Spherical Images of Continuous Convex Surfaces of Hilbert Spaces*

RUBENS LEÃO DE ANDRADE

1. Introduction and Statment of Results

A subset $M \subset H$ of a Hilbert space H is a *convex surface* if $M = \partial K$ is the (topological) boundary of a closed convex set K with non void interior $\mathring{K}$. K is called the *convex body* of M. Given a point $p \in M$ we say that a hyperplane L containing p is a *support plane* of M at p if M lies in one of the closed half-spaces determined by L. The unit vector perpendicular to L and that points to the half-space where M lies is called an *inner normal vector* ar p. We denote by $\gamma(p)$ the set of all inner normal vectors at p and define the *spherical image* of M by $\gamma(M) = \bigcup_{p \in M} \gamma(p)$. A subset $A \subset \Sigma$ of the unit sphere $\Sigma \subset H$ is *geodesically convex* if: 1) Given two points $x, y \in A$, $x \neq -y$ then the minimal geodesic segments joining x and y lies in A; 2) if x and $-x$ lie in A then at least one of the geodesic segments joining x and $-x$ lies in A. In the case that M is a convex surface of $\mathbb{R}^n \subset H$, Wu [4] proved that the closure $\overline{\gamma(M)}$ and the interior $\gamma^\circ(M)$ are geodesically convex sets. A simple proof of the fact that $\overline{\gamma(M)}$ is geodesically convex, in the case that M is a C^∞ convex surface of a Hilbert space, is given by M. do Carmo and B. Lawson [1]. In this paper we extend Wu's result to a convex surface of a Hilbert space, under the hypothesis that $\gamma^\circ(M) \neq \emptyset$. We remark that in the finite dimensional case this hypothesis is not restrictive, because every convex surface of $\mathbb{R}^n$ is isometric, by an isometry of $\mathbb{R}^n$, to a product $\mathbb{R}^m \times N$, where N is a convex surface of $\mathbb{R}^{n-m}$ and $\gamma(N) = \gamma(M)$ (as a subset of the unit sphere of $\mathbb{R}^{n-m}$) has a non void interior. In the case that M is a convex surface of a Hilbert space, the above argument can not be applied, because, as we show in [3], we may have $\gamma^\circ(M) = \emptyset$ and $\overline{\gamma(M)} = \Sigma$. We don't know, except in the case that M is a C^∞ manifold, if Wu's result remains true in the case that M is a convex surface of a Hilbert space and $\gamma^\circ(M) = \emptyset$.

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We now state the theorem that we prove here:

THEOREM. *Let $M \subset H$ be a convex surface of a Hilbert space H. Suppose that $\gamma^\circ(M) \neq \emptyset$. Then*

1) $\overline{\gamma(M)} = \gamma^\circ(M)$;

2) $\overline{\gamma^\circ(M)} = \overline{\gamma(\wedge\ )}$:

3) *$\gamma^\circ(M)$ and $\overline{\gamma(M)}$ are geodesically convex sets of the unit sphere $\Sigma \subset H$.*

2. Technical Lemmas

Given a subset $A \subset \Sigma$ we say that a point $v \in \Sigma$ is a *pole* of A if A is contained in the hemisphere $E_v = \{x \in \Sigma;\ \langle x, v\rangle \geq 0\}$. We will denote by $\mathscr{P}(A)$ the set of poles of A and by $h(M)$ the set of points $v \in \Sigma$ such that the height function $h_v(x) = \langle v, x\rangle$ is bounded below on M.

LEMMA 1. $\mathscr{P}(h(M)) = \mathscr{P}(\gamma(M)) = \{v \in \Sigma;\ \{p + tv,\ t \geq 0\} \subset \mathring{K}$ for every $p \in \mathring{K}\}$, *where K is the convex body of M.*

PROOF. Since $\gamma(M) \subset h(M)$, we have that $\mathscr{P}(\gamma(M)) \supset P(h(M))$. Set

$$A = \{v \in \Sigma;\ \{p + tv,\ t \geq 0\} \subset \mathring{K} \quad \text{for every} \quad p \in \mathring{K}\}.$$

We will prove that: 1) $A \subset \mathscr{P}(h(M))$; 2) $\mathscr{P}(\gamma(M)) \subset A$.

1) Suppose that $v \in A$ and that there exists $w \in h(M)$ such that $\langle v, w\rangle < 0$. Since the height function h_w is bounded below on M, we have that there exists a hyperplane L perpendicular to w and such M is in the half-space determined by L to where w points. Take $p \in \mathring{K}$ and consider the 2-dimensional plane containing p and parallel to $\{v, w\}$. Since w is perpendicular to L we have that this plane P intersects L at a line $\{q + tu;\ t \in \mathbb{R}\}$, where q is the intersecion $\{p + tw;\ t \in \mathbb{R}\} \cap L$.Consider the equation $p + tv = q + su$. We assert that this equation has a (unique) solution (t_0, s_0) with $t_0 > 0$. Indeed, since $\{v, u\}$ are linearly independent, because $\langle w, u\rangle = 0$ and $\langle w, v\rangle < 0$, and $p - q$ is in the plane generated by $\{v, u\}$ there exists (a unique) (t_0, s_0) such that $p - q = s_0 u - t_0 v$. Moreover, $-t_0 = \langle p - q, w\rangle / \langle v, w\rangle$. This implies that $t_0 > 0$ and our assertion in proved. Since M is convex we

have that $p + tv \notin K$ for $t > t_0$. This contradicts the fact that $v \quad A$ and we have proved that $A \subset \mathscr{P}(h(M))$.

2) Suppose that $v \in \mathscr{P}(\gamma(M))$ and that for some $p \in \mathring{K}$ the half-line $\{p + tv;\ t \geq 0\}$ intersects M at a point $q = p + t_0 v \in M$. Since $p \in \mathring{K}$ we have that for every $w \in \gamma(p)$,

$$p < \langle p - q, w\rangle = \langle -t_0 v, w\rangle = -t_1\langle v, w\rangle \leq 0.$$

This is a contradiction and the lemma is proved.

LEMMA 2. $\gamma^\circ(M) = h^\circ(M)$.

PROOF. Since $\gamma(M) \subset h(M)$, we have only to prove that $h^\circ(M) \subset \gamma(M)$. Take $v \in h^\circ(M)$ and let L be a hyperplane perpendicular to v that intersects the interior of the convex body K of M. First we will prove that $S = K \cap L$ is bounded. We may suppose, without loss of generality, that L is a (co-dimension one) subspace of H. Denote by Σ' the unit sphere of L. Given $w \in \Sigma'$, take $u \in h^\circ(M)$ such that $u = \alpha v + \beta w$ with $a, \beta > 0$. For every $x \in S$, we have that $\langle u, x\rangle = \beta\langle w, x\rangle$. From this we conclude that the height function h_w is bounded below on S for every $w \in \Sigma'$. Since $h_{-w} = -h_w$, we conclude that the height function h_w is bounded on S for every $w \in \Sigma'$. It follows from the Uniform Boundness Theorem [2], that S is bounded. Next we will prove that the part of K below L is bounded, that is $K_1 = \{x \in K;\ \langle v, x\rangle \leq 0\}$ is bounded. Suppose that K_1 is unbounded and take $a \in \mathring{K} - K_1$. For every positive integer n, let $x_n \in K_1$ be such that $\|x_n - a\| > n$. Denote by $y_n \in S$ the point of intersection of L and the segment joining a and x_n. Since S is closed, convex and bounded we have that S is weakly compact. Let y_0 be a limit point (with respect to the weak topology) of the sequence $\{y_n\}$. From the fact that K is weakly closed (because K is closed and convex) it is not difficult to prove that the half-line $\{a + tw;\ t \geq 0\}$, where $w = y_0 - a / \|y_0 - a\|$, is contained in $\mathring{K}$. It follows from Lemma 1 that w is a pole of $h(M)$ and, since $v \in h^\circ(M)$, we have that $\langle v, w\rangle > 0$. On the other hand, since $\langle v, y_n\rangle \leq 0$, we have that $\langle v, w\rangle \leq 0$. This is a contradiction and we conclude that K_1 is bounded. Since K_1 is closed and convex we have that K_1 is weakly compact and the height function $h_v(x)$ assumes its minimum at point $p \in K_1$. It is clear that $p \in M$ and hence $v \in \gamma(M)$. The lemma is proved.

LEMMA 3. $\overset{\circ}{\overline{\gamma(M)}} \subset h^\circ(M)$.

PROOF. Take $v \parallel \overset{\circ}{\overline{\gamma(M)}}$ and let $exp_v : T\Sigma_v \longrightarrow \Sigma$ be the exponential map. Let $B_r(-v)$ be a closed ball of Σ with center $-v$ and radius r such that $\gamma^\circ(M) - B_r(-v) \neq \emptyset$ and $\gamma^\circ(M) \cap B_r(-v)$. Set

$$A = \{w \in S(v);\ exp_v((\pi - r)w) \in \gamma^\circ(M)\},$$

where $S(v)$ is the unit sphere of $T\Sigma_v$. It is clear that A is an open non void set of $S(v)$. Since $v \in \overline{\gamma(M)}$ we have that there exist t, $0 < t < r$, and $w \in A$ such that $exp_v(-tw) \subset \gamma(M)$. It follows that there exist $\alpha, \beta > 0$ and $w, u \in \gamma(M)$ such that $v = \alpha w + \beta u$ and, hence, the height function h_v is bounded below on M. This proves that $\overset{\circ}{\overline{\gamma(M)}} \subset h^\circ(M)$, as was to be shown.

3. Proof of the Theorem

1) From Lemmas 2 and 3 we have that $\overset{\circ}{\overline{\gamma(M)}} \subset h^\circ(M) = \gamma^\circ(M)$. It follows then that $\overset{\circ}{\overline{\gamma(M)}} = \gamma^\circ(M)$.

2) It is easy to prove that $h(M)$ is a geodesically convex set of Σ. Indeed, suppose that $v, w \in h(M)$, $v \neq -w$ and let u be a point on the minimal geodesic segment joining v and w. We have then that $u = \alpha v + \beta w$ with $\alpha, \beta > 0$. It follows that $u \in h(M)$. If $v = -w$ then at least one of the geodesic segment joining v and w contains a point $u \neq \pm v$. By the above argument, this segment is contained in $h(M)$ and we have proved that $h(M)$ is geodesically convex. We will prove now that $\overline{\gamma^\circ(M)} = \overline{\gamma(M)}$. To prove this, we have only to show that $\gamma(M) \subset \overline{\gamma^\circ(M)}$. Take $v \in \gamma(M)$ and let $U \subset \gamma^\circ(M)$ be an open set such that $-v \notin U$. For each $x \in U$ denote by g_x the minimal geodesic segments joining v and x. Since $h(M)$ is geodesically convex, we have that $A = \bigcup_{x \in U} g_x \subset h(M)$. Since $A - \{v\}$ is open we have that $v \in \overline{h^\circ(M)} = \overline{\gamma^\circ(M)}$.

3) First observe that if $C \subset \Sigma$ is geodesically convex then $\overset{\circ}{C}$ and $\overline{C}$ are geodesically convex. By Lemma 2 we have that $\gamma^\circ(M) = h^\circ(M)$ and, from the fact that $h(M)$ is geodesically convex, we have that $\gamma^\circ(M)$ and $\overline{\gamma(M)} = \overline{\gamma^\circ(M)}$ are geodesically convex.

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Instituto de Matemática Pura e Aplicada
Rio de Janeiro - BRASIL