \mathrm{o}(1) \\ &\leq \|T(u)\|_{\ell_q}^p + \|T\|^p \cdot \|w^n - u\|_{\ell_p}^p + \mathrm{o}(1) \\ &\leq \|T(u)\|_{\ell_q}^p + \|T\|^p \cdot \left[1 - \|u\|_{\ell_p}^p + \mathrm{o}(1)\right] + \mathrm{o}(1). \end{aligned} \tag{8}$$

Letting $n \to \infty$, we finally obtain

$$\|T(u)\|_{\ell_q} \geq \|T\| \cdot \|u\|_{\ell_p},$$

which finishes the proof of Theorem 1. □

3 A disguise of Theorem 1: Pre-compactness of maximizing sequences

Proof of Theorem 2. Let $T : \ell_p \to \ell_q$, with $T \neq 0$ be a norm attaining operator and x^n a maximizing sequence on $\mathbb{S}_{\ell_p}$ that does not converge weakly to zero. Up to a subsequence, x^n converges weakly to a point x_0. By uniform convexity of ℓ_p (or if you prefer, equation (7)) it suffices to show $x_0 \in \mathbb{S}_{\ell_p}$.

When $p > q$, our thesis is a consequence of Pitt's Theorem. Indeed, since T is a compact operator, $T(x^n)$ converges strongly to $T(x_0)$ in ℓ_q.

Since $x^n \rightharpoonup x_0$ it follows that $T(x^n) \to T(x_0)$ and $\|x_0\| \leq 1$. But $\|T(x^n)\|_{\ell_q} \to \|T\|$; so we conclude that

$$\|T\| = \|T(x_0)\|_{\ell_q}$$

and hence $\|x_0\|_{\ell_p} = 1$.

When $p < q$, we argue as follows: we may assume $x_0 \neq 0$. As before, we have to verify that $x_0 \in \mathbb{S}_{\ell_p}$. Reasoning as in the proof of Theorem 1 (see (8)), we can write

$$\|T(x^n)\|_{\ell_q}^q \leq \|T(x_0)\|_{\ell_q}^q + \|T\|^q \left(1 - \|x_0\|_{\ell_p}^p\right)^{q/p} + \mathrm{o}(1). \tag{9}$$

Since $\|T(x^n)\|_{\ell_q} = \|T\| + \mathrm{o}(1)$ and (from the proof of Theorem **1**) $\|T(x_0)\|_{\ell_q} = \|T\| \cdot \|x_0\|_{\ell_p}$, equation (9) leads to

$$1 \leq \|x_0\|_{\ell_p}^q + \left(1 - \|x_0\|_{\ell_p}^p\right)^{q/p}. \tag{10}$$

Finally, since $q/p > 1$, equation (10) implies $1 - \|x_0\|^p = 0$; otherwise the strict inequality would hold

$$\begin{aligned} 1 &= \left[\|x_0\|_{\ell_p}^p + (1 - \|x_0\|_{\ell_p}^p)\right]^{q/p} \\ &> \|x_0\|_{\ell_p}^q + \left(1 - \|x_0\|_{\ell_p}^p\right)^{q/p}. \end{aligned}$$

which drives us to a contradiction. □

4 Two Applications of Theorem 1

Proof of Theorem 4. Let $T : \ell_p \to \ell_q$ be monotone with respect to non-increasing permutation and consider x^n a maximizing sequence for T. We may and will assume T is not the zero operator. Since $\|T(\sigma(x^n))\|_{\ell_q} \geq \|T(x^n)\|_{\ell_q}$, and $\|\sigma(x^n)\|_{\ell_p} = \|x^n\|_{\ell_p} = 1$, $y^n := \sigma(x^n)$ is too a maximizing sequence for T. In view of Theorem 1 it suffices to verify y^n is not weakly null. For that we argue as follows: suppose, for sake of contradiction, that y^n does converge weakly to zero. Since y^n is in non-increasing order, it would imply $\|y^n\|_{\ell_\infty} = \mathrm{o}(1)$ as $n \to \infty$, and therefore

$$\|y^n\|_{\ell_{p+\epsilon}} \leq \|y^n\|_{\ell_\infty}^{\frac{\epsilon}{p+\epsilon}} \cdot \|y^n\|_{\ell_p}^{\frac{p}{p+\epsilon}} = \mathrm{o}(1) \ \text{ as } \ n \to \infty. \tag{11}$$

By continuity, (11) would lead us to

$$\|T\| = \lim_{n\to\infty} \|Ty^n\|_q = 0,$$

which is a contradiction to our earlier assumption, $T \not\equiv 0$. □

Proof of Proposition 5. Assume $T : \ell_p \to \ell_q$ does not attain its norm. From Theorem 1, for any maximizing sequence u^n, one has

$$u^n \rightharpoonup 0 \text{ in } \ell_p.$$

Therefore, because of Bessaga-Pelczyński selection principle, see, e.g., [12], there exists a infinite subset of the natural numbers, $\mathbb{N}_1 \subseteq \mathbb{N}$, such that $\{u^n\}_{n\in\mathbb{N}_1}$ is a basic sequence equivalent to a block basic sequence of the canonical basis of ℓ_p. But now it is simple to show that actually

$$\left\{u^n\right\}_{n\in\mathbb{N}_1} \text{ is equivalent to the canonical basis of } \ell_p.$$

Furthermore, $\overline{\text{span}}\,\{u^n\}_{n\in\mathbb{N}_1}$ is isometric to ℓ_p. Indeed, because $\|u^n\|_p = 1$ and $\{u^n\}_{n\in\mathbb{N}_1}$ is equivalent to a block basic sequence of $\{e_i\}$, we can find scalars γ_i such that

$$u^i = \sum_{k=r_i+1}^{r_{i+1}} \gamma_k e_k, \ \forall i \in \mathbb{N}_1,$$

with

$$\sum_{k=r_i+1}^{r_{i+1}} |\gamma_k|^p = 1, \ \forall i \in \mathbb{N}_1.$$

Now,

$$\left\| \sum_{i=1}^{M} a_i u^i \right\|_p = \left(\sum_{i=1}^{M} \sum_{k=r_i+1}^{r_{i+1}} |a_i|^p |\gamma_k|^p \right)^{1/p}$$
$$= \left(\sum_{i=1}^{M} |a_i|^p \sum_{k=r_i+1}^{r_{i+1}} |\gamma_k|^p \right)^{1/p}$$
$$= \left\| \sum_{i=1}^{M} a_i e_i \right\|_p .$$

Now, as long as $T \not\equiv 0$, the sequence $\{T(u^n)\}_{n\in\mathbb{N}_1}$ is weakly null but

$$\|T(u^n)\|_q \to \|T\| \neq 0.$$

Thus, applying the same argument as before to the sequence $\{T(u^n)\}_{n\in\mathbb{N}_1}$, we find a $\mathbb{N}_2 \subseteq \mathbb{N}_1$, such that the sequence $\{T(u^n)\}_{n\in\mathbb{N}_2}$ is equivalent to the canonical basis of ℓ_q and the proof of the proposition is complete. □

5 Lineability of the set of norm attaining operators at a fixed point

Proof of Proposition 6. Our first observation is that it suffices to prove Proposition 6 for $F = \ell_q$. Using Hahn-Banach Theorem it is not difficult to show that $\mathcal{NA}^{x_0}(E; \ell_q) \neq \{0\}$.

Hereafter we fix a nonzero element $T \in \mathcal{NA}^{x_0}(E; \ell_q)$. We can write the set of positive integers $\mathbb{N}$ as

$$\mathbb{N} = \bigcup_{k=1}^{\infty} A_k,$$

where each

$$A_k := \left\{ a_1^{(k)} < a_2^{(k)} < \dots \right\} \tag{12}$$

has the same cardinality as $\mathbb{N}$ and the sets A_k are pairwise disjoint. For each positive integer k, we define

$$\ell_q^{(k)} := \left\{ x \in \ell_q : x_j = 0 \text{ if } j \notin A_k \right\}.$$

In the sequel, for each k fixed, let $T_k : E \to \ell_q^{(k)}$ be given by

$$\left(T_k(x)\right)_{a_j^{(k)}} = \left(T(x)\right)_j, \quad \forall j, k \in \mathbb{N}.$$

Finally, for k fixed, let $V_k : E \to \ell_q$ be given by

$$V_k = I_k \circ T_k,$$

where $I_k : \ell_q^{(k)} \to \ell_q$ is the canonical inclusion. Note that

$$\| V_k(x)\| = \| T_k(x)\| = \| T(x)\|$$

for every positive integer k and $x \in E$. Thus, each V_k attains its norm at x_0. From the fact that the operators V_k have disjoint supports it is easy to verify that

$$\{V_1, V_2, \dots\}$$

is a linearly independent set. It just remains to verify that any operator in

$$\text{span}\{V_1, V_2, \dots\}$$

attains its norm at x_0. For notation convenience, we will show that $aV_1 + bV_2$ attains its norm at x_0 for any choice of scalars a, b. We compute

$$\begin{aligned} \|aV_1 + bV_2\|^q &= \sup_{\|x\|\le 1} \|aV_1(x) + bV_2(x)\|^q \\ &\overset{(*)}{=} \sup_{\|x\|\le 1} \left(\sum_k |a(V_1(x))_k|^q + \sum_k |b(V_2(x))_k|^q \right) \\ &= \sup_{\|x\|\le 1} \left(|a|^q \sum_k |(V_1(x))_k|^q + |b|^q \sum_k |(V_2(x))_k|^q \right) \\ &= |a|^q \sum_k |(V_1(x_0))_k|^q + |b|^q \sum_k |(V_2(x_0))_k|^q \\ &= \|aV_1(x_0) + bV_2(x_0)\|^q . \end{aligned}$$

Thus, indeed $aV_1 + bV_2$ attains its norm at x_0. Equality $(*)$ holds since V_1 and V_2 have disjoint supports. □

6 Lineability of sets of non-norm-attaining operators

Proof of Proposition 7. We just need to deal with the case $F = \ell_q$. Let $T : E \to \ell_q$ be a non-norm-attaining operator. Again, we write $\mathbb{N}$ as

$$\mathbb{N} = \bigcup_{k=1}^{\infty} A_k,$$

with the A_k as in (12). Again, for each positive integer k, let

$$\ell_q^{(k)} := \left\{x \in \ell_q; x_j = 0 \text{ if } j \notin A_k\right\}.$$

For each k, we consider $T_k : E \to \ell_q^{(k)}$ defined as

$$\left(T_k(x)\right)_{a_j^{(k)}} = \left(T(x)\right)_j, \quad \forall j, k \in \mathbb{N}.$$

For every k, let $V_k : E \to \ell_q$ given by

$$V_k = I_k \circ T_k,$$

where $I_k : \ell_q^{(k)} \to \ell_q$ is the canonical inclusion. So, as in the previous proof, each V_k does not attain its norm and

$$\left\{V_1, V_2, \dots\right\}$$

is a linearly independent set. It remains to be shown that any operator in

$$\operatorname{span}\left\{V_1, V_2, \dots\right\}$$

does not attain their norms. Again, for notation convenience, let us restrict our computation to $aV_1 + bV_2$, for any choice of scalars a, b (of course, at least one of them is chosen to be different from zero). To show that $aV_1 + bV_2$ does not attain its norm we argue as follows:

$$\begin{aligned} \|aV_1 + bV_2\|^q &= \sup_{\|x\|\le 1} \|aV_1(x) + bV_2(x)\|^q \\ &= \sup_{\|x\|\le 1} \left(\sum_k |a(V_1(x))_k|^q + \sum_k |b(V_2(x))_k|^q\right) \\ &\le |a|^q \, \|V_1\|^q + |b|^q \, \|V_2\|^q . \end{aligned}$$

On the other hand, for every natural number n and $\varepsilon = \frac{1}{n}$ we can find $x_n \in \mathbb{S}_E$ so that

$$\left\|V_j(x_n)\right\| \ge \left\|V_j\right\| - \varepsilon, \ j = 1, 2$$

and hence

$$\begin{aligned} \|(aV_1 + bV_2)(x_n)\|^q &= |a|^q \, \|V_1(x_n)\|^q + |b|^q \, \|V_2(x_n)\|^q \\ &\ge |a|^q \, (\|V_1\| - \varepsilon)^q + |b|^q \, (\|V_2\| - \varepsilon)^q . \end{aligned}$$

So we conclude that

$$\|aV_1 + bV_2\|^q = |a|^q \, \|V_1\|^q + |b|^q \, \|V_2\|^q \, .$$

Besides, if $\|x\|_E = 1$, since V_1 and V_2 do not attain their norms, we get

$$\begin{aligned} \|(aV_1 + bV_2)(x)\|^q &= |a|^q \, \|V_1(x)\|^q + |b|^q \, \|V_2(x)\|^q \\ &< |a|^q \, \|V_1\|^q + |b|^q \, \|V_2\|^q \\ &= \|aV_1 + bV_2\|^q \, . \end{aligned}$$

We conclude that $aV_1 + bV_2$ belongs to $\mathcal{L}(E; \ell_q) \smallsetminus \mathcal{NA}(E; \ell_q)$. The general case is similar. The proof of Proposition 7 is completed. □

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