A Property of the Chow-Robbins Procedure for Fixed Length Confidence Intervals*

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common distribution F. Let $EX_i = \xi$ and $Var\ X_i = \sigma^2 < \infty$.

Suppose a confidence interval of given length $2d$ and coverage probability α is required for ξ.

If F is normal with variance σ^2 unknown it is a well known result that this cannot be accomplished by a non-sequential procedure.

Several authors, including Stein [1], Chow and Robbins [2], have contributed to the solution of this problem. A two-stage procedure was proposed by Stein in [1], to solve the problem in the normal case. In [2] Chow and Robbins introduced a truly sequential procedure to find confidence intervals for the mean of an arbitrary population.

The following simple considerations motivate the procedure proposed in [2]: If the variance σ^2 of the population is known if d is small compared to σ^2, define

$$\bar{X}_n = \frac{1}{n}\sum_{i=0}^{n} X_i \qquad \text{and} \qquad S_n = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

and choose a to satisfy

$$(2\pi)^{-1/2} \int_{-a}^{+a} e^{-u_2/2}\, du = \alpha.$$

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Let n = smallest integer $\geq \frac{a^2\sigma^2}{d^2}$. Then take n independent observations from the population and form the interval $I_n = [\bar{X}_n - d, \bar{X}_n + d]$. It is easy to show that $\lim_{d\to 0} P(\xi \in I_n) = \alpha$. Suppose now $0 < \sigma^2 < \infty$ is unknown. In this case Chow and Robbins proposed the following *C.R.* procedure. Define N^* = smallest $n \geq n_0$ such that $n \geq \frac{a^2 S_n^2}{d^2}$, and form the interval $I_{N^*} = [\bar{X}_{N^*} - d, \bar{X}_{N^*} + d]$.

They investigated the asymptotic behavior of this procedure as $d \longrightarrow 0$ and proved:

(a) $\lim_{d\to 0} EN^* = \infty$;

(b) $\lim_{d\to 0} P[\bar{X}_{N^*} - d < \xi < \bar{X}_{N^*} + d] = \alpha$ (asymptotic consistency);

(c) $\lim_{d\to 0} \frac{d^2 EN^*}{a^2\sigma^2} = 1$ (asymptotic efficiency).

We show that, in the class of procedures satisfying (b), the *C.R.* procedure is asymptotically minimax, as $d \longrightarrow 0$. For this, we compare expected sample sizes for d small.

2. A result on C.R. Procedure

We denote by S a procedure for obtaining a confidence interval of length $2d$ for ξ. Specifically, this is a pair $(N, \{Y_n\}_{n>1})$ where N is a stopping time and $Y_n = Y_n(X_1, \ldots, X_n)$ is an statistics based upon the first n observations. If $N = n$ the procedure estimates ξ by $(Y_n - d, Y_n + d)$. Let's suppose F normal with mean ξ and variance $\sigma^2 < \infty$ and consider ξ a random variable with prior distribution μ having density $N(0, \tau^2)$. Next we present some results from [3], needed for the proof of theorem 1:

LEMMA 1. *Let $S = (N, \{Y_n\}_{n>1})$ be a procedure describe above. Then we have for each σ:*

(i) $$\int P[Y_N - d < \xi < Y_N + d]\, d\mu(\xi) \leq \sum_1^\infty \bar{p}_m(\tau, S)\, H(C_m)$$

where

$$\bar{p}_m(\tau, S) = \int P[N = m]\, d\mu(\xi),$$

$$H(C_m) = (2\pi)^{-1/2} \int_{-C_m}^{C_m} e^{-\frac{1}{2}u^2}\, du$$

and

$$C_m = \left(\frac{m\tau^2 + d^2}{\tau^2\sigma^2}\right)^{\frac{1}{2}} d.$$

(ii) *Let* $\bar{E}N = \int EN d\mu(\xi)$ *and* v *be an integer, then*

$$\bar{E}N \leq v \Rightarrow \sum_1^\infty \bar{p}_m(\tau, S)\, H(C_m) \leq H(C_v).$$

PROOF. Result (1) is inequality (21) in [3]. For (ii) see inequalites (31) and (32) in [3].

THEOREM 1. *Consider the class of all procedures $S = (N, \{Y_n\}_{n>1})$ satisfying*

$$\lim_{d\to 0} P_{(\tau,\sigma^2)} [Y_N - d < \xi < Y_N + d] = \alpha \qquad \forall(\xi, \sigma^2).$$

In this class the C.R. procedure satisfies for each σ:

$$\liminf_{d\to 0} \frac{sup_\xi EN}{EN^*} \geq 1.$$

PROOF. First, by property (b) the C.R. procedure $S^* = (N^*, \{\bar{X}_n\})$ belongs to the class defined above. Let $S = (N, \{Y_n\}_{n>1})$ be any procedure in this class. By the Dominated Convergence theorem,

$$\lim_{d\to 0} \int P_{(\xi,\sigma^2)} [Y_N - d < \xi < Y_N + d]\, d\mu(\xi) = \alpha.$$

Given $\varepsilon > 0$ choose $\delta(\varepsilon) > 0$ such that for $d < \delta(\varepsilon)$, we have $\Sigma \bar{p}_m(\tau, S)\, H(C_m) > \alpha - \varepsilon$ (part (i) of Lemma 1). Let $a_\varepsilon = H^{-1}(\alpha - \varepsilon)$ and take γ = greatest integer $\leq \frac{a_\varepsilon^2\sigma^2}{d^2} - \frac{\sigma^2}{\tau^2}$, then $H(C_\gamma) \leq a_\varepsilon$. Therefore, we have by part (ii) of Lemma 1:

$$sup_\xi EN \geq \bar{E}N > \gamma > \frac{a_\varepsilon^2\sigma^2}{d^2} - \frac{\sigma^2}{\tau^2} - 1.$$

Using properties (a) and (c) of the C.R. procedure, we get

$$\liminf_{d\to 0} \frac{sup_\xi EN}{EN^*} \geq \frac{a_\xi^2}{a^2} \qquad \forall \varepsilon > 0,$$

which implies

$$\liminf_{d\to 0} \frac{sup_\xi EN}{EN^*} \geq 1.$$

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