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Morse 2-jet space and h-principle

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Abstract. A section in the 2-jet space of Morse functions is not always homotopic to a holonomic section. We give a necessary condition for being the case and we discuss the sufficiency.

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1 Introduction

Given a submanifold Σ in an r-jet space (of smooth sections of a bundle over a manifold M), it is natural to look at the associated *differential relation* $\mathcal{R}(\Sigma)$ formed by the $(r+1)$-jets *transverse* to Σ. For $j^r f$ being transverse to Σ at $x \in M$ is detected by $j_x^{r+1} f$. This is an open differential relation in the corresponding $(r+1)$-jet space. One can ask whether the Gromov *h-principle* holds: is any section with value in $\mathcal{R}(\Sigma)$ homotopic to a *holonomic* section of $\mathcal{R}(\Sigma)$? (We recall that a holonomic section of a $(r+1)$-jet space is a section of the form $j^{r+1} f$.)

According to M. Gromov, the answer is yes when M is an open manifold and Σ is *natural*, that is, invariant by a lift of $Diff(M)$ to the considered jet space (see [3] p. 79, [1] ch. 7).

The answer is also yes when the codimension of Σ is higher than the dimension n of M; this case follows easily from Thom's transversality theorem in jet spaces (see [6]). In the case of jet space of functions and when Σ is natural and codim $\Sigma \geq n+1$, it also can be seen as a baby case of a theorem of Vassiliev [9].

In this note we are interested in a codimension n case when M is a compact n-dimensional manifold. Let $J^r(M)$ denote the space of r-jets of real functions; when the boundary of M is not empty, it is meant that we speak of jets of

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functions which are locally constant on the boundary. We take $\Sigma \subset J^1(M)$ the set of critical 1-jets. Then $\mathcal{R}(\Sigma) \subset J^2(M)$ is the open set of 2-jets of Morse functions. We shall analyze the obstructions preventing the h-principle to hold with this differential relation.

2 Index cocycles

It is more convenient to work with the *reduced* jet spaces $\widetilde{J}^r(M)$, quotient of $J^r(M)$ by $\mathbb{R}$ which acts by translating the value of the jet. It is a vector bundle whose linear structure is induced by that of $C^\infty(M)$. For instance, $\widetilde{J}^1(M)$ is isomorphic to the cotangent space T^*M. Let $\mathcal{M}$ denote the reduced 2-jets of Morse functions, that is the 2-jets which are transverse to the zero section 0_M of T^*M (in the sequel, *jet* will mean *reduced jet*). Let $\pi : \widetilde{J}^2(M) \to \widetilde{J}^1(M)$ be the projection and $\pi_0 : \widetilde{J}^2_0(M) \to 0_M$ be its restriction over the zero section of the cotangent space. Since it is formed of critical 2-jets, it is a vector bundle whose fiber is the space of quadratic forms, $S^2(T^*_x M)$, $x \in M$. Let $\mathcal{M}_0 := \mathcal{M} \cap \widetilde{J}^2_0(M)$; it is a bundle over 0_M whose fiber consists of non-degenerate quadratic forms. Its complement in $\widetilde{J}^2_0(M)$ is denoted $\mathcal{D}$ (like discriminant); it is formed of 2-jets which are not transverse to 0_M. When M is connected, $\mathcal{M}_0$ has a connected component $\mathcal{M}^i_0$ for each index $i \in \{1, \dots, n\}$ of quadratic forms.

2.1 Tranverse orientation

Each $\mathcal{M}^i_0$ is a proper submanifold of codimension n in $\mathcal{M}$. Moreover the differential $d\pi$ gives rise to an isomorphism of normal fiber bundles

$$\nu(\mathcal{M}^i_0, \mathcal{M}) \cong \pi^*(\nu(0_M, T^*M))|\mathcal{M}^i_0 .$$

Of course, $\nu(0_M, T^*M)$ is canonically isomorphic to the cotangent bundle τ^*M, whose total space is T^*M. When M is oriented, so is the bundle τ^*M. When M is not orientable, one has a local system of orientations of τ^*M. Pulling it back by π yields a local system of orientations of $\nu(\mathcal{M}^i_0, \mathcal{M})$ (that is, co-orientations of $\mathcal{M}^i_0$). Let us denote $\mathcal{M}^{\text{even}}_0$ (resp. $\mathcal{M}^{\text{odd}}_0$) the union of the $\mathcal{M}^i_0$'s for i even (resp. odd). We endow $\mathcal{M}^{\text{even}}_0$ with the above local system of co-orientations. For reasons which clearly appear below, it is more natural to equip $\mathcal{M}^{\text{odd}}_0$ with the opposite system of co-orientations.

Lemma 2.2. *Let $s = j^2 f$ be a holonomic section of $\mathcal{M}$ meeting $\mathcal{M}_0$ transversally. Then each intersection point of $s(M)$ with $\mathcal{M}_0$ is positive. The same statement holds when s is a local holonomic section only.*

Proof. Let a be such an intersection point in $s(M) \cap \mathcal{M}_0^i$; so i is the index of the corresponding critical 2-jet. We can calculate in local coordinates (x, y', y''), where $x = (x_1, \dots, x_n)$ are local coordinates of M,

$$y' = \left(y'_1, \dots, y'_n\right) \quad \left(\text{resp. } y'' = \left(y''_{jk}\right)_{1 \le j \le k \le n}\right)$$

are the associated coordinates of T_x^*M (resp. $S^2 T_x^* M$). Since f is holonomic, we have

$$y''_{jk}(a) = \frac{\partial y'_j}{\partial x_k}(a).$$

Finally, the sign of $\det y''(a)$ (positive if i is even and negative if not) gives the sign of the Jacobian determinant at a of the map $x \mapsto y'(x)$, that is the sign of the intersection point when $\mathcal{M}_0^i$ is co-oriented by the canonical orientation of the y'-space. As we have reversed this co-orientation when i is odd, the intersection point is positive whatever the index is. $\square$

Proposition 2.3.

1) *Each $\mathcal{M}_0^i$ defines a degree n cocycle of $\mathcal{M}$ with coefficients in the local system $\mathbb{Z}^{or}$ of integers twisted by the orientation of M. Let μ_i be its cohomology class in $H^n(\mathcal{M}, \mathbb{Z}^{or})$; in particular, if $s\colon M \to \mathcal{M}$ is a section, $< \mu_i, [s] >$ is an integer.*

2) *When s is homotopic to a holonomic section $j^2 f$, then $< \mu_i, [s] >$ is positive and equals the number $c_i(f)$ of critical points of the Morse function f. In particular the total number $|Z|$ of zeroes of the section $\pi \circ s$ (which, by construction, is transverse to the 0-section) satisfies:*

$$|Z| \geq \sum_{i=0}^{n} c_i(f)\,.$$

Proof.

1) Let σ be a singular n-cycle with twisted coefficients of $\mathcal{M}$. It can be C^0-approximated by σ', an n-cycle which is transverse to $\mathcal{M}_0^i$. As $\mathcal{M}_0^i$ is a proper submanifold, there are finitely many intersections points in $\sigma' \cap \mathcal{M}_0^i$, each one having a sign with respect to the local system of coefficients. The algebraic sum of these signs defines an integer $c(\sigma')$. One easily checks that $c(\sigma') = 0$ if σ' is a boundary. As a consequence, if σ'_0

and σ'_1 are two approximations of σ, as $\sigma'_1 - \sigma'_0$ is a boundary, we have $c(\sigma'_1) - c(\sigma'_0) = 0$ which allows us to uniquely define $c(\sigma)$ as the value of an n-cocycle on σ. Typically, the image of a section carries an n-cycle with twisted coefficients and this algebraic counting applies.

2) Since c defined in 1) is a cocycle, it takes the same value on s and on $j^2 f$. According to lemma 2.2, it counts +1 for each intersection point in $j^2 f \cap \mathcal{M}_0^i$, that is, for each index i critical point of f. □

Corollary 2.4. *If s is a section of $\mathcal{M}$ which is homotopic to a holonomic section, the integers $m_i :=< \mu_i, [s] >$ fulfill the Morse inequalities*

$$\begin{aligned} m_0 &\geq \beta_0(F) \\ m_1 - m_0 &\geq \beta_1(F) - \beta_0(F) \\ \cdots & \quad \cdots \\ m_0 - m_1 + \cdots + (-1)^n m_n &= \beta_0(F) - \beta_1(F) \cdots + (-1)^n \beta_n(F) =: \chi(M) \end{aligned}$$

where F is a field of coefficients, $\beta_i(F) = \dim_F H_i(M, F^{or})$ is the i-th Betti number with coeffcients in F^{or} (F twisted by the orientation) and $\chi(M)$ is the Euler characteristic (independent of the field F).

Corollary 2.5. *The h-principle does not hold true for the sections of $\mathcal{M}$.*

Proof. It is sufficient to construct a section s of $\mathcal{M}$ which violates the Morse inequalities, for example a section which does not intersect $\mathcal{M}_0^0$. Leaving the case of the circle as an exercise, we may assume $n > 1$. One starts with a section s_1 of T^*M tranverse to O_M. Each zero of s_1 has a sign (if the local orientation of M is changed, so are both local orientations of s_1 and 0_M the sign of the zero in unchanged). For each zero a, one can construct a homotopy fixing a, with arbitrary small support, which makes s_1 linear in a small neighborhood of a. As $GL(n, \mathbb{R})$ has exactly two connected components, one can even suppose that after the homotopy, s_1 is near a the derivative of a non degenerate quadratic fonction whose index can be chosen arbitrarily provided it is even (resp. odd) if a is a positive (resp. a negative) zero. Finally, one can achieve by homotopy that near each zero a, one has $s_1 = df$ with a a non-degenerate critical point of f of index 2 (resp. 1) if a is a positive (resp. negative) zero.

Near the zeroes s_1 has a canonical lift to $\mathcal{M}$ by $s_2 = j^2 f$. Away from the zeroes, the lift s_2 extends as a lift of s_1 since the fibers of π are contractible over $T^*M \setminus 0_M$. By construction, we have $< \mu_0, [s_2] >= 0$, violating the first Morse inequality. □

Remark 2.6. *Denote $\mu_{even} = \mu_0 + \mu_2 + \ldots$ and $\mu_{odd} = \mu_1 + \ldots$. The following statement holds true: $\mu_{even} = \mu_{odd}$ if and only if the Euler characteristic vanishes.*

Proof. Assume first $\mu_{\text{even}} = \mu_{\text{odd}}$. Proposition 2.3 yields for any holonomic section in $\mathcal{M}$: $m_{\text{even}} = m_{\text{odd}}$, that is $\chi(M) = 0$. Conversely, if $\chi(M) = 0$, there exists a non-vanishing 1-form on M and hence, by lifting it to $\widetilde{\mathcal{J}}^2(M)$, a section v_0 in $\mathcal{M}$ avoiding $\mathcal{M}_0$. We form

$$W = \left\{ z \in \widetilde{\mathcal{J}}^2(M) \mid z = z_0 + t v_0, z_0 \in \mathcal{M}_0, t \geq 0 \text{ or } z_0 \in \mathcal{D}, t > 0 \right\}.$$

It is a proper submanifold in $\mathcal{M}$ whose boundary (with orientation twisted coefficients) is $\mathcal{M}_0^{\text{even}} - \mathcal{M}_0^{\text{odd}}$. Therefore, every cycle c satisfies $< \mu_{\text{even}}, c >=\ < \mu_{\text{odd}}, c >$, which implies the wanted equality. □

3 Are Morse inequalities sufficient?

This question is closely related to the problem of minimizing the number of critical points of a Morse function. This problem was solved by S. Smale in dimension higher than 5 for simply connected manifolds, as a consequence of the methods he developped for proving his famous h-cobordism theorem (see [8] or chapter 2 in [2]). Under the same topological assumptions we can answer our question positively. But there are other cases, discussed later, where the answer is negative.

Proposition 3.1. *Two sections s, t of $\mathcal{M} \subset \widetilde{\mathcal{J}}^2(M)$ are homotopic as sections of $\mathcal{M}$ if and only if their algebraic intersection numbers m_i with $\mathcal{M}_0^i$ are the same.*

Proof. According to proposition 2.3 1), the condition is necessary. Let us prove that it is sufficient. Leaving the 1-dimensional case to the reader, we assume $\dim M \geq 2$. Denote $s^1 = \pi \circ s$. Each zero of s^1 is given an index due to its lifting by s to a point of some $\mathcal{M}_0^i$. For each index i choose $|m_i|$ zeroes of s^1, $a_i^1, \ldots, a_i^{|m_i|}$, among its zeroes of index i; when $m_i > 0$ (resp. $m_i < 0$), we choose the a_i^j so that the corresponding intersection points of $s(M)$ with $\mathcal{M}_0^i$ are positive (resp. negative). When $m_i = 0$, no points are selected. In the same way, $|m_i|$ zeros $b_i^1, \ldots, b_i^{|m_i|}$ of t^1 are chosen.

The intersection signs being the same, one can find a homotopy of t in $\mathcal{M}$, which brings the b_i^j to coincide with the a_i^j and makes the two sections s and t coincide in the neighborhood of these points.

The other zeroes of s^1 of index i can be matched into pairs of points $\{a_i^{j+}, a_i^{j-}\}$ of opposite sign. A Whitney type lemma allows us to cancel all these pairs by a suitable homotopy of s in $\mathcal{M}$, reducing to the case when s^1 has no other zeroes than the a_i^j's, $j = 1, \dots, |m_i|$. A similar reduction may be assumed for t. Let us finish the proof in this case before stating and proving this lemma.

Both sections s^1 and t^1 of T^*M are homotopic (among sections) by a homotopy which is stationary on a neighborhood $N(a_i^j)$. Making this homotopy $h\colon M \times [0,1] \to T^*M$ transverse to the zero section, the preimage of 0_M consists of arcs $\{a_i^j\} \times [0,1]$ and finitely many closed curves γ_k. Each of these closed curves can be arbitrarily decorated with an index i. This choice allows us to lift h to $\widetilde{\mathcal{J}}^2(M)$ as a homotopy $\tilde{h}$ from s to t; this $\tilde{h}$ is the desired homotopy. More precisely, we proceed as follows for getting $\tilde{h}$. First $h|\gamma_k$ is lifted to $\mathcal{M}_0^i$ by using that the fiber of $\pi : \mathcal{M}_0^i \to 0_M$ is connected. The transversality of h to 0_M allows us to extend this lifting to a neighborhood of γ_k, making $\tilde{h}$ transverse to $\mathcal{M}_0^i$. Now it is easy to extend $\tilde{h}$ to $M \times [0,1]$, since the fiber of π over any point outside 0_M is contractible.

A Whitney type lemma. *Let (b^+, b^-) be a pair of transverse intersection points of s with $\mathcal{M}_0^i$ having opposite sign when they are thought of as zeroes of s_1 in M. Let α be a simple path in M joining them avoiding the other zeroes of s^1 and let N be a neighborhood of α. Then there exists a homotopy*

$$S = (s_u)_{u \in [0,1]}$$

of $s_0 = s$ into $\mathcal{M}$, supported in N and cancelling the pair (b^+, b^-), that is, $\pi \circ s_1$ has no zeroes in N.

Proof. We choose an embedded 2-disk (with corners) Δ in $N \times [0, 1[$ meeting $N \times \{0\}$ transversally along α. We first construct the homotopy $S^1 := \pi \circ S$ of s^1 among the sections of T^*M, following the cancellation process of Whitney which we are going to recall. We require S^1 to be transverse to 0_M with $(S^1)^{-1}(0_M) = \beta$, where $\alpha \cup \beta = \partial\Delta$. Using a trivialization of $T^*M|N$, $S^1|N \times [0,1]$ reads $S^1(x,u) = (x, g(x,u))$. The requirement is that g vanishes transversally along the arc β; it is possible exactly because $\dim M \geq 2$ and the end points have opposite signs. Let T be a small tubular neighborhood of β; its boundary traces an arc β' on Δ, "parallel" to β. Let α' be the subarc of α whose end points are those of β'. The restriction $g|T$ is required to be a trivialization of T, but this latter may be chosen freely. We choose it so that the loop $(g|\beta') \cup (s^1|\alpha')$ be homotopic to 0 in $(\mathbb{R}^n)^* \setminus \{0\}$; of course, when $n > 2$ this condition is automatically fulfilled. Now g can be extended to the rest of Δ as a non-vanishing map. As $N \times [0,1]$ collapses onto $N \times \{0\} \cup \Delta \cup T$, the extension of g can be completed without adding zeroes outside β, yielding the desired homotopy S^1.

It remains to lift S^1 to $\mathcal{M}$. The lifting is first performed along β with value in $\mathcal{M}_0^i$. Then it is globally extended in the same way as in the above lifting process. $\square$

Corollary 3.2. *Let s be a section of $\mathcal{M} \subset \widetilde{\mathcal{J}}^2(M)$ and m_i be its algebraic intersection number with $\mathcal{M}_0^i$. Let $f\colon M \to \mathbb{R}$ be a Morse function whose number $c_i(f)$ of critical points of index i satisfies*

$$c_i(f) = m_i$$

for all $i \in \{0, \dots, n\}$. Then s and $j^2(f)$ are homotopic as sections of $\mathcal{M}$.

Corollary 3.3. *We assume* $\dim M \geq 6$ *and $\pi_1(M) = 0$. Let s be a section of $\mathcal{M} \subset \widetilde{\mathcal{J}}^2(M)$ whose algebraic intersection numbers m_i fulfills the Morse inequalities for every field of coefficients. In particular, they are non-negative. Then s is homotopic through sections in $\mathcal{M}$ to a holonomic section.*

Proof. Under these topological assumptions the following result holds true: *For any set of non-negative integers $\{c_0, c_1, \dots, c_n\}$ satisfying the Morse inequalities for any field of coefficients, there exists a Morse function on M with c_i critical points of index i* (see theorem 2.3 in [2]). So we have a Morse function $f\colon M \to \mathbb{R}$ with m_i critical points of index i. According to corollary 3.2, s is homotopic in $\mathcal{M}$ to $j^2 f$. $\square$

3.4. We end this section by recalling that the Morse inequalities are not sharp for estimating the number of critical points of a Morse function on a non-simply connected closed manifold. Typically when $\pi_1(M)$ equals its subgroup of commutators (perfect group), some critical points of index 1 are required for generating the fundamental group, but the Morse inequalities allow $c_1 = 0$ (see [7] for more details). On the other hand, the only constraint for a section of $\mathcal{M}$ with intersection numbers m_i is the Euler-Poincaré identity:

$$m_0 - m_1 + \cdots = \chi(M).$$

So it is possible to find a section s whose intersection number m_i is the minimal rank in degree i of a free complex whose homology is $H_*(M, \mathbb{Z})$, that is,

$$m_i = \beta_i + \tau_i + \tau_{i-1},$$

where β_i stands for the rank of the free quotient of $H_i(M, \mathbb{Z})$ and τ_i denotes the minimal number of generators of its torsion subgroup ([2] p. 15). Such a set of integers satisfies the Morse inequalities but is far from being realizable by a Morse function. Finally this section s is not homotopic in $\mathcal{M}$ to a holonomic section.

4 Failure of the 1-parametric version of the h-principle

We thank Yasha Eliashberg who pointed out to us the failure of the h-principle in the 1-parametric version of the problem under consideration.

Here M is assumed to be a product $M = N \times [0, 1]$. Let $f_0 \colon M \to [0, 1]$ be the projection. When M is not 1-connected and $\dim M \geq 6$, according to Allen Hatcher the so-called *pseudo-isotopy problem* has always a negative answer: there exists f without critical points which is not joinable to f_0 among the Morse functions (see [4]). But $j^2 f$ can be joined to $j^2 f_0$ by a path γ in $\mathcal{M}$. Indeed, take a generic homotopy γ^1 joining df to df_0; then arguing as in the proof of proposition 3.1 it is possible to lift it to $\mathcal{M}$. When M is the n-torus $\mathbb{T}^n$, A. Douady showed very simply the stronger fact that the path γ^1 can be taken among the non-singular 1-forms (see appendix to [5]). This γ is not homotopic in $\mathcal{M}$ with end points fixed to a path of holonomic sections.

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