On the Rauch Comparison Theorem for Volumes*

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It is already known, that given a Riemannian manifold M, whose sectional curvature K_M satisfies $L \leq K_M \leq H$, then it is possible to compare the volume of a normal ball in M, with the volume of a normal ball with the same radius, contained on a space form with constant curvature L or H ([1], [3], [4]). In this paper we investigate the case when the Ricci curvature on M satisfies $Ricc_M \geq H$.

Given an n-dimensional Riemannian manifold M, with curvature tensor R, consider the transformation $R_X : Y \longrightarrow R(X, Y)X$. If X is a unitary vector tangent to M, $Ricc(X) = \frac{1}{n-1} tr\, R_X$. Let γ be a geodesic on M, such that $\gamma(0) = m$. We denote by $\mu(t)$ the Jacobian determinant of exp_m at $t\gamma'(0)$. We will prove the following:

THEOREM 1. *Let M be a complete, n-dimensional, Riemannian manifold and γ: $[0, a] \longrightarrow M$ a geodesic on M such that $\gamma(0) = m$. If m has no conjugate point on $\gamma(0, a]$ and $Ricc(\gamma'(t)) \geq H$, then*

$$\mu(t) \leq \begin{cases} \left(\dfrac{\operatorname{sen} \sqrt{H}\, t}{\sqrt{H}\, t}\right)^{n-1}, & if \quad H > 0 \\ 1, & if \quad H = 0 \\ \left(\dfrac{\operatorname{senh} \sqrt{-H}\, t}{\sqrt{-H}\, t}\right)^{n-1}, & if \quad H < 0 \end{cases}$$

for all $t \in (0, a]$. If equality holds, then $Ricc(\gamma'(t)) \equiv H$.

As a consequence of this theorem we get the following:

COROLLARY 1. *Let M be a complete, n-dimensional, Riemannian manifold, such that $Ricc_M \geq H$. If $B(m, r)$ and $B(\tilde{m}, r)$ are normal balls with radius r, centered*

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respectively at $m \in M$ and $\tilde{m} \in M$, where $\tilde{M}$ is the simply connected n-dimensional manifold with constant curvature H, then

$$v(B(m, r)) \leq v(B(\tilde{m}, r)).$$

If equality holds, then $Ricc(\gamma'(t)) \equiv H$, $t < r$, for all radial geodesic γ on $B(m, r)$.

The purpose of this paper is to present a counter-example, which shows that, in contrast with what happens when sectional curvature is concerned ([4], Corol. 1), $Ricc(\gamma'(t)) \equiv H$ in Theorem 1 does not imply equality for the Jacobian determinant $\mu(t)$. Consequently, in Corollary 1, we may have $Ricc_M \equiv H$ without obtaining equality for the volumes of the normal balls. Moreover this counter-example also shows, that it is not possible to obtain results analogous to Theorem 1 and Corollary 1, when $Ricc_M \leq H$.

In section 1 we prove Theorem 1 an Corollary 1, based essentially in the proof given in ([1], pg. 253). In section 2, we present the counter-example mentioned above.

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§1. We use the same notation as in [2]. Every geodesic is parametrized by arc length. M denotes a complete Riemannian manifold and S^n the unitary n-dimensional sphere contained in the Euclidean space $\mathbb{R}^{n+1}$. In the following proofs, we need the index form of a geodesic and its properties (see [2]). If W is a vector field along a geodesic $\gamma: [0, a] \longrightarrow M$, we denote

$$I(W, r) = \int_0^r \{\langle W', W'\rangle - \langle R(\gamma', W)\gamma', W\rangle\}\, dt,$$

where $r \in (0, a]$ and R is the curvature tensor on M. Moreover we need the following lemma:

LEMMA A. *Let $\gamma: [0, a] \longrightarrow M$ be a geodesic with no conjugate points to $\gamma(0)$ along $\gamma(0, l]$, $l \leq a$. Let J be a Jacobi field along γ and W a vector field along γ such that $\langle J, \gamma'\rangle = \langle W, \gamma'\rangle = 0$. If $J(0) = W(0) = 0$ and $J(l) = W(l)$, then*

$$I(J, l) \leq I(W, l).$$

Equality holds if and only if $J = W$.

A proof of this lemma can be found in ([2], pg. 205).

Consider a geodesic γ on an n-dimensional, Riemannian manifold, such that $\gamma(0) = m$. If $v_i(t)$, $i = 1, 2, \ldots, n-1$, are linearly independent vectors, normal to $\gamma'(0)$ at $t\gamma'(0)$, then the Jacobian determinant of exp_m at $t\gamma'(0)$, is

$$\mu(t) = \frac{|d\, exp_m v_1 \wedge \ldots \wedge d\, exp_m v_{n-1}|}{|v_1 \wedge \ldots \wedge v_{n-1}|},$$

where $v_1 \wedge \ldots \wedge v_{n-1}$ is the exterior product of the vectors $v_1, \ldots, v_{n-1}$. We can now prove Theorem 1.

Proof of Theorem 1: Let $\tilde{M}$ be the n-dimensional, simply connected space, with constant curvature $H > 0$. Let $\tilde{\gamma}: [0, a] \longrightarrow \tilde{M}$ be a geodesic on $\tilde{M}$, such that $\tilde{\gamma}(0) = \tilde{m}$. Denote by $\tilde{\mu}(t)$ the Jacobian determinant of exp_m at $t\tilde{\gamma}'(0)$. We will prove that, for all $t \in (0, a]$.

$$\mu(t) \leq \tilde{\mu}(t) = \left(\frac{\text{sen}\sqrt{H}\, t}{\sqrt{H}\, t}\right)^{n-1}.$$

Fix $r \in (0, a]$. Let $J_i(t)$, $i = 1, 2, \ldots, n-1$, be Jacobi fields along γ, orthogonal to $\gamma'(t)$ and such that $J_i(r)$ are orthonormal. For each i, consider $J_i(t) = d\, exp_m tA_i$, where A_i is a parallel vector field along $t\gamma'(0)$. Then

$$\mu(t) = \frac{|J_1 \wedge \ldots \wedge J_{n-1}|}{t^{n-1}|A_1 \wedge \ldots \wedge A_{n-1}|}.$$

Differentiating $\mu^2(t)$ at $t = r$, we get

$$2\mu(r)\,\mu'(r) = \frac{2\sum_{i=1}^{n-1}\langle J_i'(r), J_i(r)\rangle\, r^{2(n-1)} - 2(n-1)\, r^{2n-3}}{r^{4(n-1)}|A_1 \wedge \ldots \wedge A_{n-1}|^2}$$

Hence

$$\text{(1)} \qquad \frac{\mu'(r)}{\mu(r)} = \sum_{i=1}^{n-1}\langle J_i'(r), J_i(r)\rangle - \frac{n-1}{r} = \sum_{i=1}^{n-1} I(J_i, r) - \frac{n-1}{r}.$$

Consider $\{\tilde{Z}_1, \ldots, \tilde{Z}_n\}$ orthonormal vector fields, parallel along $\tilde{\gamma}$, such that $\tilde{Z}_n(t) = \tilde{\gamma}'(t)$ and let $\tilde{J}_i(t)$, $i = 1, \ldots, n-1$, be the Jacobi fields along $\tilde{\gamma}$, defined by

$$\tilde{J}_i(t) = \frac{\operatorname{sen}\sqrt{H}\,t}{\operatorname{sen}\sqrt{H}\,r}\,\tilde{Z}_i(t), \qquad i = 1, \ldots, n-1$$

With the same reasoning we did for J_i we get

$$\frac{\tilde{\mu}'(r)}{\tilde{\mu}(r)} = \sum_{i=1}^{n-1} I(\tilde{J}_i, r) - \frac{n-1}{r}.$$

If we denote $f(t) = \dfrac{\operatorname{sen}\sqrt{H}\,t}{\operatorname{sen}\sqrt{H}\,r}$ we conclude that

$$\frac{\tilde{\mu}'(r)}{\tilde{\mu}(r)} = (n-1)\int_0^r ((f')^2 - f^2 H)\,dt - \frac{n-1}{r}. \tag{2}$$

Now consider $\{Z_1, \ldots, Z_n\}$ orthonormal vector fields, parallel along γ, generated by $J_i(r)$ and $\gamma'(r)$. For each vector field

$$\tilde{W}(t) = \sum_{i=1}^{n} g_i(t)\,\tilde{Z}_i(t)$$

along $\tilde{\gamma}$, we associate a vector field

$$\psi(\tilde{W})(t) = \sum_{i=1}^{n} g_i(t)\,Z_i(t)$$

along γ. Then if $\tilde{W}_1$ and $\tilde{W}_2$ are vector fields along $\tilde{\gamma}$,

$$\langle \psi(\tilde{W}_1), \psi(\tilde{W}_2) \rangle = \langle \tilde{W}_1, \tilde{W}_2 \rangle,$$
$$(\psi(W))' = \psi(W').$$

It follows from these properties, Lemma A and from the hypothesis $Ricc(\gamma'(t)) \geq H$, that

$$\sum_{i=1}^{n-1} I(J_i, r) \leq \sum_{i=1}^{n-1} I(\psi(\tilde{J}_i), r) \tag{3}$$

$$= (n-1)\int_0^r ((f')^2 - f^2\,Ricc(\gamma'(t)))\,dt$$

$$\leq (n-1)\int_0^r ((f')^2 - f^2 H)\,dt.$$

From (1), (2) and (3) we obtain

$$\frac{\mu'(r)}{\mu(r)} \leq \frac{\tilde{\mu}'(r)}{\tilde{\mu}(r)}.$$

The point $r \in (0, a]$ was arbitrarily fixed, hence

$$\frac{d}{dt}\frac{\mu(t)}{\tilde{\mu}(t)} \leq 0, \qquad \text{for all} \qquad t \in (0, a].$$

Since $\mu(0) = \tilde{\mu}(0) = 1$, we conclude that

$$\mu(t) \leq \tilde{\mu}(t), \qquad \text{for all} \qquad t \in (0, a].$$

In order to prove that

$$\tilde{\mu}(t) = \left(\frac{\operatorname{sen}\sqrt{H}\,t}{\sqrt{H}\,t}\right)^{n-1},$$

we substitute f and f' in (2), and integrate from $\varepsilon > 0$ to $t \in (0, a]$ obtaining

$$\log\frac{\tilde{\mu}(t)}{\tilde{\mu}(\varepsilon)} = (n-1)\,\log\left(\frac{\operatorname{sen}\sqrt{H}\,t}{t} \times \frac{\varepsilon}{\operatorname{sen}\sqrt{H}\,\varepsilon}\right).$$

Hence

$$\frac{\tilde{\mu}(t)}{\tilde{\mu}(\varepsilon)} = \left(\frac{\operatorname{sen}\sqrt{H}\,t}{t} \times \frac{\varepsilon}{\operatorname{sen}\sqrt{H}\,\varepsilon}\right)^{n-1}$$

Considering the limit, when $\varepsilon \longrightarrow 0^+$, we conclude that

$$\tilde{\mu}(t) = \left(\frac{\operatorname{sen}\sqrt{H}\,t}{\sqrt{H}\,t}\right)^{n-1}.$$

Finally if $\mu(t) = \tilde{\mu}(t)$, for all $t \in [0, a]$, then equality holds in (3) and hence

$$Ricc(\gamma'(t)) \equiv H.$$

This completes the proof of Theorem 1, when $H > 0$.

If $H = 0$ or $H < 0$, we have similar proofs, considering the Jacobi fields

$\tilde{J}_i$ along $\tilde{\gamma}$, defined by $\tilde{J}_i(t) = \frac{t}{r}\tilde{Z}_i(t)$ when $H = 0$, and

$$\tilde{J}_i(t) = \frac{\operatorname{senh}\sqrt{-H}\,t}{\operatorname{senh}\sqrt{-H}\,r}\tilde{Z}_i(t)$$

when $H < 0$. q.e.d.

Proof of Corollary 1. It follows from Theorem 1, since the volume of a normal ball is obtained by integrating the Jacobian determinant of the exponencial map, on a ball with the same radius on the tangent space. q.e.d.

§2. In this section we give a counter-example, which shows that in Theorem 1, $Ricc(\gamma'(t)) \equiv H$ does not imply equality for the Jacobian determinant $\mu(t)$, and also in Corollary 1, $Ricc_M \equiv H$ does not imply equality for the volumes of the normal balls. First we summarize some properties of the product manifolds, which will be necessary for the counter-example.

Let M_1 and M_2 be Riemannian manifolds and $M = M_1 \times M_2$ the Riemannian product of M_1 and M_2. If γ_1 and γ_2 are curves in M_1 and M_2, X_1 and X_2 are parallel vector fields along γ_1 and γ_2 respectively, then $X_1 + X_2$ is a parallel vector field along the curve $(\gamma_1(t), \gamma_2(t))$ in M. Conversely a parallel vector field on M is obtained in this form. Hence the geodesics on M have the form $(\gamma_1(t), \gamma_2(t))$, where γ_1 and γ_2 are geodesics on M_1 and M_2 respectively. Moreover, the curvature tensor R of M at a point (m_1, m_2) satisfies the following equality

$$R_{(m_1,m_2)}(X_1 + X_2, Y_1 + Y_2) = R_{m_1}(X_1, Y_1) + R_{m_2}(X_2, Y_2),$$

where $X_1, Y_1 \in T_{m_1}M_1$, $X_2, Y_2 \in T_{m_2}M_2$ and R_{m_1}, R_{m_2} are the curvature tensors of M_1 and M_2 at m_1 and m_2 respectively. It is clear that if $X_1 \in T_{m_1}M_1$ and $X_2 \in T_{m_2}M_2$, then

$$R_{(m_1,m_2)}(X_1, X_2) = R_{(m_1,m_2)}(X_2, X_1) = 0.$$

LEMMA 1. *Let $M = M_1 \times M_2$ be the n-dimensional Riemannian product of M_1 and M_2. If M_1 and M_2 have dimensions n_1 and n_2 respectively and the Ricci curvatures satisfy*

$$(n_1 - 1)Ricc_{M_1} = (n_2 - 1)Ricc_{M_2} = \lambda,$$

then

$$(n-1)Ricc_M = \lambda.$$

PROOF. Let X be a unitary vector, tangent to M at (m_1, m_2). Consider $Z_1, \ldots, Z_{n_1}, Z_{n_1+1}, \ldots, Z_n$ an orthonormal frame at $(m_1, m_2) \in M$, such that $Z_1, \ldots, Z_{n_1} \in T_{m_1}M_1$ and $Z_{n_1+1}, \ldots, Z_n \in T_{m_2}M_2$. Then

$$X = \sum_{i=1}^{n} x_i Z_i \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = 1.$$

Hence

$$(n-1)\,Ricc(X) = \sum_{i,j=1}^{n}\sum_{k=1}^{n} x_i x_j \langle R_{(m_1,m_2)}(Z_i, Z_k)Z_j, Z_k\rangle =$$

$$= \sum_{i,j=1}^{n_1}\sum_{k=1}^{n_1} x_i x_j \langle R_{m_1}(Z_i, Z_k)Z_j, Z_k\rangle + \sum_{i,j=n_1+1}^{n}\sum_{k=n_1+1}^{n} x_i x_j \langle R_{m_2}(Z_i, Z_k)Z_j, Z_k\rangle$$

If we consider $i, j = 1, \ldots, n_1$, $i \neq j$, then using the fact that

$$(n_1 - 1)\,Ricc\left(\frac{1}{\sqrt{2}}Z_i + \frac{1}{\sqrt{2}}Z_j\right) = \lambda,$$

we get

$$\sum_{k=1}^{n_1} \langle R_{m_1}(Z_i, Z_k)Z_j, Z_k\rangle = 0.$$

Similarly,

$$\sum_{k=n_1+1}^{n} \langle R_{m_2}(Z_i, Z_k)Z_j, Z_k\rangle = 0 \text{ for all } i,j = n_1+1, \ldots, n \text{ and } i \neq j.$$

Hence

$$(n-1)\,Ricc(X) = \sum_{i=1}^{n_1}\sum_{k=1}^{n_1} x_i^2 \langle R_{m_1}(Z_i, Z_k)Z_i, Z_k\rangle +$$

$$+ \sum_{i=n_1+1}^{n}\sum_{k=n_1+1}^{n} x_i^2 \langle R_{m_2}(Z_i, Z_k)Z_i, Z_k\rangle$$

$$= \sum_{i=1}^{n_1} x_i^2\lambda + \sum_{i=n_1+1}^{n} x_i^2\lambda = \sum_{i=1}^{n} x_i^2\lambda = \lambda.$$

q.e.d.

COUNTER-EXAMPLE. Let $M_1 = S^2$ and $M_2 = S^2$. Consider $M = M_1 \times M_2$ the Riemannian product of M_1 and M_2.

It follows from Lemma 1, that $Ricc(X) = \frac{1}{3}$, for any unitary vector X tangent to M. Consider a geodesic $\gamma(t) = (m_1, \gamma_2(t))$ on M, such that $m_1 \in M_1$ and $\gamma_2(t)$ is a geodesic on M_2, defined for $t \in [0, \pi\sqrt{3}/2]$. It is not difficult to verify that $0 \leq K_M \leq 1$ and hence it follows from Rauch Comparision theorem ([2], pg. 206), that two consecutive, conjugate points occur at a a distante $\geq \pi$. Consequently there is no conjugate point of $\gamma(0)$ along γ. We shall prove that the Jacobian determinant of $exp_{\gamma(0)}$ is not equal to

$$\left(\frac{\text{sen}\, t\sqrt{3}/3}{t\sqrt{3}/3}\right)^3.$$

Fix $r \in (0, \pi\sqrt{3}/2]$ and consider Jacobi fields $J_i(t)$, $i = 1, 2, 3$, along γ, orthogonal to $\gamma'(t)$ and such that $J_i(r)$ are orthonormal vectors, $J_1(r)$ and $J_2(r)$ tangent to M_1 and $J_3(r)$ tangent to M_2. Let $\{Z_1, \ldots, Z_4\}$ be parallel vector fields along γ, generated by $J_i(r)$ and $\gamma'(r)$. Consider $\tilde{M} = S^4_{\sqrt{3}}$ a sphere of radius, $\sqrt{3}$, contained in the Euclidean space $\mathbb{R}^5$. Let $\tilde{\gamma}(t)$, $t \in [0, \pi\sqrt{3}/2]$ be a geodesic on $\tilde{M}$, and $\{\tilde{Z}_1, \ldots, \tilde{Z}_4\}$ parallel vector fields, orthonormal along $\tilde{\gamma}$, such that $\tilde{Z}_4(t) = \tilde{\gamma}'(t)$. Let $\tilde{J}_i(t)$, $i = 1, 2, 3$, Jacobi fileds along $\tilde{\gamma}$ defined by

$$\tilde{J}_i(t) = \frac{\text{sen}\, t\sqrt{3}/3}{\text{sen}\, r\sqrt{3}/3}\tilde{Z}_i(t), \qquad i = 1, 2, 3.$$

From (1), (2) and (3) in the proof of Theorem 1, we know that

$$(4) \qquad \frac{\mu'(r)}{\mu(r)} = \sum_{i=1}^{3} I(J_i, r) - \frac{3}{r} \leq \sum_{i=1}^{3} I(\psi(\tilde{J}_i), r) - \frac{3}{r} = \frac{\tilde{\mu}'(r)}{\tilde{\mu}(r)}$$

where

$$\psi(\tilde{J}_i)(t) = \frac{\text{sen}\, t\sqrt{3}/3}{\text{sen}\, r\sqrt{3}/3} Z_i(t).$$

We claim that $\psi(\tilde{J}_3)$ is not a Jacobi field along γ. In fact

$$\psi(\tilde{J}_3)'' + R(\gamma', \psi(\tilde{J}_3))\gamma' \neq 0,$$

since

$$\langle \psi(\tilde{J}_3)'', Z_3\rangle + \langle R(\gamma', \psi(\tilde{J}_3))\gamma', Z_3\rangle = \frac{2}{3}\frac{\text{sen}\, t\sqrt{3}/3}{\text{sen}\, r\sqrt{3}/3}.$$

It follows from Lemma A, that

$$I(J_3, r) < I(\psi(\tilde{J}_3), r)$$

and hence from (4), we conclude that

$$\frac{\mu'(r)}{\mu(r)} < \frac{\tilde{\mu}'(r)}{\tilde{\mu}(r)}.$$

Since r was fixed arbitrarily, we have that

$$\frac{d}{dt}\frac{\mu(t)}{\tilde{\mu}(t)} < 0, \qquad \text{for} \qquad t \in (0, \pi\sqrt{3}/2].$$

That is

$$\mu(t) < \tilde{\mu}(t) = \left(\frac{\text{sen}\, t\sqrt{3}/3}{t\sqrt{3}/3}\right)^3, \qquad \text{for all} \qquad t \in (0, \pi\sqrt{3}/2].$$

This justifies the remark about Theorem 1.

If $B((m_1, m_2), r)$ and $B(\tilde{m}, r)$ are normal balls centered respectively at $(m_1, m_2) \in M$ and $\tilde{m} \in \tilde{M}$, with radius r, then it follows from what we have just seen, about the Jacobian determinant of $exp_{(m_1, m_2)}$, that

$$v(B(m_1, m_2), r) < v(B(\tilde{m}, r)).$$

Remark. It is clear from this counter-example, that it is not possible to obtain results analogous to Theorem 1 and Corollary 1, when $\underset{M}{Ricc} \leq H$.

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