BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 40(3), 433-454

About a family of deformations of the Costa-Hoffman-Meeks surfaces

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Abstract. We show the existence of a family of minimal surfaces obtained by deformations of the Costa-Hoffman-Meeks surface of genus $k \geqslant 1$, M_k. These surfaces are obtained varying the logarithmic growths of the ends and the directions of the axes of revolution of the catenoidal type ends of M_k. Also we obtain a result about the non degeneracy property of the surface M_k.

Keywords: deformation, Jacobi operator, Costa-Hoffmann-Meeks surfaces.

Mathematical subject classification: 53A10.

Introduction

C. Costa in [1, 2] described a genus one minimal surface with two ends asymptotic to the two ends of a catenoid and a middle end asymptotic to a plane. D. Hoffman and W.H. Meeks in [5], [6] and [7] proved the global embeddedness for the Costa surface, and generalized it for higher genus. We will denote the Costa-Hoffman-Meeks surface of genus $k \geqslant 1$ by M_k. For each $k \geqslant 1$ is a properly embedded minimal surface and has three ends of finite total curvature.

J. Pérez and A. Ros in [11] studied the space $\mathcal{M}$ of minimal surfaces of finite total curvature, genus k and r ends, properly immersed in $\mathbb{R}^3$ and with embedded horizontal ends. Given $M \in \mathcal{M}$, the infinitesimal deformations of M are generated by the elements of $J(M)$, the space of the Jacobi functions u on M, that is functions such that $Lu = 0$, where L denotes the Jacobi operator of M, which have logarithmic growth at the ends. They showed that $\dim\ J(M) \geqslant r + 3$. They denoted by $\mathcal{M}^* = \{M \in \mathcal{M} :\ \dim\ J(M) = r + 3\}$ the subspace of non degenerate surfaces and proved

Received 6 March 2008.

Theorem 1 (th. 6.7, [11]). *$\mathcal{M}^*$ is an open subset of $\mathcal{M}$ and is a $(r+3)$-dimensional real analytic manifold.*

The dimension of the space $J(M)$ just introduced is known for $M = M_k$ for $k \geqslant 1$. Indeed thanks to the works [9] and [10] by S. Nayatani, dim $J(M_k) = 6$, since $r = 3$, but only for $1 \leqslant k \leqslant 37$. Recently this result has been proved also for $k \geqslant 38$ (see [8]). The elements of $J(M_k)$ are the Jacobi fields associated with the horizontal translations, the rotation about the vertical direction and three functions (one for each end) whose form in a neighbourhood of an end is $a \ln |w|$, being a the logarithmic growth. Thus, the one parameter family of deformations of these surfaces, described by D. Hoffman and H. Karcher in [4], contains all the embedded surfaces nearby M_k with a symmetry group generated by k vertical planes, up to dilations preserving the vertical direction.

In this work, following [11], we show the existence of a bigger family of immersed minimal deformations of M_k for $k \geqslant 1$ having three embedded ends. These surfaces do not enjoy any property of symmetry. In fact we admit the possibility to rotate, translate and dilate any of the three ends of the surface and, in addition, to bend the two catenoidal type ends and to transform the middle end from a planar type end into a catenoidal type end (we recall that the planar end can be thought as a catenoidal type end with null vertical flux). We will prove the following result.

Theorem 2. *For each possible choice of the limit values of the normal vectors of the three ends, there is, up to isometries, a* 1*-dimensional real analytic family of smooth minimal deformations of M_k, for $k \geqslant 1$, letting the middle planar end horizontal.*

Our result is a consequence of the moduli space theory and of the implicit function theorem. We do not treat the case where also the middle planar end is not horizontal because it can be reconduced to the previous one by an isometry.

The family of surfaces described in the statement of the theorem here, contains the 1-parameter family of deformations of M_k, for $1 \leqslant k \leqslant 37$, obtained by L. Hauswirth and F. Pacard in [3] bending the top and the bottom end and letting horizontal the middle planar end. All the surfaces of this family are not embedded and are symmetric with respect to the vertical plane $\{x_2 = 0\}$ that in particular contains the axis of the catenoidal type ends (it is assumed to be the same for the two ends). The parameter is the angle between this axis and the vertical direction. This family is used in the same work to construct some new examples of minimal surfaces by a gluing technique.

Degeneracy and non degeneracy of the Costa-Hoffman-Meeks surfaces

Let K_0 denote the $C^{2,\alpha}$ elements of the kernel of the Jacobi operator about the compactification of $M \in \mathcal{M}$, *Killing* the space of the Jacobi fields induced by the isometries of the ambient space and r the number of the ends. We set $Killing_0 = Killing \cap K_0$. In [11] the authors give the following definition of non degeneracy.

Definition 3 ([11]). *A minimal surface is non degenerate if* $Killing_0 = K_0$.

J. Pérez and A. Ros show in Proposition 5.3 of [11] that the non degeneracy of M is equivalent to the equality $\dim J(M) = r + 3$ and obtain Theorem 1. So if a minimal surface M is non degenerate then the set of the minimal immersions near M with horizontal ends has a nice behaviour.

Remark 4. The works [9] and [10] by S. Nayatani and [8] by the author about the number of bounded Jacobi functions ensure that M_k is non degenerate for $1 \leqslant k < +\infty$.

In section 2.2 we will prove that M_k is non degenerate with respect to the Definition 3 but in a more general setting. We remind that the minimal surfaces considered in [11] (where Definition 3 is introduced) have horizontal embedded ends. On the contrary here we suppose the surfaces can have non horizontal embedded ends.

Now we are going to explain why M_k is degenerate with respect to the different definition used by L. Hauswirth and F. Pacard in [3]. They studied the mapping properties of the Jacobi operator of M_k acting on the space of the $C^{2,\alpha}_\delta$ functions defined on M_k and that are invariant under the action of the symmetry with respect to the vertical plane $\{x_2 = 0\}$. In particular if $f \in C^{2,\alpha}_\delta(M_k)$, then $f = O(e^{\delta s})$ on the catenoidal type ends. The mapping properties of the Jacobi operator (denoted by L_δ) acting on functions of $C^{2,\alpha}_\delta(M_k)$ depend on the choice of δ. Their definition of non degeneracy of M_k is the following one.

Definition 5 ([3]). *The surface M_k is non degenerate if the operator L_δ is injective for all $\delta < -1$.*

Thanks to the works [9] and [10] by S. Nayatani and [8] by the author, the space $K \subset J(M_k)$ of the bounded Jacobi functions, is known to be generated by the functions $\langle N, e_1\rangle$, $\langle N, e_2\rangle$ and $\langle N, e_3\rangle$, $\langle N, e_3 \times p\rangle$, where N denotes the normal vector field about M_k, (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ and p the position vector on M_k. These functions are associated with 4 isometries of the ambient space: the three translations and the rotation about the e_3-axis. In [3] the authors remark that the Jacobi function $\langle N, e_3 \times p\rangle$ associated with the rotation about the e_3-axis and $\langle N, e_2\rangle$ associated with the translation along the e_2-axis do not respect the mirror symmetry described above, that is they are not invariant with respect to the action of the map $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$. So they did not taken into account them and could conclude that M_k is non degenerate, in the sense of their definition.

The surfaces of the family described in our work do not enjoy any property of symmetry, since we admit to bend the catenoidal type ends in arbitrary directions. Then the Jacobi functions described above must be taken into account. Since the Jacobi function $\langle N, e_3 \times p\rangle$ belongs to the space $C^{2,\alpha}_\delta(M_k)$ for $\delta = -k-1 \leqslant -2$, the property of non degeneracy does not hold any more. Actually the operator L_δ acting on $C^{2,\alpha}_\delta(M_k)$ is no more injective for all $\delta < -1$. As consequence, we can state that for all $k \geqslant 1$ the Costa-Hoffman-Meeks surface M_k is degenerate in the sense of Definition 5.

1 Preliminaries and notation

We denote by $X\colon M_k \to \mathbb{R}^3$ the conformal minimal immersion of the Costa-Hoffman-Meeks surface M_k in $\mathbb{R}^3$. If g and η are the Weierstrass data of M_k, we can write:

$$X(z) = \left(\frac{1}{2}\overline{\int g^{-1}\eta} - \frac{1}{2}\int g\eta, \operatorname{Re}\int \eta\right) \in \mathbb{C}\times\mathbb{R} = \mathbb{R}^3. \tag{1}$$

The meromorphic function g is the stereographic projection from the north pole of the Gauss map $N\colon M_k \to \mathbb{S}^2$. The total curvature is finite and M_k is conformally diffeomorphic to $\overline{M_k} \setminus \{p_t, p_b, p_m\}$, being $\overline{M_k}$ a compact surface and p_i three points. The Weierstrass data extend in a meromorphic way at each puncture p_i. In particular the Gauss map of $X(z)$ is well defined at p_i. The points p_i are identified with the ends and a neighbourhood of a puncture will parametrize the corresponding end. In the sequel we will refer to various quantities related to the three ends of the surface using the index t for the top end, the index b for the bottom end and the index m for the planar end. The Gauss map N takes the limit values $(0, 0, 1)$ at the ends p_t and p_b and $(0, 0, -1)$ at the end p_m.

We parametrize the ends p_i in the graph coordinate $x = x_1 + ix_2$ on $D_i^*(\varepsilon_i) = \{x \in \mathbb{C};\ 0 < |x| \leqslant \epsilon_i\}$ by the immersions

$$X_i(x) = \left(\frac{1}{x}, -\tilde{a}_i \ln|x| + h_i(x)\right) \in \mathbb{C}\times\mathbb{R} = \mathbb{R}^3$$

for $i = t, b$, where h_i is a smooth real valued function on $D_i^*(\varepsilon_i)$. The quantities $\tilde{a}_i$ and $h_i(0)$ are called the logarithmic growth and the height of the end. We can observe that, for the null flux condition, $\tilde{a}_t = -\tilde{a}_b$.

As for the planar end p_m, we will use the following parametrization

$$X_m(x) = \left(\frac{1}{x}, h_m(x)\right) \quad \text{on} \quad D_m^*(\epsilon_m).$$

So its logarithmic growth is zero.

1.1 The mean curvature operator at an end

Let us consider a not necessarily minimal immersion $E\colon D^*(\varepsilon) \to \mathbb{R}^3$ defined by

$$E(x) = \left(\frac{1}{x}, -a \ln|x| + h(x)\right).$$

We denote by ds_0^2 the flat metric of the x-plane. We set $\rho = |x|$ and $f = -a \ln \rho + h$. The induced metric $ds^2 = (g_{ij})$ is given by ([11])

$$g_{ij}(x) = \frac{1}{\rho^4}\left(\delta_{ij} + \rho^4 \partial_i f \partial_j f\right). \tag{2}$$

If we denote by dA (respectively dA_0) the area measure associated with the metric ds^2 (respectively ds_0^2), then (2) implies that

$$dA = \frac{Q^{\frac{1}{2}}}{\rho^4} dA_0$$

where $Q = 1 + |x|^2(a^2 + |x|^2|\nabla_0 h|^2 - 2a\langle x, \nabla_0 h\rangle)$ (∇_0 denotes the gradient computed with respect to the flat metric ds_0^2).

The Gauss map of E is given by ([11])

$$N(x) = Q^{-\frac{1}{2}}\left(-a\bar{x} + \bar{x}^2\nabla_0 h, 1\right), \tag{3}$$

where $\bar{x}^2\nabla_0 h$ means the product of the complex number $\bar{x}^2$ with the gradient $\nabla_0 h$.

The mean curvature of the immersion E is

$$H = \frac{\rho^4}{2} div_0 \left(\frac{\nabla_0 f}{\sqrt{1 + \rho^4|\nabla_0 f|^2}}\right).$$

2 The deformation of the surface and its Jacobi operator

In this section we describe how we deform the surface M_k following the ideas of [11]. In subsection 2.1 we introduce the Jacobi operator of M_k and we study its kernel and its range.

We will construct a family of deformations of M_k using in particular Theorem 4.1 of [11]. The first step is the construction of a new immersion of M_k in $\mathbb{R}^3$ starting from $X(z)$ given by (1). Using a smooth cut-off function we glue $X\colon M_k \backslash (D_t^*(\varepsilon_t) \cup D_b^*(\varepsilon_b) \cup D_m^*(\varepsilon_m)) \to \mathbb{R}^3$ with the parametrizations of the three ends with a different value of the logarithmic growths (that we denote by a_t, a_b, a_m). Furthermore we rotate the ends p_t and p_b, that is we change the directions of their axes of revolution. Secondly we consider small variations of the immersion just constructed with respect an appropriate smooth vector field. Thanks to Theorem 4.1 of [11], each immersed minimal surface having properly embedded ends with finite total curvature and fixed topology, that is in a neighbourhood of M_k, admits an immersion constructed in this way. We remark this theorem has been proved for minimal surfaces with horizontal ends but it holds also in our setting.

To give the details of the construction we need to introduce some notation. We denote by $F(\theta_{1,i}, \theta_{2,i})$ the frame defined by the following unit vectors:

$$\begin{aligned} e_1(\theta_{1,i}, \theta_{2,i}) &= \cos\theta_{1,i} e_1 + \sin\theta_{1,i} \sin\theta_{2,i} e_2 + \sin\theta_{1,i} \cos\theta_{2,i} e_3, \\ e_2(\theta_{1,i}, \theta_{2,i}) &= \cos\theta_{2,i} e_2 - \sin\theta_{2,i} e_3, \\ e_3(\theta_{1,i}, \theta_{2,i}) &= -\sin\theta_{1,i} e_1 + \cos\theta_{1,i} \sin\theta_{2,i} e_2 + \cos\theta_{1,i} \cos\theta_{2,i} e_3, \end{aligned} \tag{4}$$

where (e_1, e_2, e_3) denotes the canonical base of $\mathbb{R}^3$.

We define the immersions of the rotated catenoidal type ends on $D_i^*(\varepsilon_i)$ as

$$\begin{aligned} X_{i,\theta_{1,i},\theta_{2,i}}(x) \;=\; & \frac{x_1}{|x|^2} e_1(\theta_{1,i}, \theta_{2,i}) - \frac{x_2}{|x|^2} e_2(\theta_{1,i}, \theta_{2,i}) \\ & + (-a_i \ln|x| + h_i(x)) e_3(\theta_{1,i}, \theta_{2,i}), \end{aligned}$$

for $i = t, b$. As for the planar end, we consider on $D_m^*(\varepsilon_m)$ in the canonical frame (e_1, e_2, e_3) the immersion

$$X_{m,0,0}(x) = \left(\frac{1}{x}, -a_m \ln|x| + h_m(x)\right).$$

We define $y = (a_t, a_b, a_m, \theta_{1,t}, \theta_{2,t}, \theta_{1,b}, \theta_{2,b})$.

Using a smooth cut-off function we can glue the three immersions above, defined on $D_i^*(\varepsilon_i)$, $i = t, b, m$, to the restriction of $X(z)$ to $M_k \backslash \left(\cup_{i=t,b,m} D_i^*(\varepsilon_i)\right)$

and obtain a new immersion we denote by X_y. It is not necessarily minimal and depends smoothly on y.

Now let $\widetilde{N}(y) \in C^\infty(\overline{M_k}, \mathbb{R}^3)$ be a smooth vector field such that $\langle \widetilde{N}(y), N \rangle = 1$ on $\overline{M_k} \backslash (D_t^*(\varepsilon_t) \cup D_b^*(\varepsilon_b))$ and

$$\widetilde{N}(y) = \frac{e_3(\theta_{1,i}, \theta_{2,i})}{\langle N, e_3(\theta_{1,i}, \theta_{2,i}) \rangle}$$

on $D_i^*(\varepsilon_i)$ for $i = t, b$. We remark that we do not modify the normal vector field on $D_m^*(\varepsilon_m)$ because we keep the middle planar end horizontal. Let $\mathcal{A}$ be a neighbourhood of $(\tilde{a}_t, \tilde{a}_b, 0)$ (the logarithmic growths of the ends of M_k), $\mathcal{U}$ a neighbourhood of zero in $C^{2,\alpha}(\overline{M_k})$. For $y \in \mathcal{A} \times [-\varepsilon, \varepsilon]^4$ and a function $u \in \mathcal{U}$, we consider the family of immersions $\{X_{y,u}\}$ such that

$$X_{y,u} := X_y + u\widetilde{N}(y) \colon M_k \longrightarrow \mathbb{R}^3. \tag{5}$$

Such a family depends analytically on (y, u). As we have anticipated, Theorem 4.1 of [11] ensures that each immersed minimal surface having properly embedded ends with finite total curvature that is in a neighbourhood of M_k, admits an immersion in $\mathbb{R}^3$ which is in this family. In other terms each of them is the graph of a function u with respect the vector field $\widetilde{N}(y)$ about the surface whose immersion is X_y. Each immersion is determined by an element $(y, u) \in \mathcal{A} \times [-\varepsilon, \varepsilon]^4 \times \mathcal{U}$. In particular if $y = (\tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0)$ the immersion is the one of M_k (that is actually an embedding).

Now we are going to introduce the mean curvature operator of the immersion $X_{y,u}$ and its "compactification".

Let $\lambda \in C^\infty(M_k)$ be a positive function which in terms of the graph coordinate x, is defined by

$$\lambda(x) = \begin{cases} \dfrac{1}{|x|^4} & \text{on } D_t^*(\varepsilon_t),\ D_b^*(\varepsilon_b),\ D_m^*(\varepsilon_m), \\ 1 & \text{on } M_k \backslash (D_t^*(2\varepsilon_t) \cup D_b^*(2\varepsilon_b) \cup D_m^*(2\varepsilon_m)). \end{cases} \tag{6}$$

If ds^2 denotes the induced metric on M_k, then (2), which gives the expression of the metric for a catenoidal type end, implies that $d\bar{s}^2 = (1/\lambda)\, ds^2$ is a Riemannian metric on $\overline{M_k}$. We denote the associated area mesure by $d\bar{A}$. If $H(y, u)$ is the mean curvature operator of the immersion $X_{y,u} = X_y + u\widetilde{N}(y)$, we define the operator $\overline{H}(y, u) = \lambda H(y, u)$. Since at the ends p_t and p_b, the rotation does not change the value of the mean curvature, we can apply Lemma 6.4, proved in [11], at each end to conclude that there exist ε and neighbourhoods $\mathcal{A}$, $\mathcal{U}$ such that the operator

$$\overline{H}\colon \mathcal{A} \times [-\varepsilon, \varepsilon]^4 \times \mathcal{U} \to C^{0,\alpha}(\overline{M_k})$$

is real analytic.

2.1 The Jacobi operator

The Jacobi operator L of M_k is given by

$$L = \Delta_{ds^2} + |A|^2_{ds^2},$$

where Δ_{ds^2} and $|A|^2_{ds^2}$ are respectively the Laplacian and the norm of the second fundamental form with respect to the metric ds^2 (the metric on M_k).

The geometric meaning of L can be explained in the following way (see Section 5 of [11]). Let $\{M_k(t)\}_{|t|<\varepsilon}$ denote a family of smooth deformations of M_k, such that $M_k(0) = M_k$ and let $H(t)$ denote the mean curvature operator of $M_k(t)$. If $\psi_t\colon M_k \to \mathbb{R}^3$ is the immersion in $\mathbb{R}^3$ of $M_k(t)$, and $w = \langle \frac{d}{dt}_{|t=0}\psi_t, N\rangle$, then we have the equality

$$\frac{d}{dt}_{|t=0} H(t) = \frac{1}{2} L w. \tag{7}$$

If $Lu = 0$ is satisfied, u is called Jacobi function on M_k and it corresponds to an infinitesimal deformation of M_k by minimal surfaces. The operator L can be "compactified" to obtain the operator $\overline{L} = \Delta_{d\bar{s}^2} + |A|^2_{d\bar{s}^2} = \lambda L$ on $\overline{M_k}$ (the function λ is defined by (6)). It is related to the differential of $\overline{H}(t) = \lambda H(t)$ by a relation similar to (7).

In the sequel we will consider the family of deformations of M_k constructed in the following way. We define

$$\tilde{y} = \big(\tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0\big) \quad \text{and} \quad \dot{y} = \big(\dot{a}_t, \dot{a}_b, \dot{a}_m, \dot{\theta}_{1,t}, \dot{\theta}_{2,t}, \dot{\theta}_{1,b}, \dot{\theta}_{2,b}\big)$$

and consider a smooth curve

$$\gamma(t) = \big(a_t(t), a_b(t), a_m(t), \theta_{1,t}(t), \theta_{2,t}(t), \theta_{1,b}(t), \theta_{2,b}(t)\big), \tag{8}$$

for $|t| < \varepsilon$, such that $\gamma(0) = \tilde{y}$, with acceleration $\gamma'(0) = \dot{y}$ and a function

$$u(t)\colon \{t \in \mathbb{R}, |t| < \varepsilon\} \to C^{2,\alpha}(\overline{M_k})$$

such that $u(0) = 0$.

We define the family of deformations $\{M_k(t)\}_{|t|<\varepsilon}$ of M_k as the family of immersed surfaces whose immersion in $\mathbb{R}^3$ equals $X_{\gamma(t),u(t)} = X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t))$ (see (5)). We remark that in general $M_k(t)$ is not a minimal surface.

We are going to give the expression of the Jacobi functions on M_k defined by $\langle \frac{d}{dt}_{|t=0} X_{\gamma(t),u(t)}, N\rangle$. To do that we need to introduce additional notation. Let f_1, f_2, f_3 be the functions defined by:

$$f_1(x,i) = \frac{x_1}{|x|^2}\langle N, e_3\rangle - (-\tilde{a}_i \ln|x| + h_i(x))\langle N, e_1\rangle, \tag{9}$$

$$f_2(x,i) = \frac{x_2}{|x|^2}\langle N, e_3\rangle + (-\tilde{a}_i \ln|x| + h_i(x))\langle N, e_2\rangle, \tag{10}$$

$$f_3(x,\dot{a}_i) = -\dot{a}_i \ln|x|\langle N, e_3\rangle \tag{11}$$

for $x \in D_i^*(\varepsilon_i)$ with $i = t, b, m$, and $f_n = 0$, $n = 1, 2, 3$, in $M_k \backslash D_i^*(2\varepsilon_i)$. f_n are a smooth interpolation of previous values on the remaining part of M_k. We recall that $\tilde{a}_t$, $\tilde{a}_b$ are the logarithmic growths of the top and of the bottom end of M_k, and since the middle planar end is horizontal, we have $\tilde{a}_m = 0$.

Proposition 6. *The Jacobi functions about M_k have, in $D_i^*(\varepsilon_i)$, the following expression*

$$\dot{\theta}_{1,i} f_1(x,i) + \dot{\theta}_{2,i} f_2(x,i) + f_3(x,\dot{a}_i) + u_i$$

for $i = t, b, m$, *with* $\dot{\theta}_{1,m} = \dot{\theta}_{2,m} = 0$, $\tilde{a}_m = 0$ *and* $u_i \in C^{2,\alpha}(D_i(\varepsilon_i))$.

Proof. A Jacobi function is defined by

$$\langle \frac{d}{dt}_{|t=0} \left(X_{\gamma(t)} + u(t)\widetilde{N}(\gamma(t)) \right), N\rangle. \tag{12}$$

We observe that $X_{\gamma(t)}$ in $D_i^*(\varepsilon_i)$ is given by

$$\begin{aligned} &\frac{x_1}{|x|^2} e_1(\theta_{1,i}(t), \theta_{2,i}(t)) - \frac{x_2}{|x|^2} e_2(\theta_{1,i}(t), \theta_{2,i}(t)) \\ &\quad + (-a_i(t)\ln|x| + h_i(x)) e_3(\theta_{1,i}(t), \theta_{2,i}(t)), \end{aligned} \tag{13}$$

for $i = t, b$ and in $D_m^*(\varepsilon_m)$ by

$$\left(\frac{1}{x}, -a_m(t)\ln|x| + h_m(x) \right).$$

To simplify the notation we will omit the dependence on the end wherever it is possible, replacing $\theta_{k,i}(t)$ by $\theta_k(t)$ and $a_i(t)$ by $a(t)$.

To obtain $\frac{d}{dt}_{|t=0} X_{\gamma(t)}$ we need to compute

$$\dot{e}_j(0) = \frac{d}{dt}_{|t=0} e_i(\theta_1(t), \theta_2(t)), \quad j = 1, 2, 3.$$

So we suppose that $\theta_1(0) = \theta_2(0) = 0$. We observe that from equation (4) since

$$e_1(\theta_1(t), \theta_2(t)) = \cos\theta_1(t)e_1 + \sin\theta_1(t)\sin\theta_2(t)e_2 + \sin\theta_1(t)\cos\theta_2(t)e_3,$$

we have

$$\begin{aligned}\dot{e}_1(t) = &-\theta_1'(t)\sin\theta_1(t)e_1\\ &+\big(\theta_1'(t)\cos\theta_1(t)\sin\theta_2(t) + \theta_2'(t)\sin\theta_1(t)\cos\theta_2(t)\big)e_2\\ &+\big(\theta_1'(t)\cos\theta_1(t)\cos\theta_2(t) - \theta_2'(t)\sin\theta_1(t)\sin\theta_2(t)\big)e_3,\end{aligned}$$

then using the initial conditions, we obtain

$$\dot{e}_1(0) = \theta_1'(0)e_3. \tag{14}$$

In a similar way we obtain $\dot{e}_j(0)$, with $j = 2, 3$. We find

$$\dot{e}_2(0) = -\theta_2'(0)e_3, \tag{15}$$

$$\dot{e}_3(0) = -\theta_1'(0)e_1 + \theta_2'(0)e_2. \tag{16}$$

Then from equations (13), (14), (15) and (16)

$$\frac{d}{dt}_{|t=0} X_{\gamma(t)} = \frac{x_1}{|x|^2}\theta_1'(0)e_3 - \frac{x_2}{|x|^2}(-\theta_2'(0)e_3)$$

$$+(-a(0)\ln|x| + h(x))(-\theta_1'(0)e_1 + \theta_2'(0)e_2) + (-a'(0)\ln|x|)e_3.$$

Collecting the summands in terms of the $\theta_k'(0) \equiv \theta_{k,i}'(0)$ and taking into account the definitions (9), (10) and (11) of the f_k functions, we get

$$\langle \frac{d}{dt}_{|t=0} X_{\gamma(t)}, N\rangle = \theta_{1,i}'(0) f_1(x, i) + \theta_{2,i}'(0) f_2(x, i) + f_3(x, a_i'(0)).$$

As for the last term of (12), we recall that $u(0) = 0$ and, on $D_i^*(\varepsilon_i)$, from the definition of $\widetilde{N}(y)$ it holds that

$$\widetilde{N}(\gamma(t)) = \frac{e_3(\theta_{1,i}(t), \theta_{2,i}(t))}{\langle N, e_3(\theta_{1,i}(t), \theta_{2,i}(t))\rangle}.$$

Then $\frac{d}{dt}(u(t)\widetilde{N}(\gamma(t)))$, evaluated in $t = 0$, is equal to

$$u'(0)\widetilde{N}(\gamma(0)) + u(0)\frac{d}{dt}_{|t=0}\widetilde{N}(\gamma(t)) = \frac{u'(0)}{\langle N, e_3\rangle} e_3.$$

If u_i denotes the restriction of $u'(0)$ to $D_i^*(\varepsilon_i)$ for $i = t, b, m$, then the result is obvious. □

Lemma 7. *Let $U, V \in C^{2,\alpha}(M_k)$ be the functions defined in $D_i^*(\varepsilon_i)$, for $i \in \{t, b, m\}$, by*

$$U_i(x) = \dot\theta_{1,i} f_1(x, i) + \dot\theta_{2,i} f_2(x, i) + f_3(x, \dot a_i) + u_i(x)$$

and

$$V_i(x) = \dot\varphi_{1,i} f_1(x, i) + \dot\varphi_{2,i} f_2(x, i) + f_3(x, \dot b_i) + v_i(x),$$

with $\dot\theta_{j,i}, \dot\varphi_{j,i} \in \mathbb{R}$ and $\tilde a_m = 0, \dot\theta_{j,m} = \dot\varphi_{j,m} = 0$, $j = 1, 2$, and $u_i, v_i \in C^{2,\alpha}(D_i(\varepsilon_i))$. Then we have

$$\int_{\overline{M_k}} (U\bar L V - V\bar L U) d\bar A = \int_{M_k} (U L V - V L U) dA$$

$$= 2\pi \sum_{i\in\{t,b\}} \left[\langle \dot\Phi_i, \nabla u_i(0)\rangle - \langle \dot\Theta_i, \nabla v_i(0)\rangle\right] + 2\pi \sum_{i\in\{t,b,m\}} [\dot b_i u_i(0) - \dot a_i v_i(0)],$$

with $\dot\Theta_i = (\dot\theta_{1,i}, \dot\theta_{2,i})$, $\dot\Phi_i = (\dot\varphi_{1,i}, \dot\varphi_{2,i})$ *and* $\nabla\cdot = (\partial_{x_1}\cdot, \partial_{x_2}\cdot)$.

Proof. The proof makes use of the Green identity, so first of all we obtain the local expression of the Weiertrass representation of a properly embedded end p with finite total curvature of a minimal immersion, ψ, in $\mathbb{R}^3$, in terms of a conformal coordinate $z = re^{i\alpha}$ centered at p.

Let's suppose that the Weiestrass representation of ψ is

$$\psi = \left(\frac{1}{2}\left(\overline{\int g^{-1}\eta} - \int g\eta\right), \mathrm{Re}\int \eta\right). \tag{17}$$

It is known (see for example Section 2 of [11]) that properly embedded minimal ends with finite total curvature must be asymptotic to the end of a catenoid or of a plane. Indeed from the hypotheses on p it follows that the Weierstrass data of ψ in terms of the conformal coordinate w centered at the end are given by:

$$g(w) = \frac{q(w)}{w^k}, \quad \eta(w) = s(w)w^{k-2}dw,$$

for $w \in D^*(\varepsilon)$, $k \in \mathbb{N}^*$. The functions $q(w)$ and $s(w)$ are holomorphic and satisfy $q(0) \neq 0$, $s(0) = -a \in \mathbb{R}^*$. From (17) we obtain

$$\psi(w) = \left(\frac{1}{2}\left(\overline{\int \frac{s(w)}{q(w)} w^{2k-2}dw} - \int \frac{q(w)s(w)}{w^2}dw\right), \mathrm{Re}\int s(w)w^{k-2}dw\right),$$

If $k = 1$, ψ is asymptotic to the end of a vertical catenoid (under the additional hypothesis $(qs)'(0) = 0$, which ensures ψ is well defined). Indeed we can write

$$\psi(w) = \left(\frac{l(w)}{w}, -a\ln|w| + v(w)\right). \tag{18}$$

So a is the logarithmic growth of the end and v is a smooth function on $D(\varepsilon)$.

Now we consider the change of coordinate $z = -\frac{w}{q(0)}e^{-\frac{v(w)}{a}}$. In the new conformal coordinate we can write:

$$g(z) = -\frac{1}{z} + \tilde{t}(z), \quad \eta(z) = -a\frac{dz}{z}.$$

Replacing $g(z)$ and $\eta(z)$ in (17) we get the expression of ψ in terms of z:

$$\psi(z) = \left(\frac{a}{2}\left(\bar{z} + \frac{1}{z}\right) + t(z), -a\ln|z| + const\right). \tag{19}$$

We denoted by x the graph coordinate around the catenoidal end p_i of M_k and by $\tilde{a}_i$ its logarithmic growth. Then, for the catenoidal type ends of M_k, we get from (19) the following equality

$$\frac{1}{x} = \frac{\tilde{a}_i}{2z}(1 + |z|^2 + zt_i(z)) = \frac{s_i(z)}{z} \tag{20}$$

with $s_i(0) = \frac{\tilde{a}_i}{2}$, $i \in \{t, b\}$.

In the case of the planar end p_m, that is for $k \geqslant 2$, the third coordinate function in (18) is bounded and ψ is asymptotic to the end of a horizontal plane. Similar arguments lead us to

$$\frac{1}{x} = \frac{s_m(z)}{z}$$

where $s_m(0) \neq 0$.

The next step is to find the expressions of U and V near the ends in terms of (r, α) coordinates. From (3) we obtain that, for an end with logarithmic growth a, it holds that:

$$\langle N, e_3\rangle = Q^{-\frac{1}{2}} = \left(1 + |x|^2(a^2 + |x|^2|\nabla_0 h|^2 - 2a\langle x, \nabla_0 h\rangle)\right)^{-\frac{1}{2}},$$

$$\langle N, e_1\rangle = \langle N, e_3\rangle \operatorname{Re}(-a\bar{x} + \bar{x}^2\nabla_0 h),$$

$$\langle N, e_2\rangle = \langle N, e_3\rangle \operatorname{Im}(-a\bar{x} + \bar{x}^2\nabla_0 h).$$

Then in a neighbourhood of each end we can write:

$$\langle N, e_3\rangle = \left(1 + O(|x|^2)\right)^{-\frac{1}{2}} = 1 + O(|x|^2), \tag{21}$$

$$\langle N, e_1\rangle = \left(1 + O(|x|^2)\right)\left(-ax_1 + O(\bar{x}^2)\right) = -ax_1 + O(|x|^2), \tag{22}$$

$$\langle N, e_2\rangle = \left(1 + O(|x|^2)\right)\left(ax_2 + O(\bar{x}^2)\right) = ax_2 + O(|x|^2). \tag{23}$$

In (r, α) coordinates, U_i and V_i have the following expressions:

$$U_i(r) = \dot{\theta}_{1,i} f_1(r, i) + \dot{\theta}_{2,i} f_2(r, i) + f_3(r, \dot{a}_i) + u_i(r),$$

$$V_i(r) = \dot{\varphi}_{1,i} f_1(r, i) + \dot{\varphi}_{2,i} f_2(r, i) + f_3(r, \dot{b}_i) + v_i(r)$$

where

$$f_1(r, i) = \frac{\tilde{a}_i \cos\alpha}{2r} + O(r \ln r),$$

$$f_2(r, i) = \frac{\tilde{a}_i \sin\alpha}{2r} + O(r \ln r),$$

$$f_3(r, a) = -a \ln r + O(r).$$

If $D_i(0, r)$ are conformal disks and $M(r) = M\backslash(\cup_{i\in\{t,b,m\}} D_i(0, r))$, then the conformal invariance of the integral implies:

$$\begin{aligned} I(r) &= \int_{M(r)} (ULV - VLU)\, dA = \int_{\partial M(r)} \left(U\frac{\partial V}{\partial \eta} - V\frac{\partial U}{\partial \eta}\right) ds \\ &= -\sum_{i\in\{t,b,m\}} \int_{\partial D_i(0,r)} \left(U_i\frac{\partial V_i}{\partial r} - V_i\frac{\partial U_i}{\partial r}\right) |dz|, \end{aligned} \tag{24}$$

where dA is the area measure associated with ds^2, η is the exterior conormal field to the immersion along $\partial M(r)$ and $|dz| = r d\alpha$. To get the lemma it will be sufficient to let r go to zero.

Of course we have for $i \in \{t, b, m\}$:

$$\frac{\partial U_i}{\partial r} = \dot{\theta}_{1,i}\frac{\partial f_1(r)}{\partial r} + \dot{\theta}_{2,i}\frac{\partial f_2(r)}{\partial r} + \frac{\partial f_3(r, \dot{a}_i)}{\partial r} + \frac{\partial u_i(r)}{\partial r}$$

and a similar expression for $\frac{\partial V_i}{\partial r}$:

$$\frac{\partial V_i}{\partial r} = \dot{\varphi}_{1,i}\frac{\partial f_1(r)}{\partial r} + \dot{\varphi}_{2,i}\frac{\partial f_2(r)}{\partial r} + \frac{\partial f_3(r, \dot{b}_i)}{\partial r} + \frac{\partial v_i(r)}{\partial r}.$$

Given a $C^{2,\alpha}$ function l we will write it by its Taylor expansion in the coordinate $z = z_1 + iz_2 = re^{i\alpha}$, i.e.,

$$l = l(0) + r\cos\alpha(\partial_{z_1} l)(0) + r\sin\alpha(\partial_{z_2} l)(0) + O(r^2). \tag{25}$$

Now we proceed with the evaluation of the limit as $r \to 0$ of each summand that appears in (24). For $i \in \{t, b, m\}$ we have (to simplify the notation, we will omit the dependence on r and i)

$$\lim_{r\to 0}\int_{\partial D_i(0,r)} \left(U_i \frac{\partial V_i}{\partial r} - V_i \frac{\partial U_i}{\partial r}\right)|dz|$$

$$= \lim_{r\to 0}\int_{\{|z|=r\}} \Bigg[\left(\dot\varphi_{1,i} u_i(z)\frac{\partial f_1}{\partial r} - \dot\theta_{1,i} v_i(z)\frac{\partial f_1}{\partial r}\right) + \left(\frac{\partial v_i}{\partial r}\dot\theta_{1,i} f_1 - \frac{\partial u_i}{\partial r}\dot\varphi_{1,i} f_1\right)$$

$$+ \left(\dot\varphi_{2,i} u_i(z)\frac{\partial f_2}{\partial r} - \dot\theta_{2,i} v_i(z)\frac{\partial f_2}{\partial r}\right) + \left(\frac{\partial v_i}{\partial r}\dot\theta_{2,i} f_2 - \frac{\partial u_i}{\partial r}\dot\varphi_{2,i} f_2\right)$$

$$+ \left(u_i(z)\frac{\partial f_3(\dot b_i)}{\partial r} - v_i(z)\frac{\partial f_3(\dot a_i)}{\partial r}\right) + \left(u_i\frac{\partial v_i}{\partial r} - v_i\frac{\partial v_i}{\partial r}\right)\Bigg]|dz|.$$

We define (the expression of l is given by (25)):

$$G(l) = \lim_{r\to 0}\int_{\{|z|=r\}} l(r)\frac{\partial f_1}{\partial r}|dz|$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \left(l(0) + r(\cos\alpha(\partial_{z_1} l)(0) + \sin\alpha(\partial_{z_2} l)(0)) + O(r^2)\right)$$

$$\times\left(\frac{\tilde a_i\cos\alpha}{2r^2} + O(\ln r)\right) r\,d\alpha$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \left(l(0) + r(\cos\alpha(\partial_{z_1} l)(0) + \sin\alpha(\partial_{z_2} l)(0))\right)\frac{\tilde a_i\cos\alpha}{2r}d\alpha + O(r\ln r)$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{l(0)}{r} + \sin\alpha(\partial_{z_2} l)(0)\right)\frac{\tilde a_i\cos\alpha}{2}d\alpha$$

$$- \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde a_i(\partial_{z_1} l)(0)}{2}\cos^2\alpha\,d\alpha + O(r\ln r).$$

Then, since

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{l(0)}{r} + \sin\alpha(\partial_{z_2} l)(0)\right)\frac{\tilde a_i\cos\alpha}{2}d\alpha$$

$$= \lim_{r\to 0}\frac{\tilde a_i l(0)}{2r}\int_0^{2\pi}\cos\alpha\,d\alpha + \frac{\tilde a_i(\partial_{z_2} l(0))}{2}\int_0^{2\pi}\cos\alpha\sin\alpha\,d\alpha = 0$$

and

$$\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i(\partial_{z_1}l)(0)}{2}\cos^2\alpha d\alpha = \frac{\tilde{a}_i(\partial_{z_1}l)(0)}{2}\int_0^{2\pi}\cos^2\alpha d\alpha = \frac{\pi}{2}\tilde{a}_i(\partial_{z_1}l)(0),$$

we obtain

$$G(l) = -\frac{\pi}{2}\tilde{a}_i(\partial_{z_1}l(0)).$$

In a similar way:

$$T(l) = \lim_{r\to 0}\int_{\{|z|=r\}} l(r)\frac{\partial f_2}{\partial r}|dz|$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \big(l(0) + r(\cos\alpha(\partial_{z_1}l)(0) + \sin\alpha(\partial_{z_2}l)(0)) + O(r^2)\big)$$

$$\times\left(\frac{\tilde{a}_i\sin\alpha}{2r^2} + O(\ln r)\right) r d\alpha$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i\sin^2\alpha}{2}(\partial_{z_2}l)(0)d\alpha = -\frac{\pi}{2}\tilde{a}_i(\partial_{z_2}l)(0).$$

Then we can conclude that for $i \in \{t, b, m\}$:

$$\lim_{r\to 0}\int_{\{|z|=r\}} \big(\dot{\varphi}_{1,i}u_i(z) - \dot{\theta}_{1,i}v_i(z)\big)\frac{\partial f_1}{\partial r}|dz|$$

$$= \dot{\varphi}_{1,i}G(u_i) - \dot{\theta}_{1,i}G(v_i) = \frac{\pi}{2}\tilde{a}_i\big(\dot{\theta}_{1,i}(\partial_{z_1}v_i)(0) - \dot{\varphi}_{1,i}(\partial_{z_1}u_i)(0)\big).$$

In the same way we get

$$\lim_{r\to 0}\int_{\{|z|=r\}} \big(\dot{\varphi}_{2,i}u_i(z) - \dot{\theta}_{2,i}v_i(z)\big)\frac{\partial f_2}{\partial r}|dz|$$

$$= \dot{\varphi}_{2,i}T(u_i) - \dot{\theta}_{2,i}T(v_i) = \frac{\pi}{2}\tilde{a}_i\big(\dot{\theta}_{2,i}(\partial_{z_2}v_i)(0) - \dot{\varphi}_{2,i}(\partial_{z_2}u_i)(0)\big).$$

We define another couple of functions:

$$R(l) = \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\partial l}{\partial r}f_1|dz|$$

$$= \lim_{r\to 0}\int_{\{|z|=r\}} \big(\cos\alpha(\partial_{z_1}l)(0) + \sin\alpha(\partial_{z_2}l)(0) + O(r)\big)$$

$$\times\left(\frac{\tilde{a}_i\cos\alpha}{2r} + O(r\ln r)\right) r d\alpha$$

$$= \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i\cos^2\alpha}{2}(\partial_{z_1}l)(0)d\alpha = \frac{\pi}{2}\tilde{a}_i(\partial_{z_1}l)(0)$$

and

$$F(l) = \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\partial l}{\partial r} f_2 |dz|$$

$$= \lim_{r\to 0}\int_{\{|z|=r\}} (\cos\alpha(\partial_{z_1} l)(0) + \sin\alpha(\partial_{z_2} l)(0) + O(r))$$

$$\times\left(\frac{\tilde{a}_i \sin\alpha}{2r} + O(r\ln r)\right) r\,d\alpha$$

$$= \lim_{r\to 0}\int_{\{|z|=r\}} \frac{\tilde{a}_i \sin^2\alpha}{2}(\partial_{z_2} l)(0)d\alpha = \frac{\pi}{2}\tilde{a}_i(\partial_{z_2} l)(0).$$

Then we find:

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{\partial v_i}{\partial r}\dot{\theta}_{1,i} f_1 - \frac{\partial u_i}{\partial r}\dot{\varphi}_{1,i} f_1\right)|dz|$$

$$= \dot{\theta}_{1,i} R(v_i) - \dot{\varphi}_{1,i} R(u_i) = \frac{\pi}{2}\tilde{a}_i\big(\dot{\theta}_{1,i}(\partial_{z_1} v_i)(0) - \dot{\varphi}_{1,i}(\partial_{z_1} u_i)(0)\big).$$

Analogously:

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(\frac{\partial v_i}{\partial r}\dot{\theta}_{2,i} f_2 - \frac{\partial u_i}{\partial r}\dot{\varphi}_{2,i} f_2\right)|dz|$$

$$= \dot{\theta}_{2,i} F(v_i) - \dot{\varphi}_{2,i} F(u_i) = \frac{\pi}{2}\tilde{a}_i\big(\dot{\theta}_{2,i}(\partial_{z_2} v_i)(0) - \dot{\varphi}_{2,i}(\partial_{z_2} u_i)(0)\big).$$

As for the fifth summand, we have

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(u_i(z)\frac{\partial f_3(\dot{b}_i)}{\partial r} - v_i(z)\frac{\partial f_3(\dot{a}_i)}{\partial r}\right)|dz|$$

$$= -\lim_{r\to 0}\int_{\{|z|=r\}} \Bigg(\big(u_i(0) + r\cos\alpha(\partial_{z_1} u_i)(0) + r\sin\alpha(\partial_{z_2} u_i)(0) + O(r^2)\big)\frac{\dot{b}_i}{r}$$

$$- \big(v_i(0) + r\cos\alpha(\partial_{z_1} v_i)(0) + r\sin\alpha(\partial_{z_2} v_i)(0) + O(r^2)\big)\frac{\dot{a}_i}{r}\Bigg) r\,d\alpha$$

$$= -2\pi\big(\dot{b}_i u_i(0) - \dot{a}_i v_i(0)\big).$$

To finish we show that

$$\lim_{r\to 0}\int_{\{|z|=r\}} \left(u_i\frac{\partial v_i}{\partial r} - v_i\frac{\partial v_i}{\partial r}\right)|dz| = 0.$$

In fact:

$$\lim_{r\to 0}\int_{\{|z|=r\}} u_i \frac{\partial v_i}{\partial r}|dz| = \lim_{r\to 0}\int_{\{|z|=r\}} \big((u_i(0)+O(r))(\cos\alpha(\partial_{z_1}v_i)(0) \\ +\sin\alpha(\partial_{z_2}v_i)(0)+O(r))\big)\,r\,d\alpha = 0.$$

If we collect the previous results, we find that for $i = t, b, m$:

$$\lim_{r\to 0}\int_{\partial D_i(0,r)} \left(U_i\frac{\partial V_i}{\partial r} - V_i\frac{\partial U_i}{\partial r}\right)|dz| \\ = -\pi\tilde{a}_i\left[(\dot\varphi_{1,i}(\partial_{z_1}u_i)(0) - \dot\theta_{1,i}(\partial_{z_1}v_i)(0)) + (\dot\varphi_{2,i}(\partial_{z_2}u_i)(0) - \dot\theta_{2,i}(\partial_{z_2}v_i)(0))\right] \\ -2\pi\left(\dot b_i u_i(0) - \dot a_i v_i(0)\right) = 0,$$

with $\dot\theta_{j,m} = \dot\varphi_{j,m} = 0$ for $j = 1, 2$.

In conclusion we have:

$$\lim_{r\to 0}\int_{M(r)} (ULV - VLU)\,dA = \pi\tilde{a}_i \sum_{i\in\{t,b\}} \left[\dot\varphi_{1,i}(\partial_{z_1}u_i)(0) - \dot\theta_{1,i}(\partial_{z_1}v_i)(0)\right] \\ +\pi\tilde{a}_i \sum_{i\in\{t,b\}} \left[\dot\varphi_{2,i}(\partial_{z_2}u_i)(0) - \dot\theta_{2,i}(\partial_{z_2}v_i)(0)\right] + 2\pi \sum_{i\in\{t,b,m\}} \left[\dot b_i u_i(0) - \dot a_i v_i(0)\right].$$

We must do a change of variables to return in the graph coordinate. It is sufficient to observe that from (20) it follows

$$\frac{1}{x} = \frac{\tilde{a}_i}{2z} + O(1)$$

at each catenoidal type end. Then we get

$$\partial_{z_1}u_i(0) = \frac{2}{\tilde{a}_i}\partial_{x_1}u_i(0), \quad \partial_{z_2}u_i(0) = \frac{2}{\tilde{a}_i}\partial_{x_2}u_i(0)$$

and the same equations involving the functions v_i. After a change of sign we can conclude

$$\int_{M_k}(ULV - VLU)dA \\ = 2\pi \sum_{i\in\{t,b\}} \left[\dot\varphi_{1,i}(\partial_{z_1}u_i)(0) - \dot\theta_{1,i}(\partial_{z_1}v_i)(0)\right] \\ +2\pi \sum_{i\in\{t,b\}} \left[\dot\varphi_{2,i}(\partial_{z_2}u_i)(0) - \dot\theta_{2,i}(\partial_{z_2}v_i)(0)\right] + 2\pi \sum_{i\in\{t,b,m\}} \left[\dot b_i u_i(0) - \dot a_i v_i(0)\right].$$

Reordering the terms, we get the statement of lemma. □

2.2 The properties of the kernel and of the range of the Jacobi operator

Let $B = B(M_k) \subset C^{2,\alpha}(M_k)$ be the space of functions v such that their expression in a neighbourhood of p_i, with $i = t, b, m$, is

$$\dot{\theta}_{1,i} f_1(x, i) + \dot{\theta}_{2,i} f_2(x, i) + f_3(x, \dot{a}_i) + v_i(x), \tag{26}$$

in the graph coordinate x (here and in the sequel we use the same notation of Lemma 7) with $v_i \in C^{2,\alpha}(D(\varepsilon))$.

We are interested to study the kernel and the image of the "compactified" Jacobi operator $\bar{L}$. We define the following subspaces of the Banach space B :

$$J = J(M_k) = \ker(\bar{L}), \quad K = K(M_k) = J \cap C^{2,\alpha}(\overline{M_k}),$$
$$K_0 = K_0(\overline{M_k}) = \bar{L}(B)^{\perp}.$$

The elements of the space K are the Jacobi functions on M_k bounded at the ends. From previous definitions it follows that

$$\bar{L} \colon B = J \oplus J^{\perp} \to \bar{L}(B) \oplus K_0.$$

Lemma 8. *In the situation described above, it holds that:*

1. $K_0 = \big\{v \in K;\, \partial_{x_i} v_j(0) = 0,$ *for* $i = 1, 2,$ $j = t, b,$ *and* $v_j(0) = 0,$ *for* $j = t, b, m\big\} = \emptyset,$
2. $\dim J = 7.$

Proof.

1. Given $v \in K$, we have $v \in K_0$ if and only if $\int_{\overline{M_k}} v\bar{L}U d\bar{A} = 0\ \forall U \in B$. Then we suppose that U on a neighbourhood of the end p_i, $i = t, b, m$, has the following expression:

$$\dot{\varphi}_{1,i} f_1(x, i) + \dot{\varphi}_{2,i} f_2(x, i) + f_3(x, \dot{b}_i) + u_i(x).$$

Then, if $\dot{\Phi}_i = (\dot{\varphi}_{1,i}, \dot{\varphi}_{2,i})$ and $\dot{\Theta}_i = (\dot{\theta}_{1,i}, \dot{\theta}_{2,i})$, then by Lemma 7 we get

$$\int_{\overline{M_k}} U\bar{L}v d\bar{A} = \int_{M_k} ULv dA = 2\pi \sum_{i\in\{t,b\}} \big[\langle\dot{\Phi}_i, \nabla u_i(0)\rangle - \langle\dot{\Theta}_i, \nabla v_i(0)\rangle\big]$$
$$+ 2\pi \sum_{i\in\{t,b,m\}} \big[\dot{a}_i u_i(0) - \dot{b}_i v_i(0)\big] = 0$$

for each u_i. This is equivalent to

$$2\pi \sum_{i\in\{t,b\}} \langle \dot{\Theta}_i, \nabla v_i(0)\rangle + 2\pi \sum_{i\in\{t,b,m\}} \dot{b}_i v_i(0) = 0.$$

This gives K_0. Now we have to determine the Jacobi fields that are the generators of the space K_0. It is well known that vector fields in $\mathbb{R}^3$ whose flow consists in isometries (Killing fields) or dilations induce Jacobi functions. Thanks to the works [9] and [10] by S. Nayatani and result shown in [8], for all $k \geqslant 1$, the bounded Jacobi functions on M_k are associated with the following isometries of the ambient space: the three translations along the coordinate axes and the rotation about the vertical axis e_3. The space K_0 is generated only by the Jacobi functions which satisfy the conditions just proved. Making use of (21), (22) and (23) with the appropriate values of the logarithmic growths, we want to determine which of the following functions belongs to K_0: $\langle N, e_3\rangle$, $\langle N, e_1\rangle$, $\langle N, e_2\rangle$. We find:

$$\langle N, e_3\rangle(p_t) = \langle N, e_3\rangle(p_b) = -\langle N, e_3\rangle(p_m) = 1,$$

$$\partial_{x_j}\langle N, e_3\rangle(p_i) = 0, \quad \partial_{x_j}\langle N, e_1\rangle(p_i) = -\tilde{a}_i\delta_{1,j}, \quad \partial_{x_j}\langle N, e_2\rangle(p_i) = \tilde{a}_i\delta_{2,j},$$

with $j = 1, 2$. So we can conclude that these functions are not in K_0.

Now we consider the Jacobi function associated with the rotation about the vertical axis, that is $\langle N, e_3 \times p\rangle = \det(e_3, p, N)$, where $p = (s_1, s_2, s_3)$ denotes the position vector. We observe that its expression in $D_i^*(\varepsilon_i)$ is given by

$$s_1\langle N, e_2\rangle - s_2\langle N, e_1\rangle$$

$$\frac{x_1}{|x|^2}(\tilde{a}_i x_2 + O(|x|^2)) - \left(-\frac{x_2}{|x|^2}\right)(-\tilde{a}_i x_1 + O(|x|^2)) = O(x).$$

If we evaluate the derivatives of this function in $x = 0$, it is clear that K_0 cannot be generated by this Jacobi function. So K_0 is empty.

2. We consider the space $V \subset B$ of the functions defined on the disks $D_i^*(\varepsilon_i)$ by

$$\dot{\theta}_{1,i} f_1(x, i) + \dot{\theta}_{2,i} f_2(x, i) + f_3(x, \dot{a}_i).$$

It is a 7-dimensional space: in fact a function in V is determined by the values of the following parameters: $\dot{a}_t, \dot{a}_b, \dot{a}_m,\ \dot{\theta}_{1,t}, \dot{\theta}_{1,b},\ \dot{\theta}_{2,t}, \dot{\theta}_{2,b}$.

The spaces B and V can be decomposed in the following way:

$$B = V \oplus C^{2,\alpha}(\overline{M}_k), \qquad V = V_1 \oplus V_2,$$

where $V_1 = \{f \in V : \bar{L}f \in \bar{L}(C^{2,\alpha})\}$ and V_2 is a supplementary space. Then we have $\bar{L}(B) = \bar{L}(V_2) \oplus \bar{L}(C^{2,\alpha})$. Since $K_0 = \bar{L}(B)^{\perp}$ and $K = \bar{L}(C^{2,\alpha})$ we deduce

$$\begin{aligned} \dim\, K_0 &= \text{codim}\, \bar{L}(B) = \text{codim}\, \bar{L}(C^{2,\alpha}) - \dim\, \bar{L}(V_2) \\ &= \dim\, K - \dim\, V_2 = \dim\, K - \dim\, V + \dim\, V_1. \end{aligned}$$

that is

$$\dim\, K_0 = \dim\, K + \dim\, V_1 - 7. \tag{27}$$

Now we consider the restriction to $J = \ker(\bar{L})$ of the projection $\pi : B \to V$. It is clear that $\ker(\pi_{|J}) = K = J \cap C^{2,\alpha}$, then given a function $f \in J$ such that $f = v + u$ with $v \in V$ and $u \in C^{2,\alpha}$, we have

$$0 = \bar{L}f = \bar{L}v + \bar{L}u$$

and $\pi(f) = v \in V_1$. Furthermore, for any $v \in V_1$ there exists $\bar{v} \in C^{2,\alpha}$ such that $\bar{L}v = \bar{L}\bar{v}$, that is $v - \bar{v} \in J$. Then $\pi(J) = V_1$ and

$$\dim\, J = \dim\, \ker(\pi_{|J}) + \dim\, \text{Im}(\pi_{|J}) = \dim\, K + \dim\, V_1. \tag{28}$$

From the equations (27), (28) and $\dim\, K_0 = 0$, we get

$$\dim\, J = 7 + \dim\, K_0 = 7.$$

□

Remark 9. We remind that a minimal surface is non degenerate (see Definition 3) if the space $Killing_0 = Killing \cap K_0$ equals K_0. Thanks to Lemma 8 we can conclude that the Costa-Hoffman-Meeks surface M_k is non degenerate for all $k \geqslant 1$ with respect to this definition.

3 The proof of the main result

We consider again the immersion $X_y + u\widetilde{N}(y)$ (see (5)) and its mean curvature operator $H(y, u)$, where $y = (a_t, a_b, a_m, \theta_{1,t}, \theta_{1,b}, \theta_{2,t}, \theta_{2,b})$. We denote by $e_3(y)$ the unit vector defined in $D_i^*(\varepsilon_i)$, for $i = t, b, m$, by $e_3(\theta_{1,i}, \theta_{2,i})$ for $i = t, b$ and by $e_3(0, 0)$ for $i = m$. We recall that $\mathcal{A}$ and $\mathcal{U}$ denote, respectively a neighbourhood of $(\tilde{a}_t, \tilde{a}_b, 0)$ and a neighbourhood of zero in $C^{2,\alpha}(\overline{M}_k)$, and that $\tilde{y} = (\tilde{a}_t, \tilde{a}_b, 0, 0, 0, 0, 0)$, $\dot{y} = (\dot{a}_t, \dot{a}_b, \dot{a}_m, \dot{\theta}_{1,t}, \dot{\theta}_{2,t}, \dot{\theta}_{1,b}, \dot{\theta}_{2,b})$. We set $c = (\tilde{y}, 0)$. Let $v \in C^{2,\alpha}(\overline{M}_k)$ and let v_i denote its restrictions at the ends. We consider the function $w_{\dot{y},v} \in B$ defined in $D_i^*(\varepsilon_i)$ by $\dot{\theta}_{1,i} f_1(x, i) + \dot{\theta}_{2,i} f_2(x, i) + f_3(x, \dot{a}_i) + v_i(x)$.

Theorem 10. *For each possible choice of the limit values of the normal vectors of the three ends, there is, up to isometries, a* 1*-dimensional real analytic family of smooth minimal deformations of* M_k*, for* $k \geqslant 1$*, letting the planar end horizontal.*

Proof. We consider the map

$$\begin{array}{rcl} F\colon \mathcal{A} \times [-\varepsilon, \varepsilon]^4 \times \mathcal{U} & \longrightarrow & C^{0,\alpha}(\overline{M_k}) \\ (y, u) & \longrightarrow & \overline{H}(y, u). \end{array}$$

The map F is real analytic. Since the values $(y, u) = (\tilde{y}, 0) = c$ parametrize the Costa-Hoffman-Meeks surface M_k, the differential of F at c,

$$DF_c\colon B(M_k) \longrightarrow C^{0,\alpha}(\overline{M_k})$$

is given by

$$DF_c(w_{\dot{y},v}) = \frac{1}{2}\bar{L}(w_{\dot{y},v}).$$

Since $\bar{L}(B)^{\perp} = K_0 = \emptyset$, the differential for $(y, u) = c$ is surjective and its kernel has dimension equal to 7, being $Ker\ DF = J$. Using the implicit function theorem we find a neighborhood $\mathcal{W}$ of c in $\mathcal{A} \times [-\varepsilon, \varepsilon]^4 \times \mathcal{U}$ such that $\mathcal{V} = F^{-1}(0) \cap \mathcal{W}$ is a real analytic 7-dimensional manifold which contains only minimal immersions.

To complete the proof, it remains to observe that up to now we have considered arbitrary the choice of the logarithmic growths a_t, a_b, a_m and of the angles $\theta_{1,t}, \theta_{2,t}, \theta_{1,b}, \theta_{2,b}$ which determine the direction of the axis of revolution (or equivalently of the limit normal vector) of the top and bottom ends of the deformation of M_k. Actually it's necessary that the null flux condition is satisfied. In our case the flux is given by the sum of the flux of three catenoidal ends. So the sum of the three vectors must equal the null vector. The direction and the length of each vector are respectively given by the direction of axis of revolution and by the logarithmic growth of the respective catenoidal end. It's easy to understand that these three vectors must belong to a same vertical plane, that is we must have always $\theta_{2,t} = \theta_{2,b}$. The common value of these angles determines the orientation of this plane (see (4)). Furthermore the flux triangle described by the three vectors is uniquely determined by three of the remaining parameters (the logarithmic growths a_t, a_b, a_m and the angles $\theta_{1,t}, \theta_{1,b}$). It is clear that the choice of the limit values of the normal vectors (in other words of the angles $\theta_{1,t}, \theta_{1,b}$) of the two catenoidal type ends determines in unique way, up to a dilation, the flux triangle. So we can conclude that for each possible choice of the flux triangle, there exists a smooth 1-parameter family of minimal surfaces that are deformations of the surface M_k. □

Acknowledgements. The author wishes to thank L. Hauswirth for useful discussions and the referee for his suggestions.

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