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The influence of $\mathcal{M}$-supplemented subgroups on the structure of finite groups*

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Abstract. A subgroup H of a group G is said to be $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that $G = HB$ and $TB < G$ for every maximal subgroup T of H. Moreover, a subgroup H is called c-supplemented in G if there exists a subgroup K such that $G = HK$ and $H \cap K \leq H_G$ where H_G is the largest normal subgroup of G contained in H. In this paper we give some conditions of supersolvability of finite group under assumption that some primary subgroups have some kinds of supplements, which are generalizations of some recent results.

Keywords: primary subgroups, $\mathcal{M}$-supplemented subgroup, c-supplemented subgroup, supersolvable group.

Mathematical subject classification: 20D10, 20D20.

1 Introduction

A subgroup H of a group G is called to be *supplemented* in G if there exists a subgroup K of G such that $HK = G$ and K is called a *supplement* of H in G. Obviously every subgroup of G is supplemented in G as G can be one of its supplements. Hence we should give some other restricted conditions. The relationship between the properties of subgroups of the Sylow subgroups of G and the structure of G has been investigated extensively by a number of authors. For instance, Hall [6] proved that a group G is solvable if and only if every Sylow subgroup of G is complemented in G. Arad and Ward [1] proved that a group G is solvable if and only if every $Sylow$ 2-subgroup and every $Sylow$ 3-subgroup of G are complemented in G. A. Ballester-Bolinches and

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Guo Xiuyun [2] proved that the class of all finite supersolvable groups with elementary abelian Sylow subgroups is just the class of all finite groups for which every minimal subgroup is complemented. Srinivassan [10] proved that a finite group is supersolvable if every maximal subgroup of every Sylow subgroup is normal. Recently, by considering some special supplements (c-supplement) of some primary subgroups, Wang [12] obtained some new conditions for the solvability and supersolvability of a group. More recently, Miao and Guo [8] proved that G is supersolvable if and only if every maximal subgroups of the Sylow subgroup of G is supersolvable s-supplemented in G. In this paper we want to continue these works and obtain some sufficient conditions for a saturated formation containing all supersolvable groups.

Now we will analyze the structure of finite groups with the following concept.

Definition 1.1. *A subgroup H is called $\mathcal{M}$-supplemented in a finite group G, if there exists a subgroup B of G such that $G = HB$ and TB is a proper subgroup of G for every maximal subgroup T of H.*

Throughout this paper, all groups are finite groups. Our terminology and notation are standard, see [4] and [9]. In particular, let G denote a finite group, $M < \cdot G$ indicates that M is a maximal subgroup of G. $|G|$ denotes the order of G. G_p is the Sylow p-subgroup of G. $\mathcal{U}$ denotes the class of all supersolvable groups. $\pi(G)$ denotes the set of all prime divisor of G.

Let π be a set of primes. We say that $G \in E_\pi$ if G has a Hall π-subgroup. We say that $G \in C_\pi$ if $G \in E_\pi$ and any two Hall π-subgroups of G are conjugate in G. We say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G. We denote by $[H]K$ the semidirect of H and K; $|G|$ denotes the order of a group G; H char G denotes that H is a characteristic subgroup of G.

Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is said to be a formation provided that (1) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$. It is clear that for a formation, every group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient $G/G^{\mathcal{F}}$ is in $\mathcal{F}$. The normal subgroup $G^{\mathcal{F}}$ is called the $\mathcal{F}$-residual of G. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is well known that the class of all supersolvable groups and the class of all p-nilpotent groups are saturated formations (cf. [5]).

2 Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1. *Let G be a group. Then*

(1) *If H is $\mathcal{M}$-supplemented in G, $H \leq M \leq G$, then H is $\mathcal{M}$-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. If H is $\mathcal{M}$-supplemented in G, then H/N is $\mathcal{M}$-supplemented in G/N.*

(3) *Let π be a set of primes. Let K be a normal π'-subgroup and H be a π-subgroup of G. Then H is $\mathcal{M}$-supplemented in G if and only if HK/K is $\mathcal{M}$-supplemented in G/K.*

Proof.

(1) If H is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H. So we may set $L = B \cap M$. Clearly, $B \cap M \leq M$ and $M = M \cap HB = H(M \cap B)$. Since $H_1B < G$ for every maximal subgroup H_1 of H, we have $M \cap H_1B = H_1(M \cap B)$ is a proper subgroup of M.

(2) If H is $\mathcal{M}$-supplemented in G, then there exists a subgroup B such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H. So we have $BN < G$. Otherwise we choose T be a maximal subgroup of H which contain N, then $T = T \cap BN = N(T \cap B)$ and hence $TB = N(T \cap B)B = NB = G$, a contradiction. It is easy to have $(H/N)(BN/N) = G/N$. For any maximal subgroup H_1 of H which contain N, we have $(H_1/N)(BN/N) = H_1B/N < G/N$. Therefore H/N is $\mathcal{M}$-supplemented in G/N.

(3) If H is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H. Clearly, $(HK/K)(BK/K) = G/K$. For any maximal subgroup T/K of HK/K, since K is a normal π'-subgroup and H is a π-subgroup of G, we have $T = H_1K$ where H_1 is a maximal subgroup of H. Therefore $(H_1K/K)(BK/K) = H_1BK/K < G/K$. Otherwise, if $H_1BK = G$, then $|G : H_1B| = |K : K \cap H_1B|$ is a π'-number, on the other hand, $|G : H_1B| = |HB : H_1B|$ is a π-number, a contradiction.

Conversely, if HK/K is $\mathcal{M}$-supplemented in G/K, we may similarly get H is $\mathcal{M}$-supplemented in G.

Lemma 2.2 [12]. *Let G be a group. Then*

(1) *If H is c-supplemented in G, $H \leq K \leq G$, then H is c-supplemented in K;*

(2) *Let $K \lhd G$ and $K \leq H \leq G$. Then H is c-supplemented in G iff H/K is c-supplemented in G/K;*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is c-supplemented in G, then HN/N is c-supplemented in G/N;*

(4) *A subgroup H of G is c-supplemented in G if and only if there exists a subgroup L of G such that $G = HL$ and $H \cap L = H_G = Core_G(H)$;*

(5) *Let R be a solvable minimal normal subgroup of a group G. If there exists a maximal subgroup R_1 of R such that R_1 is c-supplemented in G, then R is a cyclic group of prime order.*

Lemma 2.3. *Let P be a p-subgroup of G where p is a prime divisor of $|G|$. If P is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $|G\colon P_1B| = p$ where P_1 is a maximal subgroup of P.*

Proof. Since P is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = PB$ and $P_1B < G$ for every maximal subgroup P_1 of P. Then $P_1 \leq P \cap P_1B = P_1(P \cap B) < P$. Since P_1 is a maximal subgroup of P, we have $P \cap B \leq P_1$ and hence $P \cap B = P_1 \cap B$. Therefore $|G\colon P_1B| = |PB\colon P_1B| = p$.

Lemma 2.4. *Let R be a solvable minimal normal subgroup of G, R_1 be a maximal subgroup of R. If R_1 is $\mathcal{M}$-supplemented in G, then R is a cyclic group of prime order.*

Proof. Since R_1 is $\mathcal{M}$-supplement in G, there exists a subgroup B of G such that $G = R_1B$ and $TB < G$ for any maximal subgroup T of R_1. By Lemma 2.3, $|G\colon TB| = p$ and hence TB is the maximal subgroup of G. Since R is the minimal normal subgroup of G, we have $R \cap TB = R$ of $R \cap TB = 1$. If $R \cap TB = R$, then $TB = RTB = G$, a contradiction. Therefore we have $R \cap TB = 1$ and hence $|R| = p$.

Lemma 2.5 [5, Theorem 1.8.17]. *Let N be a solvable normal subgroup of a group G ($N \neq 1$). If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which is contained in N.*

Lemma 2.6 [15]**.** *If H is a subgroup of G with $|G\colon H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

Lemma 2.7 [3, Main Theorem]**.** *Suppose a finite group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.8 [11]**.** *If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.9 *Let G be a finite group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. Then G is p-nilpotent if and only if P is $\mathcal{M}$-supplemented in G.*

Proof. If G is p-nilpotent, then G has a normal p-complement D. For the Sylow p-subgroup P of G and every maximal subgroup P_1 of P, we may easily get $G = PD$ and $P_1D < G$. Therefore P is $\mathcal{M}$-supplemented in G.

Conversely, if P is $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = PB$ and $P_1B < G$ for every maximal subgroup P_1 of P. By Lemma 2.3, we have $|G\colon P_1B| = p$ and hence $P_1B \trianglelefteq G$ by Lemma 2.6. Since $|G\colon P_1B| = |PB\colon P_1B| = p$, we have $P \cap B = P_1 \cap B$ for every maximal subgroup P_1 of P. Therefore $P \cap B = \bigcap_{P_1 <\cdot P}(P_1 \cap B) = \Phi(P) \cap B$. On the other hand,

$$\bigcap_{P_1 <\cdot P}(P_1B) = \Big(\bigcap_{P_1 <\cdot P} P_1\Big)B = \Phi(P)B \quad \text{and} \quad \Phi(P)B \trianglelefteq G.$$

It follows from $P \cap \Phi(P)B = \Phi(P)(P \cap B) \leq \Phi(P)$ that we have $\Phi(P)B$ is p-nilpotent by Lemma 2.8. Let H be a normal Hall p'-subgroup of $\Phi(P)B$. Clearly, H is also the normal Hall p'-subgroup of G and hence G is p-nilpotent. The proof is over.

Lemma 2.10 *Let G be a finite group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P having no c-supplement in G, is $\mathcal{M}$-supplemented in G, then $G/O_p(G)$ is solvable p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of smallest order. Clearly, G is not a nonabelian simple group. Furthermore we have,

(1) $O_p(G) \neq 1$.

If $O_p(G) = P$, then $G/O_p(G)$ is a p'-group and of course it is p-nilpotent, a contradiction. If $1 < O_p(G) < P$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice of G implies that $G/O_p(G) \cong G/O_p(G)/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) $O_p(G) = 1$.

Let P_1 be a maximal subgroup of P. By hypotheses, if P_1 is c-supplemented in G, then there exists a subgroup K of G such that $G = P_1 K$ and $P_1 \cap K \leq (P_1)_G$. Since $O_p(G) = 1$ and $(P_1)_G \leq O_p(G)$, we have $P_1 \cap K = 1$. Therefore $|K_p| = p$ and hence K is p-nilpotent by Burnside p-nilpotent Theorem. Since K is p-nilpotent, we have $K_{p'} \trianglelefteq K$ where $K_{p'}$ is a Hall p'-subgroup of K and of course is the Hall p'-subgroup of G. Hence $G = P_1 N_G(K_{p'})$. If $P \cap N_G(K_{p'}) = P$, then $K_{p'} \trianglelefteq G$, a contradiction. If $P \cap N_G(K_{p'}) = L$ where L is the maximal subgroup of P, then $|G\colon N_G(K_{p'})| = |P\colon P \cap N_G(K_{p'})| = |P\colon L| = p$ and hence $N_G(K_{p'}) \trianglelefteq G$ by Lemma 2.6, a contradiction. So we may assume $P \cap N_G(K_{p'}) \leq L_2 < L_1$ where L_1 is the maximal subgroup of P and L_2 is the maximal subgroup of L_1. If L_1 is c-supplemented in G, then there exists a p-nilpotent subgroup H such that $G = L_1 H$. With the similar discussion we have $G = L_1 N_G(H_{p'})$ where $H_{p'}$ is the Hall p'-subgroup of H and of course of G. By Lemma 2.8, there exists an element x of P such that $N_G(K_{p'}) = (N_G(H_{p'}))^x$. Therefore $G = L_1 N_G(H_{p'}) = (L_1 N_G(H_{p'}))^x = L_1 N_G(K_{p'})$. Furthermore, $P = P \cap L_1 N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1$, a contradiction.

So we may assume L_1 is $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = L_1 B$ and $TB < G$ for any maximal subgroup T of L_1. Therefore $L_2 B < G$ and $|G\colon L_2 B| = p$ by Lemma 2.3. Since p is the smallest prime divisor of $|G|$, Lemma 2.6 implies that $L_2 B \trianglelefteq G$. We have $G = L_1 B = PB = PL_2 B$ and $P \cap L_2 B = L_2(P \cap B)$ is the Sylow p-subgroup of $L_2 B$. Clearly, $L_2(P \cap B)$ is the maximal subgroup of P. By hypotheses if $L_2(P \cap B)$ is $\mathcal{M}$-supplemented in G, then $L_2(P \cap B)$ is $\mathcal{M}$-supplemented in $L_2 B$ by Lemma 2.1 and hence $L_2 B$ is p-nilpotent by Lemma 2.9. Therefore G is p-nilpotent, a contradiction. So we may assume that $L_2(P \cap B)$ has a c-supplement R in G. With the similar discussion we have $G = L_2(P \cap B) N_G(R_{p'})$ where $R_{p'}$ is the Hall p'-subgroup of R and of course is the Hall p'-subgroup of G. By Lemma 2.7, there exists an element x of P such that $N_G(K_{p'}) = (N_G(R_{p'}))^x$. Therefore $G = L_2(P \cap B) N_G(R_{p'}) = (L_2(P \cap B) N_G(R_{p'}))^x = L_2(P \cap B) N_G(K_{p'})$. Furthermore, $P = P \cap L_2(P \cap B) N_G(K_{p'}) = L_2(P \cap B)(P \cap N_G(K_{p'})) = L_2(P \cap B)$, a contradiction.

The final contradiction completes our proof.

Lemma 2.11 [7]. *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$;*
(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G)$;*
(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;*
(4) *$C_G(F^*(G)) \leq F(G)$;*
(5) *Let $P \trianglelefteq G$ and $P \leq O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;*
(6) *If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.12 [13, Theorem 3.1]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

Lemma 2.13 [14, Theorem 1.1]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

3 Main results

Theorem 3.1. *Let G be a group having a normal subgroup N such that G/N is supersolvable. If every maximal subgroups of noncyclic Sylow subgroup of N having no c-supplement in G, is $\mathcal{M}$-supplemented in G, then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample with minimal order. Then we have following claims:

(1) G is solvable.

By hypotheses and Lemma 2.11, $N/O_p(N)$ is solvable r-nilpotent where r is the smallest prime divisor of $|N|$ and hence G is solvable. Let L be a minimal normal subgroup of G contained in N. Clearly, L is an elementary abelian p-group for some prime divisor of $|G|$.

(2) G/L is supersolvable and L is the unique minimal normal subgroup of G contained in N such that $N \cap \Phi(G) = 1$. Furthermore, $L = F(N) = C_N(L)$.

First, we check that $(G/L, N/L)$ satisfies the hypotheses for (G, N). We know that $N/L \trianglelefteq G/L$ and $(G/L)/(N/L) \cong G/N$ is supersolvable. Let $\overline{Q} = QL/L$ be a Sylow q-subgroup of N/L. We may assume that Q is a Sylow q-subgroup of N. If $p = q$, we may assume that $L \leq P$, where P is a Sylow p-subgroup of N. If $L \leq P = Q$ and hence every maximal subgroup of P/L is of the form P_1/L with P_1 a maximal subgroup of P. If P_1/L has no c-supplement in G/L, then P_1 has no c-supplement in G, by hypotheses, P_1 is $\mathcal{M}$-supplemented in G and hence P_1/L is $\mathcal{M}$-supplemented in G/L by Lemma 2.1 and Lemma 2.2. Now we assume that $p \neq q$. Let $\overline{Q_1}$ be a maximal subgroup of a Sylow q-subgroup of $\overline{N}$. Without loss of generality, we may assume that $\overline{Q_1} = Q_1L/L$ with Q_1 a maximal subgroup of a Sylow q-subgroup of N. Clearly, if Q_1L/L has no c-supplement in G/L, then Q_1L/L is $\mathcal{M}$-supplemented in G/L by Lemma 2.1 and 2.2. So G/L satisfies the hypotheses of the theorem. The minimal choice of G implies that G/L is supersolvable. Since the class of all supersolvable groups is a saturated formation, we know that L is the unique minimal normal subgroup of G which is contained in N and $L \not\leq \Phi(G)$. By Lemma 2.5 we have $F(N) = L$. The solvability of N implies that $L \leq C_N(L) = C_N(F(N)) \leq F(N)$ and so $C_N(L) = L = F(N)$.

(3) L is a Sylow subgroup of N.

Let q be the largest prime divisor of $|N|$ and Q a Sylow q-subgroup of N. Since G/L is supersolvable, we have N/L is supersolvable. Consequently, LQ/L char $N/L \trianglelefteq G/L$ and hence $LQ \trianglelefteq G$. If $p = q$, then $L \leq Q \trianglelefteq G$. Therefore $Q \leq F(N) = L$ and L is a Sylow q-subgroup of N, a contradiction.

Now we assume that $p < q$. Let P be a Sylow p-subgroup of N. Clearly, P is not cyclic. Otherwise, $G/L \in \mathcal{U}$ implies that $G \in \mathcal{U}$. Then $L \leq P$ and $PQ = PLQ$ is a subgroup of N. Note that every maximal subgroup of non-cyclic Sylow subgroup of PQ having no c-supplement in PQ, is $\mathcal{M}$-supplemented in PQ by Lemma 2.1 and Lemma 2.2. Therefore PQ satisfies the hypotheses for G. If $PQ < G$, the minimal choice of G implies that PQ is supersolvable; in particular, $Q \trianglelefteq PQ$. Hence $LQ = L \times Q$ and $Q \leq C_N(L) \leq L$, a contradiction.

Now we may assume that $G = PQ = N$ and $L < P$. Since G/L is supersolvable, $LQ \trianglelefteq G$. By the Frattini argument, $G = LN_G(Q)$. Note that $L \cap N_G(Q)$ is normalized by $N_G(Q)$ and L. We have that $L \cap N_G(Q) = 1$

since L is the unique minimal normal subgroup of G and Q is not normal in G in this case. Therefore $G = [L]N_G(Q)$. Let P_2 be a Sylow p-subgroup of $N_G(Q)$. Then LP_2 is a Sylow p-subgroup of G. Choose a maximal subgroup P_1 of LP_2 such that $P_2 \leq P_1$. Clearly, $L \nleq P_1$ and hence $(P_1)_G = 1$. By our hypotheses, if P_1 is $\mathcal{M}$-supplemented in G, that is, there exists a subgroup B of G such that $G = P_1B$ and $TB < G$ for any maximal subgroup T of P_1. So we may assume $P_2 \leq T$ for some maximal subgroup T of P_1. Otherwise, $P_2 = P_1$, then we have $|L| = p$ and hence G/L is supersolvable implies that G is supersolvable, a contradiction. By Lemma 2.3, $|G\colon TB| = p$ and hence $TB \trianglelefteq G$ by Lemma 2.6. Therefore $L \leq TB$ or $L \cap TB = 1$. If $L \cap TB = 1$, then $|G\colon TB| = |L| = p$. In this case G/L is supersolvable implies that G is supersolvable. So we may assume $L \leq TB$. Since $P_2 \leq T$, we have $LP_2 \leq TB$, contrary to $|G\colon TB| = p$. Now we may assume that P_1 is c-supplemented in G, that is, there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_G = 1$. Since $|K_p| = p$, we have K is p-nilpotent by Burnside Theorem and hence K has a normal Sylow q-subgroup Q_1 which is also a Sylow q-subgroup of G in this case. By Sylow's theorem, there exists an element $g \in L$ such that $Q_1^g = Q$. Since $P_1 \trianglelefteq LP_2$, we have that $G = P_1K = (P_1K)^g = P_1K^g$. Since $K^g \cong K$ has a normal Sylow q-subgroup and $Q = Q_1^g \leq K^g$, it follows that $K^g \leq N_G(Q)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1K^g = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleq P_2$. Otherwise $LP_2 \leq P_1P_2 = P_1$, a contradiction. Therefore P_2 is a proper subgroup of $P_3 = \langle P_2, LP_2 \cap K^g\rangle$. On the other hand, since both P_2 and K^g are contained in $N_G(Q)$, P_3 is a p-subgroup of $N_G(Q)$ which contains a Sylow subgroup P_2 of $N_G(Q)$ as a proper subgroup, a contradiction.

(4) G is supersolvable.

Let L_1 be a maximal subgroup of L. If L_1 is c-supplemented in G, then $|L| = p$ by Lemma 2.2(5) and G/L is supersolvable implies that G is supersolvable. Hence L_1 is $\mathcal{M}$-supplemented in G, that is, there exists a subgroup B of G such that $L_1B = G$ and $TB < G$ for every maximal subgroup T of L_1. By Lemma 2.4, we know that $|L| = p$. Consequently, G/L is supersolvable implies that G is supersolvable.

The final contradiction completes our proof.

Corollary 3.2. *Let G be a group. If every maximal subgroups of noncyclic Sylow subgroup of G having no c-supplement in G, is $\mathcal{M}$-supplemented in G, then G is supersolvable.*

Corollary 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup N such that $G/N \in \mathcal{F}$. If every maximal subgroups of noncyclic Sylow subgroup of N having no c-supplement in G, is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup N such that $G/N \in \mathcal{F}$. If every nonnormal maximal subgroups of noncyclic Sylow subgroup of N, is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F^*(N)$ having no c-supplement in G, is $\mathcal{M}$-supplemented in G. Then $G \in \mathcal{F}$.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order. Next we consider the following two cases.

Case 1. $\mathcal{F} = \mathcal{U}$.

1) $N = G$, $F^*(G) = F(G) \neq 1$.

By Theorem 3.1, $F^*(N)$ is supersolvable. In particular, $F^*(N)$ is solvable and hence $F^*(N) = F(N) \neq 1$ by Lemma 2.11. Since N satisfies the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable if $N < G$. Next we will prove that G is supersolvable in the case of H is solvable and divide into the following steps.

(1.1) $\Phi(G) \cap N \neq 1$.

If $\Phi(G) \cap N \neq 1$, there exists a prime p such that $p \mid |\Phi(G) \cap N|$. Let $L \in \mathrm{Syl}_p(\Phi(G) \cap N)$. Then $L \trianglelefteq G$ and $(G/L)/(N/L) \in \mathcal{U}$. By [5, P_{240} Satz 3.5] we have that $F(N/L) = F(N)/L$. Let P_1/L be a maximal subgroup of the Sylow p-subgroup of $F(N)/L$. Then P_1 is a maximal subgroup of the Sylow p-subgroup of $F(N)$. If P_1/L has no c-supplement in G/L, then P_1 has no c-supplement in G, by hypotheses P_1 is $\mathcal{M}$-supplemented in G and hence P_1/L is $\mathcal{M}$-supplemented in G/L by Lemma 2.1 and 2.2. Let Q/P be a maximal subgroup of the Sylow q-subgroup of $F(N)/L$, where $q \neq p$. It is clear that $Q = Q_1 P$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $F(N)$. With the similar discussion, if $Q_1 L/L$ has no c-supplement in G/L, then $Q_1 L/L$ is $\mathcal{M}$-supplemented in G/L by Lemma 2.1 and

Lemma 2.2. Hence, we have proved that G/L satisfies the hypotheses of the theorem. So G/L is supersolvable by the minimal choice of G. Since $P \leq \Phi(G)$ and $\mathcal{U}$ is a saturated formation, we have $G \in \mathcal{U}$, a contradiction.

(1.2) $\Phi(G) \cap N = 1$.

If $N = 1$, nothing need to prove, so we may assume that $N \neq 1$, the solvability of N implies that $F(N) \neq 1$. By Lemma 2.5, $F(N)$ is the direct product of minimal normal subgroups of G contained in N. For any maximal subgroup M of G, if $F(N) \leq M$, then $G \in \mathcal{U}$ by Lemma 2.12, a contradiction. So we may assume that there at least exists a maximal subgroup M of G not containing $F(N)$. Actually, since $F(N) \nleq M$, there at least exists a prime p of $\pi(|N|)$ with $O_p(H) \nleq M$. Then $G = O_p(H)M$ as $O_p(H) \trianglelefteq G$. If $|O_p(H)| = p$, then $|G\colon M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.12. If $|O_p(H)| > p$ and $O_p(H)$ is cyclic, then we have $\Phi(O_p(H)) \neq 1$. Clearly, $G/\Phi(O_p(H))$ satisfies the condition of the theorem, the minimal choice of G implies that $G/\Phi(O_p(H))$ is supersolvable and hence G is supersolvable since G is a saturated formation, a contradiction.

Denote $P = O_p(H)$. Then P is the direct product of some minimal normal subgroup of G. So we may assume that $P = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G, $i = 1.2 \ldots t$. If every $R_i (i = 1.2 \ldots t)$ is of prime order, there exist at least a minimal normal subgroup R_j of G contained in $O_p(H)$ such that $R_j \nleq M$. Since $G = R_j M$ and $|R_j| = p$, we get that M have a prime index in G and hence $G \in \mathcal{F}$ by Lemma 2.12, a contradiction.

Hence, we assume that there exist at least a minimal normal R_i of G contained in N which is not of prime order. Without loss of generality, suppose that $i = 1$.

Since $R_1 \nleq \Phi(G)$, there exists a maximal subgroup H of G such that $G = R_1 H$ and $R_1 \cap H = 1$. Then $G_p = R_1 H_p$ where H_p is the Sylow p-subgroup of H. Pick a maximal subgroup G_p^* of G_p containing H_p. Then $|R_1\colon G_p^* \cap R_1| = |R_1 G_p^*\colon G_p^*| = |G_p\colon G_p^*| = p$. Hence $R_1^* = G_p^* \cap R_1$ is a maximal subgroup of R_1. This implies that $P_1 = R_1^* R_2 \cdots R_t$ is a maximal subgroup of P. By hypotheses, if P_1 has a c-supplement in G, then there exists a subgroup K such that $G = P_1 K$ and $P_1 \cap K = (P_1)_G$ by Lemma 2.2(4). Evidently $(P_1)_G = R_2 \times \cdots \times R_t$. So $G = P_1 K = R_1^*(P_1)_G K = R_1^* K$, moreover $R_1^* \cap K = 1 \leq (R_1^*)_G$. Hence R_1^* is c-supplemented in G by the definition of c-supplemented subgroup. By Lemma 2.2(5), R_1 is a cyclic group of prime order, a contradiction.

So we may assume that P_1 is $\mathcal{M}$-supplemented in G, that is, there exists a subgroup B of G such that $G = P_1 B$ and $TB < G$ for any maximal subgroup

T of P_1. By Lemma 2.3, $|G\colon TB| = p$ and hence TB is the maximal subgroup of G. Therefore $R_1 \le TB$ or $R_1 \cap TB = 1$. If $R_1 \le TB$, then $TB = G$, a contradiction. If $R_1 \cap TB = 1$, then $|G\colon TB| = |R_1| = p$, a contradiction.

2) Every proper normal subgroup H of G containing $F^*(G)$ is supersolvable.

By Lemma 2.11, $F^*(G) = F^*(F^*(G)) \le F^*(H) \le F^*(G)$, so $F^*(H) = F^*(G)$. And if every maximal subgroup of noncyclic Sylow subgroups of $F^*(H)$ has no c-supplement in H, then has no c-supplement in G, by hypotheses, is $\mathcal{M}$-supplemented in G and hence is $\mathcal{M}$-supplemented in H by Lemma 2.1 and Lemma 2.2. Hence H is supersolvable by the minimal choice of G.

3) $\Phi(G) < F(G)$.

If every Sylow subgroup of $F(G)$ is cyclic, then we denote that $F(G) = H_1 \times \cdots H_r$ where $H_i (i = 1, \ldots, r)$ is the cyclic Sylow of $F(G)$ and hence $G/C_G(H_i)$ is abelian for any $i \in \{1 \cdots r\}$. Moreover, we have

$$\frac{G}{\bigcap_{i=1}^{r} C_G(H_i)} = \frac{G}{C_G(F(G))}$$

is abelian and hence $G/F(G)$ is abelian since $C_G(F(G)) = C_G(F^*(G)) \le F(G)$. Therefore G is solvable, a contradiction. Let P be a noncyclic Sylow subgroup of $F(G)$ and P_1 be a maximal subgroup of P. If P_1 is $\mathcal{M}$-supplemented in G by hypotheses. Then there exists a subgroup B in G such that $G = P_1 B$ and $TB < G$ for every maximal subgroup T of P_1. If $F(G) = \Phi(G)$, then $G = P_1 B = B$, a contradiction. So we may assume that every maximal subgroup P_1 of P has a c-supplement in G, then there exists a subgroup K of G such that $G = P_1 K$ and $P_1 \cap K = (P_1)_G$. If $\Phi(G) = F(G)$, then $G = K$ and hence P_1 is normal in G. Therefore G is supersolvable by Lemma 2.13, a contradiction.

4) $G = O_p(G)M$ where M is a maximal subgroup of G.

Since $\Phi(G) < F(G)$, for any Sylow subgroup $O_p(G)$ of $F(G)$ such that $O_p(G) \not\le \Phi(G)$, there exists the maximal subgroup M of G such that $O_p(G) \not\le M$ and $G = O_p(G)M$.

5) $|O_p(G)| = p$.

If $|O_p(G)| = p$, then set $C = C_G(O_p(G))$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then C is solvable by 2). On the other hand, since G/C is cyclic, we have G is solvable and hence G is supersolvable 1), a contradiction. So we may assume $C = G$. Now we have $O_p(G) \leq Z(G)$. Then we consider factor group $G/O_p(G)$. By Lemma 2.11, we have $F^*(G/O_p(G)) = F^*(G)/O_p(G) = F(G)/O_p(G)$. In fact, every maximal subgroup of noncyclic Sylow subgroup of $F^*(G/O_p(G))$ having no c-supplement in $G/O_p(G)$, is $\mathcal{M}$-supplemented in $G/O_p(G)$ by Lemma 2.1 and Lemma 2.2. Therefore the minimal choice of G implies that $G/O_p(G) \in \mathcal{U}$ and hence G is supersolvable, a contradiction.

6) $|O_p(G)| > p$.

So we may assume that $|O_p(G)| > p$. If $\Phi(O_p(G)) \neq 1$, then we easy to know that factor group $G/\Phi(O_p(G))$ satisfies the condition of the theorem. The minimal choice of G implies that $G/\Phi(O_p(G))$ is supersolvable and hence G is supersolvable since the class of all supersolvable groups is a saturated formation, a contradiction. Therefore, $\Phi(O_p(G)) = 1$ and $O_p(G)$ is an elementary abelian p-group. Let P_1 be a maximal subgroup of $O_p(G)$.

If P_1 is $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = P_1 B$ and $TB < G$ for every maximal subgroup T of P_1. By Lemma 2.3, $|G\colon TB| = p$ and $P_1 \cap B = T \cap B$ for every maximal subgroup T of P_1. Therefore we have

$$P_1 \cap B = \bigcap_{T <\cdot P_1} (T \cap B) = \Phi(P_1) \cap B.$$

On the other hand $P_1 \trianglelefteq O_p(G)$ and hence $\Phi(P_1) \leq \Phi(O_p(G)) = 1$. So we have $P_1 \cap B = \Phi(P_1) \cap B = 1$. Therefore, if the maximal subgroup P_1 of the Sylow subgroup of $F^*(G)$ is $\mathcal{M}$-supplemented in G, we have that P_1 is complemented in G and hence c-supplemented in G. By hypotheses and 3), we have every maximal subgroup of noncyclic Sylow subgroup is c-supplemented in G and hence G is supersolvable by Lemma 2.13, a contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By case 1, H is supersolvable. Particularly, H is solvable and hence $F^*(H) = F(H)$. Therefore $G \in \mathcal{F}$ by case 1.

The final contradiction completes our proof.

Corollary 3.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every* *maximal subgroup of noncyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 3.7. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F^*(H)$ has c-supplement in G, then $G \in \mathcal{F}$.*

Corollary 3.8. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and H be a solvable normal subgroup of G such that $G/H \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F(H)$ having no c-supplement in G is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 3.9. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F(H)$ is $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Corollary 3.10. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of noncyclic Sylow subgroup of $F(H)$ has c-supplement in G, then $G \in \mathcal{F}$.*

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