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Families of periodic orbits in resonant reversible systems

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Abstract. We study the dynamics near an equilibrium point p_0 of a $Z_2(\mathbb{R})$-reversible vector field in $\mathbb{R}^{2n}$ with reversing symmetry R satisfying $R^2 = I$ and $dim Fix(R) = n$. We deal with one-parameter families of such systems X_λ such that X_0 presents at p_0 a degenerate resonance of type $0\colon p\colon q$. We are assuming that the linearized system of X_0 (at p_0) has as eigenvalues: $\lambda_1 = 0$ and $\lambda_j = \pm i\alpha_j$, $j = 2, \dots n$. Our main concern is to find conditions for the existence of one-parameter families of periodic orbits near the equilibrium.

Keywords: equilibrium point, periodic orbit, reversibility, normal form, resonance.

Mathematical subject classification: 34C25, 37C25, 37C14.

1 Introduction

In this paper we deal with C^∞ reversible vector fields on $\mathbb{R}^{2n}$. These objects are assumed to have an equilibrium at 0 and the linearized systems (at 0) have as eigenvalues: $\lambda_1 = 0$ and $\lambda_j = \pm i\alpha_j$, $j = 2, \dots, n$. The latter assumption is not generic in the class of all reversible vector fields.

One of characteristic properties of reversible systems is that generically (symmetric) periodic orbits or invariant tori or minimal sets of such systems typically appear in one-parameter families. So a number of natural questions can be formulated, such as:

(i) how do branches of such minimal sets terminate or originate?

(ii) can one branch of minimal sets bifurcate from another such branch?

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(iii) how persistent is such branching process when the original system is slightly perturbed?

In this work we present some results in this direction, mainly extending and generalizing some issues got from [2], [5], [7], [12] and [14]. Recently, there has been a surging interest in the study of systems with time-reversal symmetries and we refer [8] for a survey in reversible systems and related problems.

We present some relevant historical facts. In 1895 Lyapunov published his celebrated center theorem, see Abraham and Marsden [1] p. 498. This theorem, for analytic Hamiltonians with n degrees of freedom, states that if the eigen-frequencies of the linearized Hamiltonian are independent over $\mathbb{Z}$, near a stable equilibrium point, then there exists n families of periodic solutions filling up smooth 2-dimensional manifolds going through the equilibrium point. This result was generalized by Weinstein [15] and Moser [10]. Weinstein considered the case where the Hamiltonian has positive definite Hessian at the equilibrium, and Moser, using Lyapunov-Schmidt reduction, extended the Weinstein's theorem for systems having an integral, not necessarily Hamiltonian. Devaney [2] proved a time-reversible version of the Lyapunov center theorem. Recently this center theorem has been generalized to equivariant systems, by Golubitsky, Krupa and Lim [3] in the time-reversible case, and by Montaldi, Roberts and Stewart [9] in the Hamiltonian case. We recall that in [3] the Devaney's theorem was extended and some extra symmetries were considered. Contrasting Devaney's geometrical approach, they used Lyapunov-Schmidt reduction, adapting an alternative proof of the reversible Lyapunov center theorem given by Vanderbauwhede [13]. In [9] the existence of families of periodic orbits around an elliptic semi-simple equilibrium is analyzed. Systems with symmetry, including time-reversal symmetry, which is anti-symplectic are studied. Their approach is a continuation of the work of Vanderbauwhede, in [13], where the families of periodic solutions correspond bijectively to solutions of a variational problem.

In this paper we study a codimension-one reversible bifurcation. Such bifurcation is characterized by the appearance of a zero eigenvalue at the linear part of the system at an equilibrium. We study the existence of families of periodic solutions near an equilibrium whose eigenvalues are near a $0: p: q$ resonance. Most of our technical analysis are based on a combined use of normal form theory and the Lyapunov-Schmidt Reduction (shortly denoted by **LSR**). The system is first subjected to the normalization procedure and the Belitiskii normal form (shortly denoted by **BNF**) plays a crucial role in our context. We focus on the 6-dimensional case.

We say that a vector field X is reversible if there exists a linear involution $R \in \mathcal{L}(\mathbb{R}^{2n})$ satisfying $RX = -XR$. We are assuming $\dim(Fix(R)) = n$. An orbit solution γ of X is called symmetric if $R\gamma = \gamma$.

So we also consider reversible systems of the form $\dot{x} = X(x, \lambda)$ with $X(Rx, \lambda) = -RX(x, \lambda)$ again with $x \in \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}^k$ and with $X(x, \lambda)$ a smooth parameter-dependent vector field.

Now we introduce some of the terminology and basic concepts for the formulation of our results.

We start by fixing, throughout the paper, the linear part of the vector field X.

$$A = DX(0) = \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & -\alpha_1 & & & \\ & & \alpha_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -\alpha_{n-1} \\ & & & & & \alpha_{n-1} & 0 \end{pmatrix}.$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_j = \pm i\alpha_j$, $j = 2, \ldots, n$.

We also fix the involution R as being

$$R\big(x_1, x_2, \ldots, x_{2n}\big) = \big(x_1, -x_2, \ldots, x_{2n-1}, -x_{2n}\big).$$

Fixed A one of the main questions one wants to answer is under which conditions, periodic solutions survive when we turn on the nonlinearities and change parameters.

Recall that the linear vector field $B = A^T$ is also R-reversible. Some methods employed in this work can be applied on reversible perturbations of B and probably similar results can be achieved. This paper does not touch this case.

Definition 1. We say that the set of eigenvalues $\{\pm\alpha_j,\ j = 2, \ldots, n\}$ satisfies the *non-resonance condition* if they are rationally independent. That is:

$$\sum_{j=2}^{n} k_j\alpha_j = 0, \quad k_j \in \mathbb{Z} \quad \Rightarrow \quad k_j = 0, \quad j = 2, \ldots, n.$$

Definition 2. The vector field X, with $X(0) = 0$ is 0-non-resonant if the set $\{\pm\alpha_j,\ j = 2, \ldots, n\}$ satisfies the non-resonance condition.

Definition 3. We say that X is $(0\colon p\colon q)$-*resonant* at 0 if there is a unique pair of indices l and k such that $\pm i\alpha_l$, $\pm i\alpha_k$ are in $p\colon q$-*resonance*, (that means that $q\alpha_l - p\alpha_k = 0$, with $p, q \in \mathbb{Z}_+$) and the others nonzero eigenvalues satisfy the *non-resonance* condition.

The most results contained in this paper can be illustrated by the following model:

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & a_1 x_1^2 + a_2\big(x_3^2 + x_4^2\big) + a_3\big(x_5^2 + x_6^2\big) \\ \dot{x}_3 = & -\alpha x_4 - b_1 x_1 x_4 \\ \dot{x}_4 = & \alpha x_3 + b_1 x_1 x_3 \\ \dot{x}_5 = & -\beta x_6 - b_2 x_1 x_6 \\ \dot{x}_6 = & \beta x_5 + b_2 x_1 x_5 \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2 \in \mathbb{R}$. Examples are given in subsection 3.2.

Denote by χ_0^{2n} (resp. $\chi^{2n}(\lambda)$) the space of all jets of R-reversible vector fields X at 0 such that $DX(0) = A$ (resp. space of one parameter families of R-reversible vector fields X_λ at $(0, 0)$ such that $X(x, 0) \in \chi_0^{2n}$ and $DX(0, 0) = A$) endowed with the C^∞ topology. Moreover we assume that the elements in χ_0^{2n} are at 0 either 0-non-resonant or $0\colon p\colon q$-resonant with $p + q > 2$.

Summarizing, in what follows we give a rough overall description of the main results of the paper.

- **(0-non-resonant normal form):** The normal form $\tilde{X}$ of X is exhibited for the 0-non-resonant case (**Theorem A**). The dynamics of any polynomial truncation of order k (or simply *k-truncated* vector field) $\tilde{X}_k$ of $\tilde{X}$ can be completely understood (**Proposition A**).

- **(Version of the Lyapunov Center Theorem):** Sufficient conditions for the existence of families of periodic solutions of $X \in \chi^{2n}(\lambda)$ are presented (**Theorems B and B***). We focus on those systems that present $0\colon p\colon q$-*resonances* with $p + q > 2$. The main discussion in this setting raises the question whether there is persistence, or birth or else disappearance of families of periodic solutions (terminating at the origin) when λ crosses the value 0. The answer to this question depends mainly on some second order coefficients in the normal form of the vector field.

Mention that the $0\colon 1\colon 1$-resonant case was discussed in [6].

The paper is organized as follows. In Section 2 the main results of the paper are stated. A **BNF** approach and the proof of Theorem A are in Section 3.

In Section 4, the **LSR** adapted to our systems is presented and Theorems B and B$*$ are proved.

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2 Statement of Main Results

Theorem A. *Let X be in χ_0^{2n}. Assume that X is 0-non-resonant at $0 \in \mathbb{R}^{2n}$. Then X is formally conjugated to*

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \varphi_2\big(x_1, |z_1|^2, \dots, |z_{n-1}|^2\big) \\ \dot{z_1} = iz_1\varphi_3\big(x_1, |z_1|^2, \dots, |z_{n-1}|^2\big) \\ \dot{\bar{z}}_1 = -i\bar{z}_1\varphi_3\big(x_1, |z_1|^2, \dots, |z_{n-1}|^2\big) \\ \vdots \\ \dot{z}_{n-1} = iz_{n-1}\varphi_{n+1}\big(x_1, |z_1|^2, \dots, |z_{n-1}|^2\big) \\ \dot{\bar{z}}_{n-1} = -i\bar{z}_{n-1}\varphi_{n+1}\big(x_1, |z_1|^2, \dots, |z_{n-1}|^2\big) \end{cases}$$

where $z_j = x_{2j+1} + ix_{2j+2}$ and φ_j are real functions.

Remark 2.0. From Theorem A we may find a coordinate system such that any $X \in \chi_0^{2n}$ is expressed by

$$X(x) = Ax + Q(x) + H(x)$$

where

$$\begin{aligned} Q(x) = \Big(0, a_1x_1^2 + \sum_{j=2}^{n-2} a_j\big(x_{j+1}^2 + x_{j+2}^2\big), -b_1x_1x_4, b_1x_1x_3, \dots \\ \dots, -b_{n-1}x_1x_{2n-1}, b_{n-1}x_1x_{2n}\Big) \end{aligned} \tag{1}$$

and $H(x) = O(|x|^3)$. It is worth mentioning that the expression still holds for the $0\colon p\colon q$ resonances with $p + q > 3$ and $n = 3$ (see Proposition 3.2).

Let $X \in \chi_0^{2n}$ and $\tilde{X}$ its normal form as presented in Theorem A. For each k, $k \geq 2$, $\tilde{X}_k$ represents the k-truncated vector field of $\tilde{X}$. Regarding the expression given in Theorem A in cylindrical coordinates, the proof of the next result is straightforward.

Proposition A: *Assume $X \in \chi_0^{2n}$ satisfying the hypothesis of Theorem A. Then:*

(I) *The system $\tilde{X}_k$ possesses n independent first integrals for each k;*

(II) *There exist an open set $\mathcal{U}_0$ in χ_0^{2n} characterized by $\mathcal{U}_0 = \big\{X \in \chi_0^{2n};\ a_1 \cdot a_j < 0,\ j = 2, \ldots, n-2$ and $b_i \neq 0,\ i = 1, \ldots, n-1\big\}$ in (1) such that any $X \in \mathcal{U}_0$ satisfies:*

- *There exist two $(n-1)$-parameter families of $(n-1)$-tori, T_μ^{n-1} and S_μ^{n-1}, both terminating at the origin;*
- *There is a one-parameter family of topological n-tori , T_μ^n containing T_μ^{n-1} and terminating at the origin;*
- *There is a two-parameter family of n-tori, $T_{\mu,\nu}^n$, terminating at the origin when $\mu \to 0$, and for each μ_0, the family originates at $T_{\mu_0}^n$ and terminates at $S_{\mu_0}^{n-1}$, when ν goes to $\pm\infty$;*

(III) *There is an integer $s < n$, depending on $Q(x)$ (given above) such that $\tilde{X}_k$ has: (a) $2s$ one-parameter families of periodic orbits terminating at the origin (with bounded periods) γ_μ^i and δ_μ^i; (b) s one-parameter families of homoclinic orbits T_μ^i terminating at origin; (c) s two-parameter families of 2-tori, $T_{\mu,\nu}^i$, terminating at the origin when $\mu \to 0$, and for each μ_0, the family originates at $T_{\mu_0}^i$ and terminates at $\delta_{\mu_0}^i$, $i = 1, 2, \ldots, s$ when ν goes to $\pm\infty$.*

Let χ_2^6 be the set contained in χ_0^6 constituted by the elements $X \in \chi_0^6$ presenting at the origin either a 0-*non-ressonance* or a $0\colon p\colon q$-resonance with $p, q > 1$.

Theorem B: *There exists an open subset $\mathcal{U} = \big\{(\mathcal{U}_1 \cup \mathcal{U}_2) \times (-\delta, +\delta)\big\}$ of $\chi_2^6(\lambda)$ such that:*

(I) *$\mathcal{U}_1 \cup \mathcal{U}_2 \subseteq \chi_2^6$ is characterized by $\mathcal{U}_1 = \big\{X \in \chi_2^6;\ a_1 \cdot a_2 < 0, a_1 \cdot a_3 < 0$ and $b_i \neq 0\big\}$ and $\mathcal{U}_2 = \big\{X \in \chi_2^6;\ a_1 \cdot a_2 > 0,\ a_1 \cdot a_3 > 0$ and $b_i \neq 0\big\}$ with $a_i's$ and $b_i's$ described in (1);*

(II) *If $X(x, 0) \in \mathcal{U}_1$ then:*

- *For $\lambda = 0$ there are four families of periodic orbits terminating at 0;*
- *For $\lambda < 0$ (resp. $\lambda > 0$) there are two symmetric equilibria (a saddle-center and a elliptic point) and two families of periodic orbits converging to each one of these points, provided that $a_1 < 0$ (resp. $a_1 > 0$);*

– *For $\lambda > 0$ (resp. $\lambda < 0$) there are no equilibria and just two families of periodic orbits walking around the origin, provided that $a_1 < 0$ (resp. $a_1 > 0$).*

(III) *If $X(x,0) \in \mathcal{U}_2$ then at $\lambda = 0$ occurs a subcritical Hopf bifurcation. So at $\lambda = 0$ there is no periodic orbit nearby the origin and for $\lambda < 0$ there are two families of periodic orbits, each one terminating at each equilibrium point that is an elliptic or a saddle-center singularity.*

Let χ_2^6* be the set contained in χ_0^6 constituted by the elements X presenting at the origin a $0\colon 1\colon 2$-resonance.

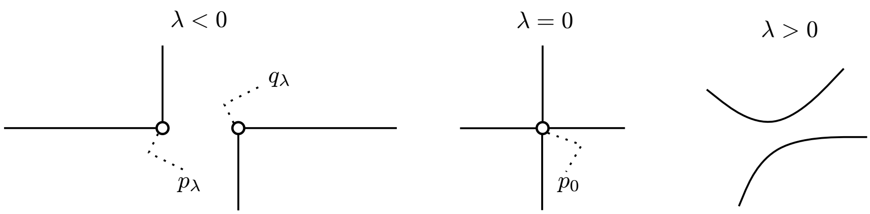

Figure 1: Bifurcation diagram illustrating case **II**, $a_1 < 0$, of Theorem B: the curves represent Lyapunov-centre families and the points are equilibria.

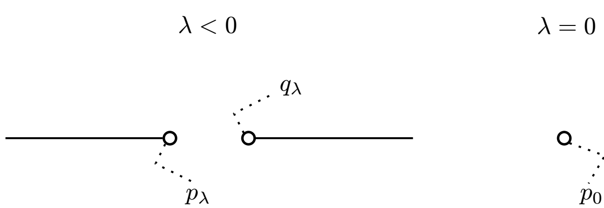

Figure 2: Bifurcation diagram illustrating case **III** of Theorem B: the curves represent Lyapunov-centre families and the points are equilibria.

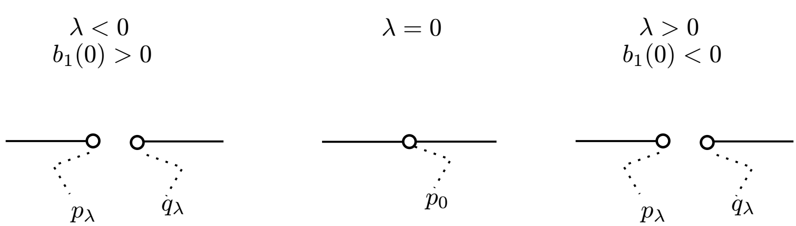

Figure 3: Bifurcation diagram illustrating case **II** of Theorem B*: the curves represent Lyapunov-centre families and the points are equilibria.

Remark 2.1. When $X \in \chi_0^6$ presents at the equilibria a $0:\ 1:\ 2$ resonance then as before we may write (see Proposition 3.2) $X(x) = Ax + S(x) + \tilde{H}(x)$ where

$$S(x) = \Big(0, a_1x_1^2 + a_2(x_3^2 + x_3^4) + a_3(x_5^2 + x_6^2), -b_1x_1x_4 - c_1(x_3x_6 - x_4x_5),\\ b_1x_1x_3 + c_1(x_3x_5 + x_4x_6), -b_2x_1x_6 - 2c_2x_3x_4, b_2x_1x_5 + c_2(x_3^2 - x_4^2)\Big) \tag{2}$$

and $\tilde{H}(x) = O(|x|^3)$. This expression will be used in the statement of the next result.

Theorem B*. *There exists a set* $\mathcal{V} = \big\{(\mathcal{V}_1 \cup \mathcal{V}_2) \times (-\delta, \delta)\big\}$ *in* $\chi_2^6 * (\lambda)$ *such that:*

(I) $\{\mathcal{V}_1 \cup \mathcal{V}_2\} \subset \chi_2^6 *$ *and* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *are characterized by* $\mathcal{V}_1 = \big\{X \in \chi_2^6 *;\ a_1 \cdot a_3 < 0$ *and* $b_2 \neq 0\big\}$ *and* $\mathcal{V}_2 = \big\{X \in \chi_2^6 *;\ a_1 \cdot a_3 > 0$ *and* $b_2 \neq 0\big\}$ *with* $a_i's$ *and* $b_i's$ *described in Remark* 2.1.

(II) *if* $X(x, 0) \in \mathcal{V}_1$ *then: for* $\lambda = 0$ *there are two families of symmetric periodic orbits with period near* 2π *converging to the equilibrium. Moreover these families are persistent for* $\lambda \neq 0$. *In this case, there are two equilibria near the origin for* $\lambda > 0$ *(resp.* $\lambda < 0$*) if* $b_1(0) < 0$ *(resp. if* $b_1(0) > 0$*). Moreover each family converges to a different equilibrium.*

(III) *If* $X(x, 0) \in \mathcal{V}_2$ *then at* $\lambda = 0$ *we get a Hopf bifurcation that is subcritical if* $b_1(0) > 0$ *and supercritical if* $b_1(0) < 0$. *We find two families of symmetric periodic orbits each one converging to a different equilibrium point.*

3 BNF and Proof of Theorem A

We say that $X(x) = Ax + h(x)$ in χ_0^{2n} is in **BNF** if the non linear term $h(x)$ satisfies $A^*h(x) = Dh(x)A^*x$ where A^* is the adjoint matrix of A.

Observe that the homological equation associated to the **BNF** is $L_{A^*} := A^*h(x) - Dh(x)A^*x$.

3.1 Proof of Theorem A.

First of all consider our system written in complex coordinates. So:

$$A = \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & i\alpha_1 & & & & \\ & & & -i\alpha_1 & & & \\ & & & & \ddots & & \\ & & & & & i\alpha_{n-1} & \\ & & & & & & -i\alpha_{n-1} \end{pmatrix}.$$

As $h(x) = (h_1(x), \ldots, h_{2n}(x))$ must satisfy $A^*h(x) = Dh(x)A^*x$, with $x = (x_1, x_2, z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1})$, then $Dh_1(x) = 0$, $Dh_2(x) = h_1$, $Dh_3(x) = -i\alpha_1 h_3$, $Dh_4(x) = i\alpha_1 h_4, \ldots, Dh_{2n-1}(x) = -i\alpha_{n-1}h_{2n-1}$ and $Dh_{2n} = i\alpha_{n-1}h_{2n}$, where

$$D := x_1\frac{\partial}{\partial x_2} - i\alpha_1 z_1\frac{\partial}{\partial z_1} + i\alpha_1\bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \cdots - i\alpha_{n-1}z_{n-1}\frac{\partial}{\partial z_{n-1}} + i\alpha_{n-1}\bar{z}_{n-1}\frac{\partial}{\partial \bar{z}_{n-1}}.$$

Hence $Dh_1(x) = 0 \Rightarrow h_1(x) = \varphi_1(x_1, |z_1|^2, \ldots, |z_{n-1}|^2)$ provided the *non-resonance* conditions are satisfied.

The reversibility of the system gives us that

$$h_1\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) = -h_1\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big).$$

Moreover the relations $Dh_2 = h_1 = 0$ imply that

$$h_2 = \varphi_2\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big).$$

Consider now $g_{2j+1} = \bar{z}_j h_{2j+1}$, $j = 1, \ldots, n-1$. We derive that:

i) $Dg_{2j+1} = 0 \Rightarrow g_{2j+1} = g_{2j+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big)$;

ii) as $g_{2j+1} = 0$ on $z_j = 0$, $j = 1, \ldots, n-1$ we have

$$g_{2j+1} = |z_j|^2\tilde{\varphi}_{2j+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big)$$

and so

$$h_{2j+1} = z_j\tilde{\varphi}_{2j+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big);$$

iv) In the same way we get the equality

$$h_{2j+2} = \bar{z}_j\tilde{\varphi}_{2j+2}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big);$$

v) The expression $\bar{\tilde{\varphi}}_{2j+2} = \tilde{\varphi}_{2j+1}$ plus the reversibility condition imply that $\tilde{\varphi}_{2j+1} = -\tilde{\varphi}_{2j+2}$. And so $\tilde{\varphi}_{2j+1} = i\varphi_{2j+1}$, with $\varphi_{2j+1} \in \mathbb{R}$;

vi) Hence

$$\begin{aligned} h_{2j+1} &= iz_j\varphi_{2j+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \quad \text{and} \\ h_{2j+2} &= -i\bar{z}_j\varphi_{2j+2}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \end{aligned}$$

where $j = 1, \ldots, n-1$.

In this way we get the R-reversible normal form of our system:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \varphi_2\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \\ \dot{z_1} = iz_1\varphi_3\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \\ \dot{\bar{z}}_1 = -i\bar{z}_1\varphi_3\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \\ \vdots \\ \dot{z}_{n-1} = iz_{n-1}\varphi_{n+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big) \\ \dot{\bar{z}}_{n-1} = -i\bar{z}_{n-1}\varphi_{n+1}\big(x_1, |z_1|^2, \ldots, |z_{n-1}|^2\big). \end{cases}$$

$\square$

Remark 3.0. The **BNF** (*) in coordinates $(x_1, \ldots, x_{2n})$ is written as:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \varphi_2\big(x_1, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2\big) \\ \dot{x_3} = -x_4\varphi_3\big(x_1, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2\big) \\ \dot{x_4} = x_3\varphi_3\big(x_1, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2\big) \\ \vdots \\ \dot{x}_{2n-1} = -x_{2n}\varphi_{n+1}\big(x_1, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2\big) \\ \dot{x}_{2n} = x_{2n-1}\varphi_{n+1}\big(x_1, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2\big). \end{cases} \tag{3}$$

Proposition 3.1. *Let $X, Y \in \chi_0^6$ presenting at the origin a 0-non-resonance and $0\colon p\colon q$-resonance with $p + q > 3$ respectively. Then the 2-jets of X and Y at 0 have similar* **BNF**.

Proof. First of all observe that the operator D is the same as before. Let us solve $Dh_1 = 0$ $(**)$.

The monomial $v = x_1^{k_1} z^{k_2} \bar{z}^{k_3} \omega^{k_4} \bar{\omega}^{k_5}$ is a solution of (**) if and only if,

$$(k_2 - k_3)\alpha + (k_4 - k_6)\beta = 0 \Rightarrow \begin{cases} k_2 = k_3 + kq \\ k_5 = k_4 + kp \end{cases}.$$

Hence

$$v = x_1^{k_1} z^{k_3+kq} \bar{z}^{k_3} \omega^{k_4} \bar{\omega}^{k_4+kp} = x_1^{k_1} (z\bar{z})^{k_3} (\omega\bar{\omega})^{k_4} (z^q \bar{\omega}^p)^k.$$

That means $h_1 = h_1(x_1, |z|^2, |\omega|^2, z^q \bar{\omega}^p)$.

In this way $O(2, h_1) = h_1(x_1, |z|^2, |\omega|^2)$.

The reversibility condition on X implies that $O(2, h_1) = 0$.

The condition $Dh_2 = h_1$ implies that que $Dh_2 = 0$ (for terms of order 2). So $h_2 = h_2(x_1, |z|^2, |\omega|^2, z^q \bar{\omega}^p)$.

When we restrict to terms of order 2 we get $O(2, h_2) = h_2(x_1, |z|^2, |\omega|^2))$.

Arguing in the same way as in the proof of Theorem A we obtain $h_3, \ldots, h_6$ and the desired proof now is straightforward. $\square$

Proposition 3.2. *Let $X \in \chi_0^6$ presenting at the origin a $0\colon 1\colon 2$-resonance. Then we can find a coordinate system where X can be written as:*

$$\begin{cases} \dot{x}_1 = x_2 + O(3) \\ \dot{x}_2 = a_1 x_1^2 + a_2\left(x_3^2 + x_4^2\right) + a_3\left(x_5^2 + x_6^2\right) + O(3) \\ \dot{x}_3 = -x_4 - b_1 x_1 x_4 - c_1\left(x_3 x_6 - x_4 x_5\right) + O(3) \\ \dot{x}_4 = x_3 + b_1 x_1 x_3 + c_1\left(x_3 x_5 + x_4 x_6\right) + O(3) \\ \dot{x}_5 = -2x_6 - b_2 x_1 x_6 - 2c_2 x_3 x_4 + O(3) \\ \dot{x}_6 = 2x_5 + b_2 x_1 x_5 + c_2\left(x_3^2 - x_4^2\right) + O(3). \end{cases}$$

Proof. First of all consider coordinates (x_1, x_2, z_1, z_2) with $z_1 = x_3 + ix_4$, $z_2 = x_5 + ix_6$. The linear part of the system is then:

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & i & & & \\ & & & -i & & \\ & & & & 2i & \\ & & & & & -2i \end{pmatrix}.$$

Let $\dot{x} = Ax + h^{(2)}(x) + O(3)$. The condition $A^* h^{(2)}(x) = Dh^{(2)}(x) A^* x$, with $x = (x_1, x_2, z_1, \bar{z}_1, z_2, \bar{z}_2)$, implies that $Dh_1^{(2)}(x) = 0$, $Dh_2^{(2)}(x) = h_1^{(2)}$, $Dh_3^{(2)}(x) = -ih_3^{(2)}$, $Dh_4^{(2)}(x) = ih_4^{(2)}$, $Dh_5^{(2)}(x) = -2ih_5^{(2)}$ and $Dh_6^{(2)}(x) = 2ih_6^{(2)}$ where

$$h^{(2)}(x) = \left(h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, h_4^{(2)}, h_5^{(2)}, h_6^{(2)}\right)$$

and

$$D := x_1 \frac{\partial}{\partial x_2} - iz_1 \frac{\partial}{\partial z_1} + i\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - 2iz_2 \frac{\partial}{\partial z_2} + 2i\bar{z}_2 \frac{\partial}{\partial \bar{z}_2}.$$

We look now for those monomials $u = x_1^{k_1} z_1^{k_2} \bar{z}_1^{k_3} z_2^{k_4} \bar{z}_2^{k_5}$ that are in normal form up to degree 2. Observe that we are assuming that $|k| = \sum_{j=1}^{5} k_j = 2$ with $k = (k_1, k_2, k_3, k_4, k_5)$.

Analysis of $h_1^{(2)}$.

$$Du = 0 \Rightarrow (-ik_2 + ik_3 - 2ik_4 + 2ik_5)u = 0 \Rightarrow (k_2 - k_3) + 2(k_4 - k_5) = 0.$$

The elements k that satisfy the cited condition plus $|k| = 2$ are:

$$(0, 1, 1, 0, 0); \quad (0, 0, 0, 1, 1); \quad (2, 0, 0, 0, 0).$$

In fact they are: $|z_1|^2$; $|z_2|^2$; x_1^2.
So $Dh_1^{(2)} = 0$ implies that $h_1^{(2)} = h_1^{(2)}(x_1, |z_1|^2, |z_2|^2)$.
The R-reversibility condition implies that $h_1^{(2)} \equiv 0$.
Hence

$$Dh_2^{(2)} = h_1^{(2)} = 0 \Rightarrow h_2^{(2)} = \varphi\big(x_1, |z_1|^2, |z_2|^2\big) = a_1 x_2^2 + a_2 |z_1|^2 + a_3 |z_2|^2.$$

Analysis of $h_3^{(2)}$.

$$Du = -iu \ \Rightarrow (k_2 - k_3 - 1) + 2(k_4 - k_5) = 0.$$

The elements that satisfy the above condition with $|k| = 2$ are $(1, 1, 0, 0, 0)$ and $(0, 0, 1, 1, 0)$ that represent the monomials $x_1 z_1$ and $\bar{z}_1 z_2$ respectively.
So $h_3^{(2)} = \tilde{b}_1 x_1 z_1 + \tilde{c}_1 \bar{z}_1 z_2$.
Similarly we obtain $h_4^{(2)} = \tilde{e}_1 x_1 \bar{z}_1 + \tilde{d}_1 z_1 \bar{z}_2$.
Now we already know that $\bar{h}_3 = h_4$. This implies that $\bar{\tilde{b}}_1 = \tilde{e}_1$ and $\bar{\tilde{c}}_1 = \tilde{d}_1$.
The R-reversibility of the system implies that $\tilde{e}_1 = -\tilde{b}_1$ and $\tilde{d}_1 = -\tilde{c}_1$.

So it follows that $\begin{cases} \tilde{b}_1 = i b_1 \\ \tilde{c}_1 = i c_1 \end{cases}, \quad b_1, c_1 \in \mathbb{R}$.

Hence $h_3^{(2)} = i[b_1 x_1 + c_1 \bar{z}_1 z_2]$ and $h_4^{(2)} = -i[b_1 x_1 z_1 + c_1 z_1 \bar{z}_2]$

Similar computations allow us to analyze the terms $h_5^{(2)}$ and $h_6^{(2)}$ and finally obtain:

$$\begin{cases} \dot{x}_1 = x_2 + o(3) \\ \dot{x}_2 = a_1 x_1^2 + a_2 |z_1|^2 + a_3 |z_2|^2 + o(3) \\ \dot{z}_1 = i z_1 + i\big[b_1 x_1 z_1 + c_a \bar{z}_1 z_2\big] + o(3) \\ \dot{z}_2 = 2i z_2 + i\big[c_2 z_1^2 + b_2 x_1 z_2\big] + o(3) \end{cases}$$

or in real coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$:

$$\begin{cases} \dot{x}_1 = x_2 + O(3) \\ \dot{x}_2 = a_1 x_1^2 + a_2\big(x_3^2 + x_4^2\big) + a_3\big(x_5^2 + x_6^2\big) + O(3) \\ \dot{x}_3 = -x_4 - b_1 x_1 x_4 - c_1\big(x_3 x_6 - x_4 x_5\big) + O(3) \\ \dot{x}_4 = x_3 + b_1 x_1 x_3 + c_1\big(x_3 x_5 + x_4 x_6\big) + O(3) \\ \dot{x}_5 = -2x_6 - b_2 x_1 x_6 - 2c_2 x_3 x_4 + O(3) \\ \dot{x}_6 = 2x_5 + b_2 x_1 x_5 + c_2\big(x_3^2 - x_4^2\big) + O(3). \end{cases}$$

□

3.2 Systems in BNF

We present a brief discussion of the systems derived from Proposition A.

Let $X \in \chi_0^{2n}$ and $\tilde{X}_k$ be its corresponding truncated system, $k > 1$. The first integrals of this system are:

$$H_1 = x_3^2 + x_4^2, \; H_2 = x_5^2 + x_6^2, \ldots, \; H_{n-1} = x_{2n-1}^2 + x_{2n}^2$$

$$\text{and} \quad H_n = x_2^2 - \int \varphi_2\big(x_1, H_1, \ldots, H_{n-1}\big) dx_1.$$

In what follows we illustrate the Proposition A in the 4- and 6-dimensional cases:

Case $n = 2$. We may assume, without loss of generality, that the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_\pm = \pm i$. In the coordinate system (x_1, x_2, r, θ) with $x_3 = r cos\theta$ and $x_4 = r sen\theta$, $\tilde{X}_k$ is represented by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varphi_2(x_1, r^2) \\ \dot{r} = 0 \\ \dot{\theta} = 1 + \tilde{\varphi}_3(x_1, r^2). \end{cases}$$

Taking θ as the time we consider the auxiliary system X_1 in $\mathbb{R}^3$ expressed by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varphi_2(x_1, r^2) \end{cases}$$

for each $r^2 = k > 0$.

The later system allows us to understand the phase portrait of the original X_0.

For example, when $\varphi_2(x_1, r^2) = a_1 x_1^2 + a_2 r^2 + O(3)$ with $a_1 \cdot a_2 < 0$ and $r = c > 0$ we observe that any equilibrium of X_1 away from $r = 0$ corresponds to a periodic orbit of $\tilde{X}_k$. So $\tilde{X}_k$ possesses:

i) two one-parameter families of periodic orbits (of saddle and elliptical types) converging to 0; as one moves along the family towards 0 the minimal period tends to 2π.

ii) two one-parameter family of homoclinic orbits at each periodic orbit of saddle type converging to 0.

For $r = 0$ we have a cusp singularity.

Case $n = 3$**.** As above consider on the (x_3, x_4, x_5, x_6)-*space* the bi-polar coordinate system (r, θ, ρ, ψ).

We get then:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \varphi_2\big(x_1, r^2, \rho^2\big) \\ \dot{r} = 0 \\ \dot{\theta} = \alpha + \eta_3\big(x_1, r^2, \rho^2\big) \\ \dot{\rho} = 0 \\ \dot{\psi} = \beta + \eta_4\big(x_1, r^2, \rho^2\big). \end{cases}$$

So on $r = k_1,\ \rho = k_2$ with $k_1, k_2 > 0$ we have:

i) The auxiliary system contains two critical points that correspond to two invariant 2-tori T_1 and T_2.

ii) Corresponding to the periodic orbits of the auxiliary system there is a one-parameter family of invariant 3-tori terminating at T_1 and originating at an invariant "Topological 3-Torus" T_3 that contains T_2.

iii) the orbits in T_3 not in T_2 are bi-asymptotic to T_2.

If $r = k_1$ and $\rho = 0$ or $r = 0$ and $\rho = k_2$ the configuration is similar to the case $n = 2$.

4 LSR and Proof of Theorem B

The main goal of this section is to verify how persistent are the one-parameter families of periodic orbits detected for the truncated system $\tilde{X}_2$ when the original vector field X or the external parameter λ are considered. This approach has a certain natural structure allowing a narrow comparison between different cases. Some of the strategies in the proofs use similar arguments and methods in many different situations.

4.1 Generic Bifurcations

Fix a coordinate system such that the 2-*jet* of any $X_0 \in \chi_0^{2n}$ is written in normal formal. Recall that we consider both cases:

i) 0: *non-resonant* or

ii) 0: p: q-*resonant* with $p + q > 2$.

It is worth to point out that the systems treated here appear generically in one-parameter families of reversible vector fields.

For $n = 2$, in the reversible universe a generic one-parameter family X_λ passing through $X \in$ at $\lambda = 0$ is expressed as:

$$\begin{cases} \dot{x}_1 = x_2 + O(3) \\ \dot{x}_2 = \lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) + O(3) \\ \dot{x}_3 = -x_4 - b_1(\lambda)x_4x_1 + O(3) \\ \dot{x}_4 = x_3 + b_1(\lambda)x_3x_1 + O(3). \end{cases}$$

Assuming for instance that $a_1 > 0,\ a_2 < 0$ ($X \in \mathcal{U}_1$) we derive that:

For $\lambda < 0$: the auxiliary system has two equilibria (a center and a saddle-center).

For $\lambda = 0$: the origin is the unique equilibrium.

For $\lambda > 0$: the system has no equilibrium nearby the origin.

The analysis when $a_1 > 0 \cdot a_2 > 0$ ($X \in \mathcal{U}_2$) can be done by solving simple algebraic equations and it will be omitted.

It seems clear that this discussion can be performed in $\mathbb{R}^{2n}$ in a very straightforward way.

4.2 LSR in $\mathbb{R}^{2n}$ (0-*non-resonant* case)

In what follows we are going to analyze the existence of families of periodic orbits for the bifurcation scenario X_λ via the **LSR**. There are a few complications which arise in our context. We must be careful when handling splittings of projections. As we are looking for periodic solutions, the (minimal) period is one of the "unknowns" of the problem. As will be seen in the treatment which follows the circle group S^1 plays an important role in the bifurcation analysis of periodic orbits.

So let

$$\dot{x} = X(x, \lambda); \quad X \in \chi^{2n}(\lambda) \tag{4}$$

Recall that $X(0,0) = 0$.

Consider

$$A_0 = DX(0,0) := \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & -\alpha_1 & & & \\ & & \alpha_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -\alpha_{n-1} \\ & & & & & \alpha_{n-1} & 0 \end{pmatrix}$$

be 0-*non-resonant*.

Denote by $C^0_{2\pi}$ the space of all 2π-periodic continuous functions $x\colon \mathbb{R} \to \mathbb{R}^{2n}$, $n \geq 2$, and let $C^1_{2\pi}$ be the correspondent C^1-subspace.

In $C^0_{2\pi}$ we define the product

$$(x_1, x_2) = \frac{1}{2\pi}\int_0^{2\pi} \langle x_1(t), x_2(t)\rangle dt$$

where $\langle .\,, .\rangle$ is an inner product in $\mathbb{R}^{2n}$.

Consider now the mappings: $F_j\colon C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C^0_{2\pi}$ defined by

$$F_j(x, \sigma, \lambda)(t) = (1+\sigma)\,\alpha_j\,\dot{x}(t) - X(x(t), \lambda), \quad j = 1, \dots, n-1 \tag{5}$$

We recall that if $(x_o, \sigma_o, \lambda_o) \in C^1_{2\pi} \times \mathbb{R} \times \mathbb{R}$ satisfies $F_j(x_o, \sigma_o, \lambda_o) = 0$ then $\tilde{x}(t) := x_o((1+\sigma_o)\,\alpha_j\,t)$ is a $\frac{2\pi}{(1+\sigma_o)\alpha_j}$-periodic solution of (4) for $\lambda = \lambda_o$.

So the problem is carried out to find the zeroes of F_j. In this way to each solution of $F_j = 0$ corresponds a periodic solution of the original system with period near $\frac{2\pi}{\alpha_j}$.

Of course $(0,0,0)$ is always a solution of (5).

Let $L_j := D_1F_j(0,0,0)\colon C^1_{2\pi} \to C^0_{2\pi}$ be given by $L_j\,x(t) = \dot{x}(t) - \frac{1}{\alpha_j}A_0 x(t)$.

Consider $A_0 = S_0 + N_0$ the unique decomposition of A_0 with (S_0) and (N_0) being the semi-simple and nilpotent parts respectively with $S_0N_0 = N_0S_0$.

Denote $V_j := ger\{e_1, e_2, e_{2j+1}, e_{2j+2}\}$, $j = 1, \dots, n-1$ where $e_k{}'s$ are elements of the canonical basis of $\mathbb{R}^{2n}$.

Take the following subspaces in $C^1_{2\pi}$:

$$\mathcal{N}_j := \left\{q;\ q(t) = \exp(tS_0/\alpha_j)v_j,\ \ v_j \in V_j\right\}, \quad j = 1, \dots, n-1.$$

We are putting the solutions of (5) in 1 : 1-correspondence with the solutions of an equation defined in $\mathcal{N}_j$. Hence for each j, define the following subspaces:

$$X_j = \left\{x \in C^1_{2\pi};\, (x, \mathcal{N}_j) = 0\right\} \text{ and}$$
$$Y_j = \left\{y \in C^0_{2\pi};\; (y, \mathcal{N}_j) = 0\right\},\; j = 1, \dots, n-1$$

as the orthogonal complements of $\mathcal{N}_j$ in $C^1_{2\pi}$ and $C^0_{2\pi}$, respectively.

Recall that the dimension of $\mathcal{N}_j$ in $C^1_{2\pi}$ is 4 for every j.

Consider $(q_{j,1}, q_{j,2}, q_{j,3}, q_{j,4})$ with $q_{j,i} = \exp(tS_0/\alpha_j)v_{j,i}$ and $\{v_{j,1} = e_1, v_{j,2} = e_2, v_{j,3} = e_{2j+1}, v_{j,4} = e_{2j+2}\}$ a basis of V_j.

Take the projections $\mathcal{P}_j : C^0_{2\pi} \to C^0_{2\pi}$ defined by

$$\mathcal{P}_j(.) = \sum_{i=1}^{4} \left(q_{j,i}\right)^* (.)q_{j,i} \tag{6}$$

where $(q_{j,i})^*(x) = (q_{j,i}\, , x)$. We get: $\mathrm{Im}(\mathcal{P}_j) = \mathcal{N}_j$, $\mathrm{Ker}(\mathcal{P}_j) = Y_j$, $C^1_{2\pi} = X_j \oplus \mathcal{N}_j$ and $C^0_{2\pi} = Y_j \oplus \mathcal{N}_j$.

Finally we define

$$F_j(x, \sigma, \lambda) = F_j(q_j + x_j, \sigma, \lambda) =: \hat{F}_j(q_j, x_j, \sigma, \lambda), \quad q_j \in \mathcal{N}_j, \quad x_j \in X_j.$$

The proof of next result is in [4].

Lemma 4.1 (Fredholm alternative)**.** *Let $A(t)$ be a matrix in C^0_T and g be in C_T. Then the equation $\dot{x} = A(t)x + g(t)$ has a solution in C_T if and only if $\int_0^T \langle y(t), g(t)\rangle dt = 0$ for every solution y of the adjoint equation $\dot{y} = -A^*(t)y$ with y in C_T.*

As $L_j\left(\mathcal{N}_j\right) \subset \mathcal{N}_j$ the last lemma implies immediately that

Lemma 4.2. *The mappings $\hat{L}_j : = L_j\,|_{X_j} : X_j \to Y_j$ are bijections for every $j = 1, \dots, n-1$.*

In what follows we establish a discussion that will be useful in the sequel.

We have the following equivalence

$$\hat{F}_j(q_j, x_j, \sigma, \lambda) = 0 \;\Leftrightarrow (I - \mathcal{P}_j) \circ \hat{F}_j(q_j, x_j, \sigma, \lambda) = 0$$
$$\mathcal{P}_j \circ \hat{F}_j(q_j, x_j, \sigma, \lambda) = 0\,.$$

So from the Implicity Function Theorem and Lemma 4.2 the equation

$$\hat{F}_j(q_j, x_j, \sigma, \lambda) = 0$$

can be solved as $x_j = x_j^*(q_j, \sigma, \lambda)$.

So (5) can be reduced to

$$\tilde{F}_j(q_j, \sigma, \lambda) := \mathcal{P}_j \circ \hat{F}_j(q_j, {x_j}^*(q_j, \sigma, \lambda), \sigma, \lambda) = 0.$$

On the other hand, it follows from (6) that this equation is satisfied if and only if

$$ {q_{j,i}}^*\big(\hat{F}_j(q_j, {x_j}^*(q_j, \sigma, \lambda), \sigma, \lambda\big) = 0, \quad i = 1, \dots, 4. \tag{7}$$

So (v_j, σ, λ), $v_j = (x_1, x_2, x_{2j+1}, x_{2j+2})$ is a solution of (5) provided that

$$B_j(v_j, \sigma, \lambda) = 0$$

with $B_j : \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ defined by

$$B_j\big(v_j, \sigma, \lambda\big) := \frac{1}{2\pi}\int_0^{2\pi} \exp\big(-tS_{0,j}/\alpha_j\big)\Pi_j \hat{F}_j\big({x_j}^*(v_j, \sigma, \lambda), \sigma, \lambda\big)dt,$$

where

$${x_j}^*\big(v_j, \sigma, \lambda\big) := \exp\big(tS_{0,j}/\alpha_j\big)v_j + {x_j}^*\big(\exp(tS_{0,j}/\alpha_j)v_j, \sigma, \lambda\big).$$

and

$$\Pi_j\big(x_1, x_2, \dots, x_{2j+1}, x_{2j+2}, \dots, x_{2n}\big) = \big(x_1, x_2, x_{2j+1}, x_{2j+2}\big)$$

with

$$S_{0,j} := \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -\alpha_j \\ & & \alpha_j & 0 \end{pmatrix}, \quad j = 1 \dots, n-1.$$

4.2.1 Properties of the mapping B_j

The proof of the following lemma is in [11] and [14].

Lemma 4.3. *Each mapping B_j satisfies:*

(i) $R'B_j(v_j,\sigma,\lambda) = -B_j(R'v_j,\sigma,\lambda)$, *and*

(ii) $s_\phi B_j(v_j,\sigma,\lambda) = B_j(s_\phi v_j,\sigma,\lambda)$, *where* $s_\phi v_j = \exp(-\phi S_{0,j})v_j$ *and*

$$R'\big(x_1, x_2, x_{2j+1}, x_{2j+2}\big) = \big(x_1, -x_2, x_{2j+1}, -x_{2j+2}\big).$$

Recall that we are assuming that the m-jet of (4) *at* 0 *is in* **BNF**. *That implies that $X(x,\lambda) = \lambda e_2 + A_0 x + \tilde{X}(x,\lambda) + r_m(x,\lambda)$, where $x \in \mathbb{R}^{2n}$, $T_m X(x,\lambda) = \lambda e_2 + A_0 x + \tilde{X}(x,\lambda)$ is in normal form and $r_m(x,\lambda) = o(|x|^{m+1})$.*

The proof of the next result will be omitted since it is a slight variation of the proof of Theorem 4.5 in [14].

Proposition 4.4.

(i) $x_j{}^*(v_j,\sigma,\lambda) = \exp(tS_{0,j}/\alpha_j)v_j + o(\|v_j\|^{m+1})$

(ii) $B_j(v_j,\sigma,\lambda) = (1+\sigma)\, S_{0,j}\, v_j - A_{0,j}\, v_j - \lambda\, e_{2,j} - \tilde{X}_j(v_j,\lambda) + o(\|v_j\|^{m+1})$, *where*

$$S_{0,j} = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -\alpha_j \\ & & \alpha_j & 0 \end{pmatrix}, \quad A_{0,j} = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & -\alpha_j \\ & & \alpha_j & 0 \end{pmatrix}, \quad e_{2,j} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\tilde{X}_j = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_{2j+1} \\ \tilde{x}_{2j+2} \end{pmatrix}.$$

Remark 4.5. It is worth mentioning that the reduction above performed can be reproduced *ipse-literis* to the $0\colon p\colon q$-*ressonant* case with $p, q > 1$ and it will be omitted. So Proposition 4.4 remains valid whether applied to the present case.

4.3 LSR in $\mathbb{R}^6$ $(0\colon 1\colon 2$-resonant case)

Let $X \in \chi^6(\lambda)$.

Recall that $X(0,0) = 0$ and so:

$$A_0 = D_1X(0,0) = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -2 \\ & & & & 2 & 0 \end{pmatrix}.$$

We emphasize that we search for periodic solutions of the original system with periods nearby π and 2π. In the first situation the same analysis done in the last section can be performed here by means of $B_2\colon \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$ given by

$$B_2(u_2,\sigma,\lambda) = (1+\sigma)\, S_{0,2}\, u_2 - A_{o,2}\, u_2 - \lambda\, e_{2,2} - \tilde{X}_2(u_2,\lambda) + o\big(\|u_2\|^{m+1}\big)$$

where

$$S_{0,2} = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -2 \\ & & 2 & 0 \end{pmatrix}, \quad A_{0,2} = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & -2 \\ & & 2 & 0 \end{pmatrix}, \quad e_{2,2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{X}_2 = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_5 \\ \tilde{x}_6 \end{pmatrix} \quad \text{and} \quad u_2 = (x_1, x_2, x_5, x_6).$$

In the second situation (periods near 2π) we argue as follows.

First of all, consider the mapping $F_1\colon C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C^0_{2\pi}$ defined by $F_1(x,\sigma,\lambda)(t) = (1+\sigma)\,\dot{x}(t) - X(x(t),\lambda)$.

As before let $L_1 := D_1F_1(0,0,0)\colon C^1_{2\pi} \to C^0_{2\pi}$ be given by $L_1\, x(t) = \dot{x}(t) - A_0 x(t)$.

Define now:

$$\mathcal{N}_1 := \big\{q \in C^1_{2\pi};\ q(t) = \exp(tS_0)u,\ u \in \mathbb{R}^6\big\},$$
$$X_1 = \big\{x \in C^1_{2\pi};\ (x,\mathcal{N}_1) = 0\big\} \quad \text{and} \quad Y_1 = \big\{y \in C^0_{2\pi};\ (y,\mathcal{N}_1) = 0\big\}.$$

Consider $(q_{1,1},\ q_{1,2}\ , q_{1,3}\ , q_{1,4}\ , q_{1,5}\ , q_{1,6})$ with $q_{1,i} = \exp(tS_0)e_i$, where $\{e_i,\ i = 1,\ldots,6\}$ is the canonical basis of $\mathbb{R}^6$.

As above we get in a similar way a reduction given by $B_1 : \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^6$ where

$$B_1(u_1, \sigma, \lambda) := \frac{1}{2\pi} \int_0^{2\pi} \exp(-tS_0) F_1(x_1{}^*(u_1, \sigma, \lambda), \sigma, \lambda) dt$$

and

$$x_1{}^*(u_1, \sigma, \lambda) = \exp(tS_0)u_1 + x_1{}^*(\exp(tS_0)u_1, \sigma, \lambda).$$

The proofs of the followings results are direct and they will be omitted.

Lemma 4.6. *The mapping B_1 satisfies: (i) $R\, B_1(u_1, \sigma, \lambda) = -B_1\,(Ru_1, \sigma, \lambda)$ and (ii) $s_\phi\, B_1(u_1, \sigma, \lambda) = B_1(s_\phi\, u_1, \sigma, \lambda)$, with $s_\phi\, u_1 = \exp\,(-\phi S_0)\, u_1$.*

Proposition 4.7. The mapping B_1 is expressed by

$$B_1(u_2, \sigma, \lambda) = (1 + \sigma)\, S_0\, u_2 - A_0\, u_2 - \lambda\, e_2 - \tilde{X}(u_2, \lambda) + o(|u_2|^{m+1})$$

provided that the *m-jet* (the *m-truncation*) of the system at 0 is in **BNF** and $u_2 = (x_1, x_2, x_3, x_4, x_5, x_6)$.

4.4 Proof of Theorem B.

(a) First we present a brief discussion on the case $n = 2$. It will be very useful in the sequel.

The 2-*truncated* **BNF** system is expressed by:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) \\ \dot{x_3} = -x_4 - b_1(\lambda)x_1x_4 \\ \dot{x_4} = x_3 + b_1(\lambda)x_1x_3. \end{cases}$$

Let's solve $B(x, \sigma, \lambda) = 0$ with $B(x, \sigma, \lambda)$ given in Theorem 4.4.
Let

$$S_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So $B(x, \sigma, \lambda) = \begin{pmatrix} -x_2 \\ -\lambda - a_1(\lambda)x_1^2 - a_2(\lambda)(x_3^2 + x_4^2) \\ -\sigma x_4 + b_1(\lambda)x_1x_4 \\ \sigma x_3 - b_1(\lambda)x_1x_3 \end{pmatrix}$ + higher order terms.

We separate the cases:

(i) $a_1(0)a_2(0) < 0$ (for abuse of terminology we say that $X(x, 0) \in \mathcal{U}_1$);

(ii) $a_1(0)a_2(0) > 0$ (we say that $X(x, 0) \in \mathcal{U}_2$).

We have then:

(a) $-x_2 + O(\|x\|^3) = 0$,

(b) $\lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) + O(\|x\|^3) = 0$,

(c) $-\sigma x_4 + b_1(\lambda)x_1x_4 + O(\|x\|^3) = 0$,

(d) $\sigma x_3 - b_1(\lambda)x_1x_3 + O(\|x\|^3) = 0$.

Consider the following auxiliary system:

$$
\begin{aligned}
&\lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) = 0\\
&-\sigma x_4 + b_1(\lambda)x_1x_4 = 0\\
&\sigma x_3 - b_1(\lambda)x_1x_3 = 0.
\end{aligned}
$$

We see that the problem is reduced to the analysis of the relative position between elements of two 2-parameter families of curves in the plane depending on λ and σ.

Case i) For simplicity assume that $a_1(0) = -a_2(0) = b_1(0) = 1$.

That means that in the (x_1, x_3, x_4)-space we have:

Subcase i1) $\lambda = 0$: the equation (b) represents a cone Σ_b centered in the origin whereas the equation (c) represents an algebraic surface $\Sigma_c = \pi_4 \cup \pi$ and equation (d) represents another algebraic variety $\Sigma_d = \{\pi_3 \cup \pi\}$ where: $\pi = \{x_1 = \sigma\}$, $\pi_j = \{x_j = 0\}$ with $j = 3, 4$.

Observe now that any pair among the following manifolds, Σ_b, π, π_3 and π_4, are in general position. So the solution of the auxiliary system is represented by $S = \Sigma_b \cap \pi \cap \pi_i$. In another words we have that $\gamma_i = \Sigma_b \cap \pi \cap \pi_i$, $i = 3, 4$ is constituted by two distinct points $p_i(\sigma) \cup q_i(\sigma)$. This situation is generic and it persists if we add higher order terms to the auxiliary system. It is worthwhile to mention that in the general case we cannot conclude *a priori* about the symmetric properties of such solutions.

The analysis of the Subcases i2) $\lambda < 0$ and i3) $\lambda > 0$ follow similarly.

Also the case ii) can be performed in a straightforward way. Such analysis will be also omitted.

In conclusion we have: if i) $X(x, 0) \in \mathcal{U}_1$ in the level $\lambda = 0$ we derive the existence of two one-parameter families of periodic orbits terminating at the origin with periods converging to 2π. When $\lambda < 0$ these families split from the origin in such a way each one of them terminates at one of the bifurcated critical points. For $\lambda > 0$ there is only one such family walking around the origin. When ii) $X(x, 0) \in \mathcal{U}_2$ then we deduce that $\lambda = 0$ is a subcritical Hopf bifurcation value since for $\lambda < 0$ we have two families of symmetric periodic orbits terminating at each one of the equilibria.

(b) $n = 3$.

Let $\dot{x} = X(x, \lambda)$ with $x \in \mathbb{R}^6$ and $X(0, 0) = 0$ be such that:

(i) $$A_0 = DX(0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 & \beta & 0 \end{pmatrix};$$

(ii) X is R-*reversible*, with

$$R(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, x_3, -x_4, x_5, -x_6);$$

(iii) $\alpha k_1 + \beta k_2 = 0, \quad k_1, k_2 \in \mathbb{Z} \Rightarrow k_1 = k_2 = 0.$

The truncated normal form is:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) + a_3(\lambda)\big(x_5^2 + x_6^2\big) \\ \dot{x_3} = -x_4\big[\alpha + b_1(\lambda)x_1\big] \\ \dot{x_4} = x_3\big[\alpha + b_1(\lambda)x_1\big] \\ \dot{x_5} = -x_6\big[\beta + b_2(\lambda)x_1\big] \\ \dot{x_6} = x_5\big[\beta + b_2(\lambda)x_1\big]. \end{cases}$$

We assume that $b_1(0) \neq 0$ and $b_2(0) \neq 0$.

Recall that:

$$\mathcal{U}_1 = \big\{X;\ a_1(0) \cdot a_2(0) < 0 \ \ and \ \ a_1(0) \cdot a_3(0) < 0\big\} \quad \text{and}$$
$$\mathcal{U}_2 = \big\{X;\ a_1(0) \cdot a_2(0) > 0 \ \ and \ \ a_1(0) \cdot a_3(0) > 0\big\}.$$

Of course we may consider $\alpha = 1$ and $\gamma = \frac{\beta}{\alpha}$, $\gamma \in \mathbb{Q}^c$. So

$$A_0 = DX(0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & \gamma & 0 \end{pmatrix}.$$

As shown in Proposition 4.4 the **LSR** in the general case is analogous to that one in dimension 4. So we just have to analyze the system:

$$\begin{cases} x_2 + O(3) = 0 & \text{(a)} \\ \lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) + O(3) = 0 & \text{(b)} \\ -\sigma x_4 + b_1(\lambda)x_1x_4 + O(3) & \text{(c)} \\ \sigma x_3 - b_1(\lambda)x_1x_3 + O(3) = 0. & \text{(d)} \end{cases}$$

Hence the discussion of the system is reduced to the 4-dimensional case analysis. So the original system is R-symmetric and at level $\lambda = 0$, when the families approach to 0 the periods converge to $\frac{2\pi}{\alpha}$. As before we may reproduce the above procedure on the generalized eigenspace: $V_1 = ger\{e_1, e_2, e_5, e_6\}$ and so obtain similar results on the space (x_1, x_2, x_5, x_6).

In this way, for $\lambda = 0$, we get four families of periodic orbits terminating at the origin. Moreover, the same considerations made before on the bifurcation phenomenon can be *ipse-literis* formulated here.

In this way, to finish the present proof is enough to appeal directly to the procedure made for $n = 2$ since the analysis was reduced to solve a $4D$ system of four equations. $\square$

4.5 Proof of Theorem B*:

In this case the system X_λ can be written as:

$$\begin{cases} \dot{x}_1 = x_2 + O(3) \\ \dot{x}_2 = \lambda + a_1(\lambda)x_1^2 + a_2(\lambda)\big(x_3^2 + x_4^2\big) + a_3(\lambda)\big(x_5^2 + x_6^2\big) + O(3) \\ \dot{x}_3 = -x_4 - b_1(\lambda)x_1x_4 - c_1(\lambda)\big(x_3x_6 - x_4x_5\big) + O(3) \\ \dot{x}_4 = x_3 + b_1(\lambda)x_1x_3 + c_1(\lambda)\big(x_3x_5 + x_4x_6\big) + O(3) \\ \dot{x}_5 = -2x_6 - b_2(\lambda)x_1x_6 - 2c_2(\lambda)x_3x_4 + O(3) \\ \dot{x}_6 = 2x_5 + b_2(\lambda)x_1x_5 + c_2(\lambda)\big(x_3^2 - x_4^2\big) + O(3). \end{cases} \tag{8}$$

The characterization of the subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ in χ_2^6* will become clear along the performance of this proof.

We search now for periodic orbits of periods nearby 2π of $X(x,\lambda)$ with $X(x,0) \in \mathcal{U}_1 \cup \mathcal{U}_2$.

As before we focus on the zeroes of:

$$\begin{cases} x_2 + O(3) = 0 \\ -\lambda - a_1(\lambda)x_1^2 - a_2(\lambda)\big(x_3^2 + x_4^2\big) - a_3(\lambda)\big(x_5^2 + x_6^2\big) + O(3) = 0 \\ -\sigma x_4 + b_1(\lambda)x_1x_4 + c_1(\lambda)\big(x_3x_6 - x_4x_5\big) + O(3) = 0 \\ \sigma x_3 - b_1(\lambda)x_1x_3 - c_1(\lambda)\big(x_3x_5 - x_4x_6\big) + O(3) = 0 \\ -2\sigma x_6 + b_2(\lambda)x_1x_6 + 2c_2(\lambda)x_3x_4 + O(3) = 0 \\ 2\sigma x_5 - b_2(\lambda)x_1x_5 - c_2(\lambda)\big(x_3^2 - x_4^2\big) + O(3) = 0. \end{cases}$$

In order to find symmetric periodic orbits with period near 2π we have to study the reduced system

$$\begin{cases} -\lambda - a_1(\lambda)x_1^2 - a_2(\lambda)x_3^2 - a_3(\lambda)x_5^2 + O(3) = 0 \\ \sigma x_3 - b_1(\lambda)x_1x_3 - c_1(\lambda)x_3x_5 + O(3) = 0 \\ 2\sigma x_5 - b_2(\lambda)x_1x_5 - c_2(\lambda)x_3^2 + O(3) = 0. \end{cases}$$

Consider, as before, the auxiliary system:

$$\begin{cases} -\lambda - a_1(\lambda)x_1^2 - a_2(\lambda)x_3^2 - a_3(\lambda)x_5^2 = 0 \\ \sigma x_3 - b_1(\lambda)x_1x_3 - c_1(\lambda)x_3x_5 = 0 \\ 2\sigma x_5 - b_2(\lambda)x_1x_5 - c_2(\lambda)x_3^2 = 0. \end{cases} \tag{9}$$

Now, with respect to this system we obtain:

If $X(x,0) \in \mathcal{V}_1 = \big\{X \textit{ given by } (8) \textit{ such that } a_1(0)a_3(0) < 0 \textit{ and } b_2(0) \neq 0\big\}$ then there are two families of symmetric periodic orbits for $\lambda = 0$ and these families are persistent for $\lambda \neq 0$. This periodic orbits are given by

$$x_\sigma^* = \left(\frac{2\sigma}{b_2(\lambda)}, 0, \sqrt{-\frac{1}{a_3(\lambda)}\left(\lambda + \frac{4a_1(\lambda)}{b_2^2}\sigma^2\right)}\right)$$

that is a solution of (9).

On the other hand, if $X(x,0) \in \mathcal{V}_2 = \big\{X \textit{ given by } (8) \textit{ such that } a_1(0)a_3(0) > 0 \textit{ and } b_2(0) \neq 0\big\}$ then we have: if $a_1(0) > 0$ a subcritical Hopf bifurcation occurs where two families of symmetric periodic orbits related to x_σ^* appear and if $a_1(0) < 0$ a supercritical Hopf bifurcation, also related to x_σ^*, occurs. In this case the number of families of periodic orbits is two.

As the intersection of (9) in x^*_σ is transversal these solutions are persistent when the $O(3)$ terms are considered.

A similar analysis can be done in the case of symmetric periodic orbits with period near π.

This ends the proof. □

References

[1] R. Abraham and J.E. Marsden. *Foundations of Mechanics*. 2nd ed., Benjamin-Cummings (1978).

[2] R.L. Devaney. *Reversible diffeomorphisms and flows*. Trans. Am. Math. Soc., **218** (1976), 89–113.

[3] M. Golubitsky, M. Krupa and C. Lim. *Time-reversibility and particle sedimentation*. SIAM J. Appl. Math., **51**(1) (1991), 49–72.

[4] J. Hale. *Ordinary Differential Equations*. New York, Wilay-Interscience (1969).

[5] G. Iooss and K. Kirchgässner. *Water Waves for Small Surface Tension: an approach via normal form*. Proc. Royal Soc. Edinburg, **112 A** (1992), 267–299.

[6] A. Jacquemard, M.F.S. Lima and M.A. Teixeira. *Degenerate resonances and branching of periodic orbits*. Annali di Matematica, **187** (2007), 105–117.

[7] J. Knobloch and A. Vanderbauwhede. *A General Reduction Method for Periodic Solutions in Conservative and Reversible Systems*. J. Dyn. Diff. Equations, **8**(1) (1996), 71–102.

[8] J.S.W. Lamb and J.A.G. Roberts. *Time-reversal symmetry in dynamical systems: A survey*. Physica D, **112** (1998), 1–39.

[9] J. Montaldi, M. Roberts and I. Stewart. *Existence of nonlinear normal modes of symmetric Hamiltonian systems*. Nonlinearity, **3** (1990), 695–730.

[10] J. Moser. *Periodic orbits near equilibria and theorem by Alan Weinstein*. Commun. Pure Appl. Math., **29** (1996), 727–747.

[11] C.W. Shi. *Bifurcations of Symmetric Periodic Orbits near Equilibrium in Reversible Systems*. Int. J. Bifurcation and Chaos, **7** (1997), 569–584.

[12] M.A. Teixeira and J. Yang. *The center-focus problem and reversibility*. J. Differential Equations, **174**(1) (2001), 237–251.

[13] A. Vanderbauwhede. *Local bifurcation and symmetry*. Res. Notes in Math., **75** (1982), Pitman, Boston.

[14] T. Wagenknecht. *An Analytical Study of a two degrees of Freedom Hamiltonian System associated the Reversible Hyperbolic Umbilic*. Thesis, University Ilmenau, Germany.

[15] A. Weinstein. *Normal modes for nonlinear Hamiltonian systems*. Invent. Math., **20** (1973), 47–57.

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