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On C^r-closing for flows on orientable and non-orientable 2-manifolds

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Abstract. We provide an affirmative answer to the C^r-Closing Lemma, $r \geq 2$, for a large class of flows defined on every closed surface.

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Mathematical subject classification: 37C10, 37C15, 37C40, 37B20.

1 Introduction

This paper addresses the open problem C^r Closing Lemma, which can be stated as follows:

Problem 1.1 (C^r Closing Lemma)**.** *Let M be a compact smooth manifold, $r \geq 2$ be an integer, $X \in \mathfrak{X}^r(M)$ be a C^r vector field on M, and $p \in M$ be a non-wandering point of X. Does there exist $Y \in \mathfrak{X}^r(M)$ arbitrarily C^r-close to X having a periodic trajectory passing through p?*

C. Pugh [24] proved the C^1 Closing Lemma for flows and diffeomorphisms on manifolds. As for greater smoothness $r \geq 2$, the C^r Closing Lemma is an open problem even for flows on the 2-torus. Concerning flows on closed surfaces, only a few, partial results are known in the orientable case (see [5, 8, 12]). No affirmative C^r-closing results are known for flows on non-orientable surfaces. In this paper, we present a class of flows defined on every closed surface supporting non-trivial recurrence for which Problem 1.1 has an affirmative answer – see Theorem A. Notice that every closed surface distinct from the sphere, from the

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projective plane and from the Klein bottle (see [18]) admits flows with non-trivial recurrent trajectories (see [14]).

To achieve our results we provide a partial, positive answer to the following local version of the C^r Closing Lemma for flows on surfaces:

Problem 1.2 (Localized C^r Closing Lemma)**.** *Let M be a closed surface, $r \geq 2$ be an integer, $X \in \mathfrak{X}^r(M)$ be a C^r vector field on M, and $p \in M$ be a non-wandering point of X. For each neighborhood V of p in M and for each neighborhood $\mathcal{V}$ of X in $\mathfrak{X}^r(M)$, does there exist $Y \in \mathcal{V}$, with $Y - X$ supported in V, having a periodic trajectory meeting V?*

Due to a C. Pugh's argument (see [8, p. 1887]), it is known that if Problem 1.2 has a positive answer for some class of vector fields $\mathcal{N} \subset \mathfrak{X}^r(M)$ then so does Problem 1.1, considering the same class $\mathcal{N}$. The approach we use to show that a flow has local C^r-closing properties is to make arbitrarily small C^r-twist-perturbations of the original flow along a transversal segment. This requires a tight control of the perturbation: it may happen that a twist-perturbation leaves the non-wandering set unchanged [13] or else collapses it into the set of singularities [5, 9]. More precisely: C. Gutierrez [9] proved that local C^2-closing is not always possible even for flows on the 2-torus; C. Carroll [5] presented a flow (having finitely many singularities) on the 2-torus with poor C^r-closing properties: no arbitrarily small C^2-twist-perturbation yields closing; C. Gutierrez and B. Pires [13] provided a flow on a non-orientable surface of genus four whose non-trivial recurrent behaviour persists under a class of arbitrarily small C^r-twist-perturbations of the original flow. Recently, S. Lloyd [17] found C^r closing perturbations of twist-type for vector fields on the 2-torus with bounded type rotation number which are area-preserving at all saddle-points.

Deeply related to Problem 1.1 is the Peixoto-Wallwork Conjecture that the Morse-Smale vector fields are C^r-dense on non-orientable closed surfaces, which is implied by the following open problem:

Problem 1.3 (Weak C^r Connecting Lemma)**.** *Let M be a non-orientable closed surface, $r \geq 2$ be an integer, and $X \in \mathfrak{X}^r(M)$ have singularities, all of which hyperbolic. Assume that X has a non-trivial recurrent trajectory. Does there exist $Y \in \mathfrak{X}^r(M)$ arbitrarily C^r-close to X having one more saddle-connection than X?*

M. Peixoto [23] gave an affirmative answer to the Weak C^r Connecting Lemma, $r \geq 1$, for flows on orientable closed surfaces whereas C. Pugh [25] solved the Peixoto-Wallwork Conjecture in class C^1.

To give a positive answer to the Peixoto-Wallwork Conjecture, it would be enough to prove either the C^r-Closing Lemma or the Weak C^r Connecting Lemma for the class $\mathscr{G}^\infty(M)$ of smooth vector fields having nontrivial recurrent trajectories and finitely many singularities, all hyperbolic. However there is not a useful classification of vector fields of $\mathscr{G}^\infty(M)$. Surprisingly, this is not contradictory with the fact that the class $\mathscr{F}^\infty(M)$ of smooth vector fields having nontrivial recurrent trajectories and finitely many singularities, all locally topologically equivalent to hyperbolic ones, is essentially classified. The vector fields that are constructed to classify $\mathscr{F}^\infty(M)$ have flat singularities [7]. The answer to either of the following questions is unknown (see [19] for related results):

(1) Given $X \in \mathscr{F}^\infty(M)$, is there a vector field $Y \in \mathscr{G}^\infty(M)$ topologically equivalent to X?

(2) Given $X \in \mathscr{G}^\infty(M)$ which is dissipative at its saddles, is there $Y \in \mathscr{G}^\infty(M)$ topologically equivalent to X but which has positive divergence at some of its saddles?

Considering vector fields of $\mathscr{G}^\infty(M)$ which are dissipative at their saddles, their existence in a broad context was considered by C. Gutierrez [10]. The motivation of this work was to find a C^r- Closing Lemma for all vector fields of $\mathscr{G}^\infty(M)$ whose existence is ensured by the work done in [10]. In this paper we have accomplished this aim. We do not know any other existence result improving that of [10].

2 Statement of the results

Throughout this paper, we shall denote by M a closed Riemannian surface, that is, a compact, connected, boundaryless, C^∞, Riemannian 2-manifold and by $\mathfrak{X}^r_H(M)$ the open subspace of $\mathfrak{X}^r(M)$ formed by the C^r vector fields on M having singularities (at least one), all of which hyperbolic. When M is neither the torus nor the Klein bottle, $\mathfrak{X}^r_H(M)$ is also dense in $\mathfrak{X}^r(M)$. To each $X \in \mathfrak{X}^r_H(M)$ we shall associate its flow $\{X_t\}_{t\in\mathbb{R}}$. Given a transversal segment Σ to $X \in \mathfrak{X}^r_H(M)$ and an arc length parametrization $\theta\colon I \subset \mathbb{R} \to \Sigma$ of Σ, we shall perform the identification $\Sigma = \theta(I) = I$, where I is a subinterval of $\mathbb{R}$. In this way, subintervals of I will denote subsegments of Σ. The transversal segments we use are assumed to be C^∞ smooth. If $P\colon \Sigma \to \Sigma$ is the forward Poincaré Map induced by X on Σ and x belongs to the domain dom (P) of P, we shall denote:

$$DP(x) = D\big(\theta^{-1} \circ P \circ \theta\big)\big(\theta^{-1}(x)\big).$$

Notice that $DP(x)$ does not depend on the particular arc length parametrization θ of Σ and may take positive and negative values. Given $n \in \mathbb{N} \setminus \{0\}$, we let

$$\mathcal{O}_n^-(\partial\Sigma) = \left\{P^{-i}(\partial\Sigma) : 0 \leq i \leq n-1\right\},$$

where $\partial\Sigma$ denotes the set of endpoints of Σ and P^0 is the identity map. In this way, the n-th iterate P^n is differentiable on dom $(P^n) \setminus \mathcal{O}_n^-(\partial\Sigma)$.

Definition 2.1 (Infinitesimal contraction)**.** *Let Σ be a transversal segment to a vector field $X \in \mathfrak{X}_H^r(M)$ and let $P : \Sigma \to \Sigma$ be the forward Poincaré Map induced by X. Given $n \in \mathbb{N} \setminus \{0\}$ and $0 < \kappa < 1$, we say that P^n is an infinitesimal κ-contraction if $|DP^n(x)| < \kappa$ for all $x \in$ dom $(P^n) \setminus \mathcal{O}_n^-(\partial\Sigma)$.*

We say that $N \subset M$ is a *quasiminimal set* if N is the topological closure of a non-trivial recurrent trajectory of X.

Definition 2.2. *We say that $X \in \mathfrak{X}_H^r(M)$ has the infinitesimal contraction property at a subset V of M if for every non-trivial recurrent point $p \in V$, for every $\kappa \in (0,1)$ and for every transversal segment Σ_1 to X passing through p, there exists a subsegment Σ of Σ_1 passing through p such that the forward Poincaré Map $P : \Sigma \to \Sigma$ induced by X is an infinitesimal κ-contraction.*

Given a transversal segment Σ to $X \in \mathfrak{X}_H^r(M)$ passing through a non-trivial recurrent point of X, we let $\mathcal{M}_P(\Sigma)$ denote the set of Borel probability measures on Σ invariant by the forward Poincaré Map $P : \Sigma \to \Sigma$ induced by X. We say that a Borel subset $B \subset \Sigma$ is *of total measure* if $\nu(B) = 1$ for all $\nu \in \mathcal{M}_P(\Sigma)$.

Definition 2.3 (Lyapunov exponents)**.** *We say that $X \in \mathfrak{X}_H^r(M)$ has negative Lyapunov exponents at a subset V of M if for each non-trivial recurrent point $p \in V$ and for each transversal segment Σ_1 passing through p, there exist a subsegment Σ of Σ_1 containing p and a total measure set $W \subset \Omega_+$ such that for all $x \in W$,*

$$\chi(x) = \liminf_{n\to\infty} \frac{1}{n} \log |DP^n(x)| < 0,$$

where $P : \Sigma \to \Sigma$ is the forward Poincaré Map induced by X and $\Omega_+ = \cap_{n=1}^{\infty}$ dom (P^n).

Now we state our results.

Theorem A. *Suppose that $X \in \mathfrak{X}_H^r(M)$, $r \geq 2$, has the contraction property at a quasiminimal set N. For each $p \in N$, there exists $Y \in \mathfrak{X}_H^r(M)$ arbitrarily C^r-close to X having a periodic trajectory passing through p.*

Theorem B. *Suppose that $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, has divergence less or equal to zero at its saddle-points and that X has negative Lyapunov exponents at a quasiminimal set N. Then X has the infinitesimal contraction property at N.*

Theorem C. *Suppose that $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, has the contraction property at a quasiminimal set N. There exists $Y \in \mathfrak{X}^r_H(M)$ arbitrarily C^r-close to X having one more saddle-connection than X.*

3 Preliminares

A transversal segment Σ to $X \in \mathfrak{X}^r_H(M)$ *passes through* $p \in M$ if $p \in \Sigma \setminus \partial\Sigma$. Given $p \in M$, we shall denote by γ_p the trajectory of X that contains p. We may write $\gamma_p = \gamma_p^- \cup \gamma_p^+$ as the union of its negative and positive semitrajectories, respectively. We shall denote by $\alpha(p)$ or $\alpha(\gamma_p)$ (resp. $\omega(p)$ or $\omega(\gamma_p)$) the α-limit set (resp. ω-limit set) of γ_p. The trajectory γ_p is *recurrent* if it is either *α-recurrent* (i.e. $\gamma_p \subset \alpha(\gamma_p)$) or *$\omega$-recurrent* (i.e. $\gamma_p \subset \omega(\gamma_p)$). A recurrent trajectory is either *trivial* (a singularity or a periodic trajectory) or *non-trivial*. A point $p \in M$ is *recurrent* (resp. *non-trivial recurrent, ω-recurrent,...*) according to whether γ_p is recurrent (resp. non-trivial recurrent, ω-recurrent...). We say that $N \subset M$ is a *quasiminimal set* if N is the topological closure of a non-trivial recurrent trajectory of X. There are only finitely many quasiminimal sets $\{N_j\}_{j=1}^m$, all of which are invariant. Furthermore, every non-trivial recurrent trajectory is a dense subset of exactly one quasiminimal set (see [7, p. 18, Structure Theorem]).

Proposition 3.1. *Let N be a quasiminimal set of $X \in \mathfrak{X}^r_H(M)$. Suppose that for some non-trivial recurrent point $p \in N$, there exist a transversal segment Σ to X passing through p, $(\kappa, n) \in (0,1) \times \mathbb{N}$, and $L > 0$ such that the forward Poincaré Map $P\colon \Sigma \to \Sigma$ induced by X has the following properties:*

(a) *The n-th iterate P^n is an infinitesimal κ-contraction;*

(b) $\sup\left\{|DP(x)|\colon x \in \operatorname{dom}(P)\right\} \leq L$.

Then X has the infinitesimal contraction property at N.

Proof. We claim that

(a) for every $K \in (0,1)$ there exists a subsegment Σ_K of Σ passing through p such that the forward Poincaré Map $P_K\colon \Sigma_K \to \Sigma_K$ induced by X is an infinitesimal K-contraction.

In fact, let $L_0 = \max\{1, L^{n-1}\}$ and $d \in \mathbb{N}$ be such that $L_0\kappa^d < K$. We shall proceed considering only the case in which p is nontrivial α-recurrent. We can take a subsegment Σ_K of Σ passing through p such that $\Sigma_K \subset \mathrm{dom}\,(P^{-dn})$ and $\Sigma_K, P^{-1}(\Sigma_K), \cdots, P^{-dn}(\Sigma_K)$ are paiwise disjoint. Hence, if $P_K\colon \Sigma_K \to \Sigma_K$ is the forward Poincaré Map induced by X, then, for all $q \in \mathrm{dom}\,(P_K)$, there exists $m(q) > dn$ such that $P_K(q) = P^{m(q)}(q)$. In this way, since the function $m\colon q \mapsto m(q)$ is locally constant, $|DP_K(q)| = |DP^{m(q)}(q)| \leq L_0\kappa^d < K$ for all $q \in \mathrm{dom}\,(P_K) \setminus P_K^{-1}(\partial\Sigma_K)$. This proves (a).

Let $q \in N$ be a nontrivial recurrent point. Now we shall shift the property obtained in (a) to any segment $\widetilde{\Sigma}$ transversal to X passing through q. We shall only consider the case in which q is non-trivial α-recurrent and so γ_q^- is dense in N.

Let $K \in (0,1)$ and take $p_1 \in (\gamma_q^- \cap \Sigma_{K/2}) \setminus \{p\}$. Select a subsegment Σ_1 of $\Sigma_{K/2}$ passing through p_1 and a subsegment $\widetilde{\Sigma}_K$ of $\widetilde{\Sigma}$ passing through q such that the forward Poincaré Map $T\colon \Sigma_1 \to \widetilde{\Sigma}_K$ is a diffeomorphism and, for all $x \in \Sigma_1$, $y \in \widetilde{\Sigma}_K$, $|DT(x)DT^{-1}(y)| < 2$. This implies that the forward Poincaré Map $\widetilde{P}_K\colon \widetilde{\Sigma}_K \to \widetilde{\Sigma}_K$ will be an infinitesimal K-contraction because

$$|D\widetilde{P}_K(y)| = |D(T \circ P_1 \circ T^{-1})(y)| \leq 2|DP_1(z)| < K,$$

where $P_1\colon \Sigma_1 \to \Sigma_1$ is the forward Poincaré Map induced by X and $T(z) = y$. □

Definition 3.2 (Flow box)**.** *Let $X \in \mathfrak{X}^r_H(M)$ and let Σ_1, Σ_2 be disjoint, compact transversal segments to X such that the forward Poincaré Map $T\colon \Sigma_1 \to \Sigma_2$ induced by X is a diffeomorphism. For each $p \in \Sigma_1$, let $\tau(p) = \min\{t > 0\colon X_t(p) \in \Sigma_2\}$. The compact region $\{X_t(p)\colon p \in \Sigma_1,\ 0 \leq t \leq \tau(p)\}$ is called a flow box of X.*

Theorem 3.3 (Flow box theorem)**.** *Let $U \subset M$ be an open set, $X \in \mathfrak{X}^r_H(U)$, $\Sigma \subset U$ be a (C^∞ smooth) compact transversal segment to X and $p \in \Sigma \setminus \partial\Sigma$. There exist $\epsilon > 0$ arbitrarily small such that $B = B(\Sigma, \epsilon) = \{X_t(p)\colon p \in \Sigma\ , t \in [-\epsilon, 0]\}$ is a flow box of X, and a C^r-diffeomorphism $h\colon B \to [-\epsilon, 0] \times [a, b]$ such that $h(p) = (0,0)$, $h(\Sigma) = \{0\} \times [a, b]$, $h|_\Sigma$ is an isometry and $h_*(X|_B) = (1,0)|_{[-\epsilon,0]\times[a,b]}$, where $a < 0 < b$, $(1,0)$ is the unit horizontal vector field in $\mathbb{R}^2$ and $h_*(X|_B)$ is the pushforward of the vector field $X|_B$ by h. The map h is denominated a C^r-rectifying diffeomorphism for B.*

Proof. See Palis and de Melo [21, Tubular Flow Theorem, p. 40]. □

Definition 3.4. *Given a compact transversal segment Σ to $X \in \mathfrak{X}^r_H(M)$, $p \in \Sigma \setminus \partial\Sigma$ and $\epsilon > 0$ small, we say that $B(\Sigma, \epsilon) = \{X_t(p) : t \in [-\epsilon, 0], p \in \Sigma\}$ is a flow box of X ending at Σ or at p. We say that $B(\Sigma, \epsilon)$ is arbitrarily thin if ϵ can be taken arbitrarily small and we say that $B(\Sigma, \epsilon)$ is arbitrarily small if $B(\Sigma, \epsilon)$ can be taken contained in any neighborhood of p.*

Next lemma will be used in the proofs of Theorem 5.5 and Theorem 6.4.

Lemma 3.5. *Suppose that $X \in \mathfrak{X}^r_H(M)$ has the infinitesimal contraction property at a non-trivial recurrent point $p \in M$ of X. There exist an arbitrarily small flow box B_0 of X ending at p and an arbitrarily small neighborhood $\mathcal{V}_0$ of X in $\mathfrak{X}^r_H(M)$ such that every $Z \in \mathcal{V}_0$, with $Z - X$ supported in B_0, has the infinitesimal contraction property at B_0.*

Proof. Let $\Sigma_1 = (a_1, b_1)$ be a transversal segment to X passing through p such that the forward Poincaré Map $P_1 : \Sigma_1 \to \Sigma_1$ induced by X is an infinitesimal κ-contraction for some $\kappa \in (0, 1)$. Let $[a, b] \subset (a_1, b_1)$ be a compact subsegment passing through p and let $B_0 = B([a, b], \epsilon)$ be a flow box. There exists a neighborhood $\mathcal{V}_1$ of X in $\mathfrak{X}^r_H(M)$ such that for every $Z \in \mathcal{V}_1$ with $Z - X$ supported in B_0 we have that B_0 is still a flow box of Z. In particular, for every $Z \in \mathcal{V}_1$ such that $Z - X$ supported in B_0, dom (P_Z) = dom (P_1), where P_Z denotes the forward Poincaré Map induced by Z on (a_1, b_1). Given $\delta > 0$ satisfying $0 < \kappa + \delta < 1$, by the continuity of the map $Z \mapsto DP_Z$, there exists a neighborhod $\mathcal{V}_0 \subset \mathcal{V}_1$ of X such that for every $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 we have that $|DP_Z(w)| < |DP_1(w)| + \delta < \kappa + \delta < 1$ for all $w \in$ dom (P_1). Hence P_Z is an infinitesimal $(\kappa + \delta)$-contraction. The rest of the proof follows as in the proof of Proposition 3.1 by recalling that the trajectory of every non-trivial recurrent point of Z in B_0 meets (a, b). □

4 Topological dynamics

Let $X \in \mathfrak{X}^r_H(M)$. We say that $N \subset M$ is an *invariant set of X* if $X_t(N) \subset N$ for all $t \in \mathbb{R}$. We say that $K \subset N$ is a *minimal set of X* if K is compact, nonempty and invariant, and there does not exist any proper subset of K with these properties. We shall need the following lemmas from topological dynamics.

As every vector field of $\mathfrak{X}^r_H(M)$ has singularities, the Denjoy-Schwartz Theorem (see [26] or [27, pp. 39–40]) implies that

Lemma 4.1. *Let $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$. Then any minimal set of X is either a singularity or a periodic trajectory.*

Definition 4.2. *A graph of $X \in \mathfrak{X}^r_H(M)$ is a connected closed subset of M consisting of saddle-points and separatrices such that:*

(a) *the ω-limit and the α-limit set of each separatrix of the graph are saddle-points;*

(b) *each saddle-point in the graph has at least one stable and one unstable separatrix in the graph.*

The proof of the following lemma can be found in [20, Theorem 2.6.1].

Lemma 4.3. *Let $X \in \mathfrak{X}^r_H(M)$ and let $p \in M$. Then $\omega(p)$ (resp. $\alpha(p)$) is exactly one of the following sets: a singularity, a periodic trajectory, a graph, or a quasiminimal set.*

Remark 4.4. It shoud be remarked that if a graph σ is contained in the ω-limit of a trajectory γ_p of $X \in \mathfrak{X}^r_H(M)$ then there exists an open annulus $A \subset M$ containing p such that σ is a connected component of the boundary of A and $\omega(p) = \sigma$ for all $p \in A$. In this case, we say that the graph is *attracting*.

Lemma 4.5. *Let N be a quasiminimal set of $X \in \mathfrak{X}^r_H(M)$. Then every trajectory of N is either a saddle-point or a saddle-connection or else a non-trivial recurrent trajectory dense in N (which may possibly be a saddle-separatrix.)*

Proof. See [20, Theorem 2.4.2, pp. 31–32]. □

Lemma 4.6. *Let $X \in \mathfrak{X}^r_H(M), r \geq 2$, and let N be a quasiminimal set of X. Then there exist saddle-separatrices $\sigma_1, \sigma_2 \subset N$, both dense in N, such that $\alpha(\sigma_1) = N = \omega(\sigma_2)$.*

Proof. Firstly let us prove that X has singularities in N and that all of them are hyperbolic saddle-points. If this was not the case, then N would contain no singularities and, by Lemma 4.5, N would be a minimal set of X contradicting Lemma 4.1.

We shall only prove that N contains dense unstable separatrices. Suppose by contradiction that

($\star$) every unstable separatrix $\sigma \subset N$ is a saddle-connection.

Let $\gamma \subset N$ be a non-trivial recurrent trajectory. Without loss of generality, we may assume that γ is ω-recurrent. Hence, γ^+ is dense in N and in particular

accumulates in all the saddle-points of N. By standard arguments of topological dynamics (see [21, p. 140]), one may show that ($\star$) implies that $\omega(\gamma^+)$ is a graph, which so is a trapping region contradicting the density of γ^+ in N. Consequently, ($\star$) is false and there exists an unstable separatrix in N which is not a saddle-connection. By Lemma 4.5, in a quasiminimal set N every separatrix which is not a saddle-connection is necessarily a non-trivial recurrent trajectory dense in N. □

Definition 4.7. *Let $X \in \mathfrak{X}^r_H(M)$ and let σ be a non-trivial recurrent unstable separatrix of a saddle-point s. We say that a transversal segment Σ to X is σ-adapted if σ (oriented as starting at s) intersects Σ infinitely many times and the first two of such intersections are precisely the endpoints of Σ.*

Lemma 4.8. *Let σ be a non-trivial recurrent unstable saddle-separatrix of $X \in \mathfrak{X}^r_H(M)$. Then every transversal segment $\Sigma_1 = (a_1, b_1)$ to X intersecting σ contains a compact subsegment $[a, b] \subset (a_1, b_1)$ which is σ-adapted.*

Proof. Orient σ so that it starts at the saddle-point $\alpha(\sigma)$. Let p_1, p_2, p_3 be the first three points at which σ intersects (a_1, b_1) and denoted in such a way that $a_1 < p_1 < p_2 < p_3 < b_1$. If σ accumulates in p_2 from above (resp. from below) then $[p_2, p_3]$ (resp. $[p_1, p_2]$) will be σ-adapted. □

Lemma 4.9. *Let $X \in \mathfrak{X}^r_H(M)$, $\Sigma = [a, b]$ be a transversal segment to X passing through a non-trivial recurrent point of X and $P\colon [a, b] \to [a, b]$ be the forward Poincaré Map induced by X. Then* dom $(P) \setminus \{a, b\}$ *is properly contained in (a, b) and consists of finitely many open intervals such that if $s \notin \{a, b\}$ is an endpoint of one of these intervals then the positive semi-trajectory γ_s^+ starting at s goes directly to a saddle-point without returning to $[a, b]$.*

Proof. The proof of this lemma can be found in Palis and de Melo [21, pp. 144–146] or in Peixoto [23]. □

5 C^r-connecting results

Definition 5.1. *Given $X \in \mathfrak{X}^r_H(M)$ and a flow box B of X, we shall denote by $\mathcal{A}(B, X)$ the set of the vector fields $Y \in \mathfrak{X}^r_H(M)$ supported in B such that for all $\lambda \in [0, 1]$, B is still a flow box of $X + \lambda Y$.*

In next lemma we assume that the domain of the forward Poincaré Map P is non-empty. In the applications of Lemma 5.2 and Theorem 5.3, p will be a non-trivial recurrent point.

Lemma 5.2. *Let $X \in \mathfrak{X}^r_H(M)$ be smooth in a neighborhood V_0 of a point $p \in M$ and let $\Sigma = [a, b] \subset V_0$, with $a < 0 < b$, be a transversal segment to X passing through $p = 0$. There exist an arbitrarily thin flow box $B = B([a, b], \epsilon)$ contained in V_0, and $Y \in \mathcal{A}(B, X) \subset \mathfrak{X}^r_H(M)$ arbitrarily C^r-close to the zero-vector-field such that for each $\lambda \in [0, 1]$ the forward Poincaré Map $P_\lambda \colon [a, b] \to [a, b]$ induced by $X + \lambda Y$ is of the form $P_\lambda = E_\lambda \circ P$, where $P = P_0$, E_0 is the identity map, $c = \min\{-a, b\}$, $\delta \in (0, c/8)$, and $E_\lambda \colon [a, b] \to [a, b]$ is a C^r diffeomorphism satisfying the following conditions:*

$$E_\lambda(x) - x = \lambda\delta, \qquad x \in [-4\delta, 4\delta], \tag{1}$$

$$E_\lambda(x) - x \leq \lambda\delta, \qquad x \in [a, b]. \tag{2}$$

Proof. By Theorem 3.3, there exist $\epsilon > 0$ arbitrarily small, a flow box $B = B([a, b], \epsilon) \subset V_0$, and a C^{r+1}-rectifying diffeomorphism $h \colon B \to [-\epsilon, 0] \times [a, b]$. Let $\phi_1 \colon [-\epsilon, 0] \to [0, 1]$ and $\phi_2 \colon [a, b] \to [0, 1]$ be smooth functions such that

$$(\phi_1)^{-1}(1) = [-8\epsilon/10, -2\epsilon/10], \quad (\phi_1)^{-1}(0) = [-\epsilon, 0] \setminus [-9\epsilon/10, -\epsilon/10],$$
$$(\phi_2)^{-1}(1) = [-6\delta, 6\delta], \quad (\phi_2)^{-1}(0) = [a, b] \setminus [-7\delta, 7\delta].$$

Let $Y_0 \colon [-\epsilon, 0] \times [a, b] \to \mathbb{R}^2$ be the smooth vector field which at each $(x, y) \in [-\epsilon, 0] \times [a, b]$ takes the value:

$$Y_0(x, y) = (1, 0) + \eta\phi_1(x)\phi_2(y)(0, \delta),$$

where $\eta > 0$ is a positive constant such that the positive semitrajectory $\gamma^+_{(-\epsilon, -4\delta)}$ of Y_0 starting at $(-\epsilon, -4\delta)$ intersects $\{0\} \times [a, b]$ at the point $(0, -3\delta)$. By construction, for each $y \in [-4\delta, 4\delta]$, the positive semitrajectory $\gamma^+_{(-\epsilon, y)}$ of Y_0 starting at $(-\epsilon, y)$ is an upward shift of $\gamma^+_{(-\epsilon, -4\delta)}$ and so intersects $\{0\} \times [a, b]$ at $(0, y + \delta)$. Define $Y \in \mathfrak{X}^r_H(M)$ to be a vector field supported in B such that $Y|_B = (h^{-1})_*(Y_0)$. Accordingly,

$$(X + \lambda Y)|_B = (h^{-1})_*((1, 0) + \lambda Y_0).$$

Recall that by Theorem 3.3, the map h takes isometrically $[a, b]$ onto $\{0\} \times [a, b]$. By construction, the one-parameter family of vector fields $X + \lambda Y$ has all the required properties. □

Theorem 5.3. *Let $X \in \mathfrak{X}^r_H(M)$, σ be a non-trivial recurrent unstable saddle-separatrix, $\Sigma = [a, b]$ be a σ-adapted transversal segment to X, $B = B([a, b], \epsilon)$ be a flow box of X and $Y \in \mathcal{A}(B, X)$. If $q \in [a, b]$ is the third intersection of σ with $[a, b]$ then either of the following alternatives happens:*

(a) *for some $\lambda \in [0, 1]$, $[a, b]$ intersects a saddle-connection of $X + \lambda Y$ or,*

(b) *for every $(\lambda, n) \in [0, 1] \times \mathbb{N}$, the point q belongs to* dom (P^n_λ) *and $P^n_\lambda(q)$ depends continuously on λ. In this case, for each $\lambda \in [0, 1]$, the sequence $\{P^n_\lambda(q)\}_{n\in\mathbb{N}}$ accumulates in a point of $[a, b]$ belonging, with respect to $X + \lambda Y$, to either a closed trajectory or to a non-trivial recurrent trajectory, where $P_\lambda : [a, b] \to [a, b]$ denotes the forward Poincaré map induced by $X + \lambda Y$.*

Proof. Assume that (a) does not happen. Let us prove that then (b) occurs. Firstly we have to show that for every $(\lambda, n) \in [0, 1] \times \mathbb{N}$, the point q belongs to dom (P^n_λ). Suppose that this does not happen. So for some $(\lambda_1, n_1) \in (0, 1] \times \mathbb{N} - \{0\}$, we have that $q \in$ dom $(P^{n_1-1}_\lambda)$ for all $\lambda \in [0, 1]$, and $q \notin$ dom $(P^{n_1}_{\lambda_1})$. Hence, we have that $P^{n_1-1}_{\lambda_1}(q)$ does not belong to dom $(P_{\lambda_1}) =$ dom (P_0) whereas $P^{n_1-1}_0(q) \in$ dom (P_0). By construction, $P^{n_1-1}_\lambda(q)$ depends continuously on λ, and so for some $\lambda_2 \in [0, \lambda_1]$, $P^{n_1-1}_{\lambda_2}(q)$ intersects the boundary of dom (P_0). By Lemma 4.9, $X + \lambda_2 Y$ has a saddle-connection intersecting $[a, b]$, which contradicts the initial assumption. Therefore, the first part of (b) is proved. The second part of (b) follows from Lemma 4.3 since the existence of an attracting graph intersecting $[a, b]$ would imply (a). □

In the proof of next lemma we shall use the fact that a transversal segment $\Sigma = [a, b]$ to $X \in \mathfrak{X}^r_H(M)$ may also be represented by $[a + s, b + s]$, for any $s \in \mathbb{R}$. Henceforth, if A is a subset of M then $\overline{A}$ will denote its topological closure.

Lemma 5.4. *Let $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, be smooth in a neighborhood V_0 of a non-trivial recurrent point $p \in M$. Assume that X has the infinitesimal contraction property at p. Given a neighborhood V of p, there exist a flow box $B \subset V$ and $Y \in \mathcal{A}(B, X)$ arbitrarily C^r-close to the zero-vector-field such that for some $\lambda \in [0, 1]$, $X + \lambda Y$ has a saddle-connection meeting B.*

Proof. By Lemma 4.6, there exist non-trivial recurrent saddle-separatrices σ_1, σ_2 such that $\omega(\sigma_2) \cap \alpha(\sigma_1) = \overline{\gamma_p}$. Let $\Sigma_1 = [a_1, b_1] \subset V_0 \cap V$ be a transversal segment to X passing through p such that P_{Σ_1} is an infinitesimal κ-contraction for some $0 < \kappa < 0.1$. By Lemma 4.8, there exists a σ_2-adapted

subsegment $\Sigma = [a, b] \subset [a_1, b_1]$. Let $\overline{p} \in (a, b)$ be the last intersection of σ_1 with (a, b). Accordingly, $\overline{p}$ is a non-trivial recurrent point. Modulo shifting the interval $[a_1, b_1]$, we may assume that $a < 0 < b$ and $\overline{p} = 0$. Let $B = B([a, b], \epsilon) \subset V_0 \cap V$ be a flow box for some $\epsilon > 0$. By Lemma 5.2, there exists $Y \in \mathcal{A}(B, X)$ arbitrarily C^r-close to the zero-vector-field such that the forward Poincaré Map $P_\lambda = E_\lambda \circ P$ induced by $X + \lambda Y$ on $[a, b]$ has the properties (1) and (2). We shall consider only the case in which 0 is an accumulation point of $\sigma_2 \cap [a, 0)$. Let $q \in \sigma_2 \cap [a, b]$ be the third intersection of σ_2 with $[a, b]$.

Suppose by contradiction that, for all $\lambda \in [0, 1]$, $X + \lambda Y$ has no saddle-connections. Then by Theorem 5.3, for all $(\lambda, n) \in [0, 1] \times \mathbb{N}$, the point q belongs to dom (P_λ^n) and $P_\lambda^n(q)$ depends continuously on λ. By (2) of Lemma 5.2 and by proceeding inductively, we may see that, for all integer $n \geq 1$,

$$|P \circ (E_\lambda \circ P)^{n-1}(q) - P^n(q)| \leq \kappa\delta\big(1 + \kappa + \cdots + \kappa^{n-2}\big) \leq \frac{\kappa\delta}{1-\kappa}.$$

As 0 is an accumulation point of $\sigma_2 \cap [a, 0)$ there exists $N \in \mathbb{N}$ such that $P^N(q) \in [-\kappa\delta, 0]$. Therefore,

$$P \circ (E_1 \circ P)^{N-1}(q) \geq P^N(q) - \frac{\kappa\delta}{1-\kappa} \geq -\kappa\delta - \frac{\kappa\delta}{1-\kappa} \geq -3\kappa\delta.$$

Hence, by (1) of Lemma 5.2 and by the fact that $0 < \kappa < 0.1$,

$$\begin{aligned}(E_1 \circ P)^N(q) &= E_1 \circ \big(P \circ (E_1 \circ P)^{N-1}\big)(q) \\ &= P \circ \big(E_1 \circ P\big)^{N-1}(q) + \delta \geq -3\kappa\delta + \delta > 0\,.\end{aligned}$$

This implies that there exists $\lambda \in [0, 1]$ such that $P_\lambda^N(q) = (E_\lambda \circ P)^N(q) = 0$ (see (b) of Theorem 5.3). That is, $X + \lambda Y$ has a saddle-connection passing through 0. This contradiction proves the lemma. □

Theorem 5.5. *Suppose that $\mathfrak{X}^r_H(M)$, $r \geq 2$, has the infinitesimal contraction property at a non-trivial recurrent point p. Then, given neighborhoods V of p in M and $\mathcal{V}$ of X in $\mathfrak{X}^r(M)$, there exist $Z \in \mathcal{V}$, with $Z - X$ supported in V, having either a periodic trajectory meeting V or a saddle-connection meeting V.*

Proof. Let be given neighborhoods V of p in M and $\mathcal{V}$ of X in $\mathfrak{X}^r_H(M)$. By Lemma 3.5, there exist a flow box $B_0 \subset V$ and a neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of X in $\mathfrak{X}^r_H(M)$ such that every $Z \in \mathcal{V}_0$, with $Z - X$ supported in B_0, has the

infinitesimal contraction property at B_0. By the proof of Lemma 3.5 and by Lemma 4.8, we may assume that $B_0 = B(\Sigma, \epsilon)$, where Σ is a σ-adapted transversal segment to X for some non-trivial recurrent unstable saddle-separatrix σ. By shrinking $\mathcal{V}_0$ if necessary, we may assume that for every $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 we have that $Z - X \in \mathcal{A}(B, X)$. Suppose, by contradiction, that every vector field in $\mathcal{V}_0$ with $Z - X$ supported in B_0 has neither periodic trajectories meeting B_0 nor saddle-connections meeting B_0. We claim that, under these assumptions, every $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has a non-trivial recurrent point in the interior of B_0. Indeed, by taking $\lambda = 1$ in (b) of Theorem 5.3, we get that every $Z = X + (Z - X) \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has a non-trivial recurrent point intersecting the boundary of B_0. Since B_0 is still a flow box of Z, we have that the interior of B_0 has non-trivial recurrent points of Z. This proves the claim. Now let $Z_1 \in \mathcal{V}_0$ be a C^r vector field which is smooth in B_0 and is such that $Z_1 - X$ supported in B_0. By the claim, Z_1 has a non-trivial recurrent point p_1 in the interior of B_0, and Z_1 has the infinitesimal contraction property at B_0. By Lemma 5.4, there exist a flow box $B \subset V$ and $Z_2 \in \mathcal{V}_0$, with $Z_2 - X$ supported in B, having a saddle-connection meeting B. This contradiction finishes the proof. □

6 C^r-closing results

An *interval exchange transformation* or simply an *iet* is an injective map $E\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ of the unit circle, differentiable everywhere except possibly at finitely many points and such that for all $x \in \text{dom}\ (E)$ (its domain of definition), $|DE(x)| = 1$. The trajectory of E passing through $x \in \mathbb{R}/\mathbb{Z}$ is the set $O(x) = \{E^n(x)\colon n \in \mathbb{Z} \text{ and } x \in \text{dom}\ (E^n)\}$. We say that E is *minimal* if $O(x)$ is dense in $\mathbb{R}/\mathbb{Z}$ for every $x \in \mathbb{R}/\mathbb{Z}$. Given a transversal circle C to $X \in \mathfrak{X}^r_H(M)$, we say that the forward Poincaré Map $P\colon C \to C$ is *topologically semiconjugate* to an iet $E\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ if there is a monotone continuous map $h\colon C \to \mathbb{R}/\mathbb{Z}$ of degree one such that $E \circ h(x) = h \circ P(x)$ for all $x \in \text{dom}\ (P)$.

We shall need the following structure theorem due to Gutierrez [7]. We should remark that in this theorem below, the item (d) although not explicitly stated in [7] follows from the proof given therein and from the fact that X has finitely many singularities.

Theorem 6.1. *Let $X \in \mathfrak{X}^r_H(M)$. The topological closure of the non-trivial recurrent trajectories of X determines finitely many quasiminimal sets $N_1, N_2 \ldots, N_m$. For each $1 \leq i \leq m$, there exists a transversal circle C_i*

to X intersecting every non-trivial recurrent trajectory of $X|_{N_i}$ such that if $P_i : C_i \to C_i$ is the forward Poincaré Map induced by X on C_i then:

(a) *Either $N_i \cap C_i = C_i$ or $N_i \cap C_i$ is a Cantor set;*

(b) *$N_j \cap C_i = \emptyset$, for all $j \in \{1, 2 \ldots, i-1, i+1, \ldots, m\}$;*

(c) *P_i is topologically semiconjugate to a minimal interval exchange transformation $E_i : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$;*

(d) *For each $q \in C_i$, $\gamma_q \cap C_i$ is an infinite set.*

We call the circle C_i a special transverse circle for N_i.

Corollary 6.2. *Let $X \in \mathfrak{X}^r_H(M)$ and let N be a quasiminimal set. Given a transversal segment Σ_1 passing through a non-trivial recurrent point $p \in N$, there exists a subsegment Σ of Σ_1 passing through p such that if $z \in \Sigma$ then either $\alpha(z) = N$ or $\omega(z) = N$. In particular, either $z \in \cap_{n=1}^{\infty} \mathrm{dom}\,(P^n)$ or $z \in \cap_{n=1}^{\infty} \mathrm{dom}\,(P^{-n})$, where $P : \Sigma \to \Sigma$ is the forward Poincaré Map induced by X.*

Proof. Let C be a special transversal circle for N. There exist a subsegment Σ of Σ_1 passing through p and a subsegment Γ of C such that the forward Poincaré Map $T : \Sigma \to \Gamma$ induced by X is a diffeomorphism. Since C is free of finite trajectories (by (d) of Theorem 6.1), so is Σ. Hence, by Lemma 4.3, either $\alpha(z)$ or $\omega(z)$ is a quasiminimal set, which by (b) of Theorem 6.1, has to be N. □

Proposition 6.3. *Suppose that $X \in \mathfrak{X}^r_H(M)$ has the infinitesimal contraction property at a non-trivial recurrent point $p \in M$. There exists an arbitrarily small flow box B_0 ending at p and an arbitrarily small neighborhood $\mathcal{V}_0$ of X in $\mathfrak{X}^r_H(M)$ such that either:*

(i) *some $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has a periodic trajectory meeting B_0 or,*

(ii) *every $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has a non-trivial recurrent point in the interior of B_0.*

Proof. By Corollary 6.2, given a transversal segment Σ_1 to X passing through p, there exists a subsegment Σ of Σ_1 passing through p such that for every $z \in \Sigma$, either $\alpha(z) = N$ or $\omega(z) = N$, where $N = \overline{\gamma_p}$. By taking a subsegment of Σ if necessary, we may assume that the forward Poincaré Map $P : \Sigma \to \Sigma$ induced by X is an infinitesimal κ-contraction for some $\kappa \in (0, 1)$.

We claim that $z \in \Sigma \setminus \cap_{n=1}^{\infty} \text{dom}\,(P^n)$ if and only if $\omega(z)$ is a saddle-point. Indeed, if $z \in \Sigma \setminus \cap_{n=1}^{\infty} \text{dom}\,(P^n)$ then there exists $m \in \mathbb{N}$ such that $z \in \text{dom}\,(P^m)$ but $z \notin \text{dom}\,(P^{m+1})$. Hence, $P^m(z) \notin \text{dom}\,(P)$ and by Lemma 4.9, either $\omega(z)$ is a saddle-point or $P^m(z)$ belongs to the open set $\Sigma \setminus \text{dom}\,(P)$. In this last case, there exists a subsegment $I \subset \Sigma$ containing z such that $P^m(I) \subset \Sigma \setminus \text{dom}\,(P)$ and $I \subset \cap_{n=1}^{\infty} \text{dom}\,(P^{-n})$ (by the first part of this proof). Of course, this is impossible since P^{-1} has a uniformly expanding behaviour and Σ has finite length. This proves the claim.

In particular, we have that dom (P) is the whole transversal segment Σ but finitely many points. Let $B_0 = B(\Sigma, \epsilon)$ be a flow box and let $\mathcal{V}_0 \subset \mathfrak{X}^r_H(M)$ be a neighborhood of X such that if $Z \in \mathcal{V}_0$ and $Z - X$ is supported in B_0 then B_0 is still a flow box of Z and so dom $(P_Z) =$ dom (P), where P_Z is the forward Poincaré Map induced by Z on Σ. Hence, for every $Z \in \mathcal{V}_0$ such that $Z - X$ is supported in B_0, dom (P_Z) is the whole transversal segment but finitely many points whose positive trajectories go directly to saddle-points. Since there are only finitely many saddle-points, we have that for each $Z \in \mathcal{V}_0$ such that $Z - X$ is supported in B_0, there exists a countable subset D of Σ such that for every $z \in \Sigma \setminus D$ the positive semitrajectory of Z starting at z intersects Σ infinitely many times. By Lemma 4.3, $\omega(z)$ is either a recurrent trajectory intersecting B_0 or an attracting graph intersecting B_0. In the second case, an arbitrarily small C^r-perturbation of Z supported in B_0 yields a vector field $\widetilde{Z} \in \mathcal{V}_0$ having a periodic trajectory meeting B_0. □

Theorem 6.4 (Localized C^r-Closing Lemma)**.** *Suppose that $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, has the contraction property at a non-trivial recurrent point $p \in M$ of X. Given neighborhoods V of p in M and $\mathcal{V}$ of X in $\mathfrak{X}^r_H(M)$, there exists $Y \in \mathcal{V}$, with $Y - X$ supported in V, such that Y has a periodic trajectory meeting V.*

Proof. Assume by contradiction that no vector field $Y \in \mathcal{V}$ with $Z - X$ supported in V has a periodic trajectory meeting V. By Proposition 6.3 and by Lemma 3.5, there exist a flow box $B_0 \subset V$ and a neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of X such that every $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has the infinitesimal contraction property at B_0 and a non-trivial recurrent point in int (B_0), the interior of B_0. Note that every vector field $Z \in \mathcal{V}_0$ with $Z - X$ supported in B_0 has at most $4N_s$ saddle-connections, where N_s is the number of saddle-points of X. Therefore, the proof will be finished if we construct a sequence $\{Z_n\}_{n=0}^{4N_s+1}$ of vector fields in $\mathcal{V}_0$ such that for each $n \in \mathbb{N}$, $Z_n - X$ is supported in B_0 and Z_{n+1} has one more saddle-connection than Z_n. Let us proceed with such a

construction. Let $p_0 \in \operatorname{int}(B_0)$ be a non-trivial recurrent point of $Z_0 = X$. By Theorem 5.5, there exist an open set $V_1 \subset B_0$ and $Z_1 \in \mathcal{V}_0$ with $Z_1 - X$ supported in V_1 having a saddle-connection σ_1 meeting V_1. By the above, Z_1 has also a non-trivial recurrent point $p_1 \in \operatorname{int}(B_0)$. Now we may repeat the reasoning. By Theorem 5.5, there exist an open set $V_2 \subset B_0 \setminus \sigma_1$ and $Z_2 \in \mathcal{V}_0$ with $Z_2 - X$ supported in V_2 having a saddle-connection σ_2 meeting V_2 (and a saddle-connection σ_1 meeting V_1). Moreover, Z_2 has a non-trivial recurrent point $p_2 \in \operatorname{int}(B_0)$. By proceeding by induction, we shall obtain a vector field $Z_{4N_s+1} \in \mathcal{V}_0$ with $Z_{4N_s+1} - X$ supported in B_0 having at least $4N_s + 1$ saddle-connections, which is a contradiction. □

Theorem A. *Suppose that $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, has the contraction property at a quasiminimal set N. For each $p \in N$, there exists $Y \in \mathfrak{X}^r_H(M)$ arbitrarily C^r-close to X having a periodic trajectory passing through p.*

Proof. That localized C^r-closing (Theorem 6.4) implies C^r-closing (Theorem A) is an elementary fact. □

7 Transverse measures

Let N be a quasiminimal set of $X \in \mathfrak{X}^r_H(M)$, Σ be a transversal segment to X such that $\Sigma \setminus \partial\Sigma$ intersects N and $P\colon \Sigma \to \Sigma$ be the forward Poincaré Map induced by X. We may consider Σ as a Borel measurable space $(\Sigma, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra on Σ. We say that a Borel probability measure is *non-atomic* if it assigns measure zero to every one-point-set. A *transverse measure on* Σ is a non-atomic P-invariant Borel probability measure which is supported in $N \cap \Sigma$. A transverse measure ν is called *ergodic* if whenever $P^{-1}(B) = B$ for some Borel set $B \in \mathscr{B}$ then either $\nu(B) = 0$ or $\nu(B) = 1$. We let $\mathscr{M}(\Sigma)$ denote the set of Borel probability measures on Σ and we let $\mathscr{M}_P(\Sigma)$ denote the subset of $\mathscr{M}(\Sigma)$ formed by the P-invariant Borel probability measures. We say that P is *uniquely ergodic* if $\mathscr{M}_P(\Sigma)$ is a unitary set. A set $W \subset \Sigma$ is called a *a total measure set* if $\nu(W) = 1$ for every $\nu \in \mathscr{M}_P(\Sigma)$. Concerning the existence of transverse measures, we have the following result.

Theorem 7.1. *Let N be a quasiminimal set of $X \in \mathfrak{X}^r_H(M)$ and let Σ_1 be a compact transversal segment to X passing through a non-trivial recurrent point $p \in N$. There exist a subsegment $\Sigma \subset \Sigma_1$ passing through p and finitely many ergodic transverse measures $\nu_1, \ldots, \nu_s \in \mathscr{M}_P(\Sigma)$ such that every $\nu \in \mathscr{M}_P(\Sigma)$ can be written in the form $\nu = \sum_{i=1}^{s} \lambda_i \nu_i$, where $\lambda_i \geq 0$ for all $1 \leq i \leq s$, and $\sum_{i=1}^{s} \lambda_i = 1$.*

Proof. The proof may be split into two parts. The first part of the proof – that every small subsegment of Σ_1 passing through p can be endowed with a transverse measure – can be found in Katok [15] and Gutierrez [11]. To prove the second part, let C be a special transversal circle to X passing through γ_p as in the Theorem 6.1. There exist subsegments $\Sigma \subset \Sigma_1$ containing p and $\Gamma \subset C$ such that the forward Poincaré Map $T\colon \Sigma \to \Gamma$ induced by X is a diffeomorphism. We claim that $\mathscr{M}_P(\Sigma)$ is made up of transverse measures, where $P\colon \Sigma \to \Sigma$ is the forward Poincaré Map induced by X. Indeed, by (d) of Theorem 6.1, Σ is free of periodic points. By Poincaré Recurrence Theorem, the set of non-trivial recurrent points in Σ is a total measure set. By (b) of Theorem 6.1, all these non-trivial recurrent points belong to the same quasiminimal set. This proves the claim. Now, every (ergodic) transverse measure on Σ corresponds, via the diffeomorphism T, to a (ergodic) transverse measure on C. By (c) of Theorem 6.1, every (ergodic) transverse measure on C corresponds to a (ergodic) Borel probability measure on $\mathbb{R}/\mathbb{Z}$ invariant by a minimal interval exchange transformation $E\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. By a result of Keane [16], which also holds for interval exchange transformations with flips [6], there exist only finitely many ergodic Borel probability measures invariant by E. Each of such E-invariant Borel probability measures on $\mathbb{R}/\mathbb{Z}$ is associated to exactly one ergodic transverse measure in $\mathscr{M}_P(\Sigma)$. Now the rest of the proof follows from the fact that $\mathscr{M}_P(\Sigma)$ is the convex hull of its ergodic measures. □

Let $P\colon \Sigma \to \Sigma$ be the forward Poincaré Map induced by X on a transversal segment Σ to $X \in \mathfrak{X}^r_H(M)$. By Lemma 4.9, the domain of P is the union of finitely many open, pairwise disjoint subintervals of Σ: dom $(P) = \cup_{i=1}^m J_i$. We say that the *lateral limits of* $|DP|$ *exist* if for every $1 \leq i \leq m$ and for every $p \in \partial J_i$, the lateral limit

$$\ell = \lim_{\substack{x \to p \\ x \in J_i}} |DP(x)|$$

exists as a point of $[0, +\infty]$.

Henceforth, till the end of this paper, we shall assume that N is a quasiminimal set, Σ is a transversal segment to X such that $\Sigma \setminus \partial\Sigma$ intersects N and $P\colon \Sigma \to \Sigma$ is the forward Poincaré Map induced by X on Σ. We shall assume that Σ is so small that the forward Poincaré Map $T\colon \Sigma \to T(\Sigma) \subset C$ induced by X is a diffeomorphism, where C is a special transversal circle for N, and that P has the following properties:

(P1) $|DP|$ is bounded from above;

(P2) The lateral limits of $|DP|$ exist.

Definition 7.2 (Almost-integrable function)**.** *We say that* $\log|DP|$ *is* ν*-almost-integrable if*

$$\min\Big\{\int \log^+|DP|\,\mathrm{d}\nu, \int \log^-|DP|\,\mathrm{d}\nu\Big\} < \infty,$$

where

$$\begin{aligned}\log^+|DP(x)| &= \max\big\{\log|DP(x)|, 0\big\},\\ \log^-|DP(x)| &= \max\big\{-\log|DP(x)|, 0\big\},\end{aligned}$$

and $\nu \in \mathscr{M}(\Sigma)$*. In this case we define*

$$\int \log|DP|\,\mathrm{d}\nu = \int \log^+|DP|\,\mathrm{d}\nu - \int \log^-|DP|\,\mathrm{d}\nu,$$

which is a well defined value of the subinterval $[-\infty,\infty)$ *of the extended real line* $[-\infty,\infty]$.

Now we present four preparatory lemmas for the proof of Proposition 7.7. We would like to point out that Proposition 7.7 in the case where P is a smooth map defined on a compact interval was presented in [1, Lemma 2.1, p. 1305] (see also [4, Lemma 2, p. 1475]), however the ideas used in that proof do not generalize immediately to the case where P has discontinuities.

Lemma 7.3. *Suppose that there exists* $K \in \mathbb{R}$ *such that* $\int \log|DP|\,\mathrm{d}\nu < K$ *for all* $\nu \in \mathscr{M}_P(\Sigma)$*. Then there exists a continuous function* $\phi\colon \Sigma \to \mathbb{R}$ *everywhere defined, with* $\log|DP(x)| < \phi(x)$ *for all* $x \in \operatorname{dom}(P) \setminus P^{-1}(\partial\Sigma)$*, such that* $\int \phi\,\mathrm{d}\nu < K$ *for all* $\nu \in \mathscr{M}_P(\Sigma)$*.*

Proof. By reasoning as in Theorem 7.1, since Σ is disjoint of periodic trajectories, we may show that $\mathscr{M}_P(\Sigma)$ is the convex hull of finitely many ergodic (non-atomic) transverse measures $\nu_1, \ldots, \nu_s$. It follows from $(P1)$ and $(P2)$ that there exists a continuous function $\overline{\phi}\colon \overline{\operatorname{dom}(P)} \to \mathbb{R}$ such that $\int \overline{\phi}\,\mathrm{d}\nu_i < K$, for all $1 \le i \le s$, and $\log|DP(x)| < \overline{\phi}(x)$ for all $x \in \operatorname{dom}(P) \setminus P^{-1}(\partial\Sigma)$. Hence, $\int \overline{\phi}\,\mathrm{d}\nu < K$ for all $\nu \in \mathscr{M}_P(\Sigma)$. Now we may take ϕ to be any continuous extension of $\overline{\phi}$ to Σ. Since every $\nu \in \mathscr{M}_P(\Sigma)$ is supported in $N \cap \Sigma \subset \overline{\operatorname{dom}(P)}$, we have that $\int \phi\,\mathrm{d}\nu = \int \overline{\phi}\,\mathrm{d}\nu < K$ for all $\nu \in \mathscr{M}_P(\Sigma)$. □

Lemma 7.4. *The following statements are equivalent:*

(a) $\liminf_{n\to\infty} \frac{1}{n}\log|DP^n(x)| < 0$ *for all* x *in a total measure set;*

(b) $\int \log |DP| \, d\nu < -c$ *for some* $c > 0$ *and for all* $\nu \in \mathscr{M}_P(\Sigma)$*;*

(c) $\liminf_{n\to\infty} \frac{1}{n} \log |DP^n(x)| < -c$ *for some* $c > 0$ *and for all* x *in a total measure set;*

Proof. Let us show that (a) implies (b). By $(P1)$, $\log |DP|$ is ν-almost integrable with respect to each $\nu \in \mathscr{M}(\Sigma)$. Hence, there exists $K \in \mathbb{R}$ such that $\int \log |DP| \, d\nu_i < K$, for all $1 \leq i \leq s$, where $\{\nu_i\}_{i=1}^s$ are the ergodic transverse measures in $\mathscr{M}_P(\Sigma)$. So either $\int \log |DP| \, d\nu_i = -\infty$ for all $i = 1, \ldots, s$ (and we are done) or there exists a non-empty subset Λ of $\{1, 2, \ldots, m\}$ such that $\log |DP|$ is ν_i-integrable for all $i \in \Lambda$. In this case, (a) and Birkhoff Ergodic Theorem yields that

$$\int \log |DP| \, d\nu_i = \lim_{n\to\infty} \frac{1}{n} \log |DP(x)| = \liminf_{n\to\infty} \frac{1}{n} \log |DP(x)| = -c_i < 0$$

for all x in a ν_i-full measure set. Now take $c = \min\{c_i : i \in \Lambda\}$. A similar reasoning shows that (b) implies (c). This finishes the proof. □

Lemma 7.5. *Let* $\{\mu_j\}_{j\in\mathbb{N}}$ *be a sequence of Borel probability measures in* $\mathscr{M}(\Sigma)$ *weakly* converging to* $\mu \in \mathscr{M}(\Sigma)$*. The following hold:*

(a) $\mu(B) = \lim_{j\to\infty} \mu_j(B)$ *for every Borel set* $B \in \mathscr{B}$ *such that* $\mu(\partial B) = 0$*, where* ∂B *denotes the topological boundary of* B*;*

(b) $\mu(J) = \lim_{j\to\infty} \mu_j(J)$ *for every open subinterval* J *of* Σ *such that* $\mu(\partial J \setminus \partial\Sigma) = 0$*.*

Proof. The item (a) is a standard theorem from measure theory (see [22, Theorem 6.1, p. 40]). Let us prove (b). Let J be an open subinteval of Σ. If $\partial J \cap \partial\Sigma = \emptyset$ then $\mu(\partial J) = \mu(\partial J \setminus \partial\Sigma) = 0$ and the result follows from (a). If $J = \Sigma$ then the indicator function χ_J is continuous and so the result follows immediately from the weak* convergence of $\{\mu_j\}_{j\in\mathbb{N}}$ to μ. Hence we may assume that $\partial J \cap \partial\Sigma$ is a one-point set such that $\mu(\partial J \setminus \partial\Sigma) = 0$. Under these assumptions, there exist monotone sequences of continuous functions $\{\varphi_K\}_{K\in\mathbb{N}}$ and $\{\psi_K\}_{K\in\mathbb{N}}$ such that $\varphi_K < \chi_J < \psi_K$ and $\int \psi_K - \varphi_K \, d\mu < \frac{1}{K}$ for each $K \in \mathbb{N}$. Since $\mu_j \xrightarrow{*} \mu$ (in the weak* topology) as $j \to \infty$ and $\psi_K - \varphi_K$ is a continuous function, we have that for each $K \in \mathbb{N}$ there exists $L_K \in \mathbb{N}$

such that $\int \psi_K - \varphi_K \, d\mu_j < \frac{2}{K}$ for all $j > L_K$. It is easy to see that for each $K \in \mathbb{N}$ and for all $j > L_K$,

$$\left| \int \chi_J \, d\mu - \int \chi_J \, d\mu_j \right| < \frac{3}{K} + \left| \int \varphi_K \, d\mu - \int \varphi_K \, d\mu_j \right|.$$

This shows that $\mu(J) = \lim_{j\to\infty} \mu_j(J)$. □

Lemma 7.6. *Let $\{x_{n_j}\}_{j=0}^{\infty}$ be a sequence in Σ such that $n_j \geq 1$ and $x_{n_j} \in$ dom (P^{n_j-1}) for all $j \in \mathbb{N}$. Any accumulation point of the sequence of Borel probability measures*

$$\mu_j = \frac{1}{n_j} \sum_{k=0}^{n_j-1} \delta_{P^k\left(x_{n_j}\right)}, \tag{3}$$

where δ_x is the Dirac probability measure on Σ concentrated at x, is a non-atomic measure.

Proof. Let $\mu \in \mathscr{M}(\Sigma)$ be an accumulation point of $\{\mu_j\}_{j\in\mathbb{N}}$. By taking a subsequence if necessary and by renaming variables, we may assume that $\mu_j \xrightarrow{*} \mu$ as $j \to \infty$. Since the set $D = \{z \in \Sigma \mid \mu(\{z\}) > 0\}$ is at most countable, for each $p \in \Sigma$, there exists an open subinterval I_p of Σ containing p of length $\ell(I_p)$ arbitrarily small such that $\mu(\partial I_p \setminus \partial\Sigma) = 0$. By (c) of Theorem 6.1 and by Lemma 3.1 of Camelier-Gutierrez [3], for each $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$, such that if $\ell(I_p) < \delta$ then for each $n \geq N$ and $x \in$ dom (P^{n-1}),

$$\frac{1}{n} \sum_{k=0}^{n-1} \chi_{I_p}\left(P^k(x)\right) < \epsilon.$$

Hence, for each $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that if $\ell(I_p) < \delta$ then for all $j \geq N$,

$$\mu_j(I_p) = \frac{1}{n_j} \sum_{k=0}^{n_j-1} \chi_{I_p}\left(P^k(x_{n_j})\right) < \epsilon.$$

By Lemma 7.5, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\ell(I_p) < \delta$ then

$$\mu\left(I_p\right) = \lim_{j\to\infty} \mu_j\left(I_p\right) \leq \epsilon.$$

Hence, $\mu(\{p\}) = 0$ and so μ is non-atomic, which finishes the proof. □

Proposition 7.7. *Suppose that there exist a constant $c > 0$ and a continuous function $\phi\colon \Sigma \to \mathbb{R}$ such that $\int \phi \, d\nu < -c$ for all $\nu \in \mathcal{M}_P(\Sigma)$. Then there exists $N \in \mathbb{N}$ such that for each $n > N$ and for all $x \in \operatorname{dom}(P^{n-1})$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi\big(P^k(x)\big) < -c.$$

Proof. Assume by contradiction that there exists a sequence $\{x_{n_j}\}_{j=0}^{\infty} \subset \Sigma$ such that for each $j \in \mathbb{N}$, $x_{n_j} \in \operatorname{dom}(P^{n_j-1})$ and

$$\frac{1}{n_j}\sum_{k=0}^{n_j-1} \phi\big(P^k(x_{n_j})\big) \geq -c.$$

The set $\mathcal{M}(\Sigma)$, endowed with the weak* topology, is a compact metric space. Consequently, the sequence of Borel probability measures

$$\mu_j = \frac{1}{n_j}\sum_{k=0}^{n_j-1} \delta_{P^k\left(x_{n_j}\right)},$$

has a subsequence that weakly* converges to a Borel probability measure $\mu \in \mathcal{M}(\Sigma)$. By renaming variables, we may assume that $\mu_j \xrightarrow{*} \mu$ as $j \to \infty$. By Lemma 7.5 and by Lemma 7.6, we have that $\mu(I) = \lim_{j\to\infty} \mu_j(I)$ for all set I which is union of finitely many intervals. On the other hand,

$$\lim_{j\to\infty} \mu_j\big(P^{-1}(B)\big) = \lim_{j\to\infty} \mu_j(B)$$

for all Borel set B. Besides, for every interval J, $P^{-1}(J)$ is union of finitely many intervals. Therefore, by the above, we have that for every interval J

$$\mu\big(P^{-1}(J)\big) = \lim_{j\to\infty} \mu_j\big(P^{-1}(J)\big) = \lim_{j\to\infty} \mu_j(J) = \mu(J).$$

This implies that μ is P-invariant and so $\mu \in \mathcal{M}_P(\Sigma)$. Since the function ϕ is continuous, we have

$$\int \phi \, d\mu = \lim_{j\to\infty} \int \phi \, d\mu_j = \lim_{j\to\infty} \frac{1}{n_j}\sum_{k=0}^{n_j-1} \phi\big(P^k(x_{n_j})\big) \geq -c,$$

by the definition of μ and by the way we have chosen the sequence $\{n_j\}_{j=0}^{\infty}$, which contradicts the initial assumption that

$$\int \phi d\nu < -c \quad \text{for all} \quad \nu \in \mathcal{M}_P(\Sigma). \qquad \square$$

Theorem B. *Suppose that X has divergence less or equal to zero at its saddle-points and that X has negative Lyapunov exponents at a quasiminimal set N. Then X has the infinitesimal contraction property at N.*

Proof. Let Σ_1 be a transversal segment to X passing through a non-trivial recurrent point $p \in N$ so small that the forward Poincaré Map $T\colon \Sigma_1 \to T(\Sigma_1) \subset C$ is a diffeomorphism, where C is a special transversal circle for N. By the hypothesis on the divergence of X at its saddle-points every forward Poincaré Map $P\colon \Sigma \to \Sigma$ induced by X on a transversal segment Σ to X has properties $(P1)$ and $(P2)$ (see [19]). By the hypothesis of negative Lyapunov exponents and by Lemma 7.4, there exist a subsegment Σ of Σ_1 passing through p and a constant $c > 0$ such that the forward Poincaré Map $P\colon \Sigma \to \Sigma$ induced by X satisfies $\int \log|DP|\,\mathrm{d}\nu < -c$ for all $\nu \in \mathscr{M}_P(\Sigma)$. By Lemma 7.3, there exists a continuous function $\phi\colon \Sigma \to \mathbb{R}$ everywhere defined, with $\log|DP(x)| < \phi(x)$ for all $x \in \operatorname{dom}(P) \setminus P^{-1}(\partial\Sigma)$, such that $\int \phi\,\mathrm{d}\nu < -c$ for all $\nu \in \mathscr{M}_P(\Sigma)$. By Proposition 7.7, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in \operatorname{dom}(P^n) \setminus \mathcal{O}_n^-(x)$,

$$\frac{1}{n}\log|DP^n(x)| = \frac{1}{n}\sum_{k=0}^{n-1}\log|DP\big(P^k(x)\big)| < \frac{1}{n}\sum_{k=0}^{n-1}\phi\big(P^k(x)\big) < -c.$$

Thus P^n is an infinitesimal contraction. By Proposition 3.1, X has the infinitesimal contraction property at N. □

To finish the paper, we now provide a sketch of the proof of Theorem C.

Theorem C. *Suppose that $X \in \mathfrak{X}^r_H(M)$, $r \geq 2$, has the contraction property at a quasiminimal set N. There exists $Y \in \mathfrak{X}^r(M)$ arbitrarily C^r-close to X having one more saddle-connection than X.*

Sketch of the proof. In the smooth case, we may use the same proof as Lemma 5.4 without any changes. In the case in which $X \in \mathfrak{X}^r(M)$ we cannot use that proof because in taking a C^r-flow box to make the perturbation, the vector field so obtained is of class C^{r-1}. Thus we have to make the perturbation directly on the surface (using bump functions defined on the surface and using also the orthogonal vector field to X) and to use the flow box coordinates only for estimation purposes. □

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