A Note on a Central Limit Theorem for Dependent Random Variables*

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1. INTRODUCTION. Let $\{\psi_n(x), n = 0, 1, \ldots, x \in \mathbb{R}_+\}$ be the set of Walsh functions, periodically extended to the non-negative real numbers. These functions are defined as products of Rademacher functions and form an orthonormal, complete set on $[0, 1]$. They assume only the values -1 and $+1$ and may be identified with the full set of characters of the dyadic group. This is the set of all sequences $\bar{x} = \{x_n\}$, where $x_n = 0$ or $x_n = 1$ and the group operation is addition modulo 2, $\dot{+}$, componentwise. For the necessary details and notation we refer to Fine [2] and Morettin [3].

Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a strictly stationary sequence with $E\{X_n\} = 0$, for all n, and covariance function, $R_{XX}(k) = E\{X_n X_{n+k}\}$, $k = 0, 1, 2, \ldots$. If we assume that $\sum_k |R_{XX}(k)| < \infty$, then we define the (Fourier) spectrum of X_n as being

$$g_{XX}(x) = (2\pi)^{-1} \sum_k R_{XX}(k) e^{-ixk}, \tag{1}$$

$-\infty < x < \infty$. This is bounded, uniformly continuous and of period 2π. Also

$$R_{XX}(k) = \int_{-\pi}^{\pi} e^{ik\alpha} g_{XX}(\alpha)\, d\alpha. \tag{2}$$

Let the cumulant of order r of X_n be denoted by

$$c_{X\cdots X}(n_1, \ldots, n_r) = cum\{X_{n_1}, \ldots, X_{n_r}\}, \tag{3}$$

$n_1, \ldots, n_r = 0, 1, 2, \ldots$, assuming $E\{|X_n|^r\} < \infty$. By stationarity,

$$c_{X\cdots X}(n_1, \ldots, n_r) = c_{X\cdots X}(n_1 + u, \ldots, n_r + u),$$

and in asymmetric notation,

$$c_{X\cdots X}(n_1, \ldots, n_{r-1}) = c_{X\cdots X}(n_1, \ldots, n_{r-1}, 0). \tag{4}$$

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Let $X_0, X_1, \ldots, X_{N-1}$ be N observed values of $\{X_n\}$ and consider the *finite Walsh transform*

(5) $$d^{(N)}(x) = \sum_{n=0}^{N-1} X_n \psi_n(x),$$

$0 < x < \infty$.

2. THEOREM. *Assume* $E\{X_0^2\} < \infty$ *and*

$$\sum_{u_1, \ldots, u_{r-1}} |c_{X \cdots X}(u_1, \ldots, u_{r-1})| < \infty$$

Suppose also that

(6) $$\lim_{N \to \infty} N^{-1} \sum_{n=0}^{N-k} \sum_{k=0}^{N-1} \psi_{n \dot{+} (n \dot{+} k)}(x) R_{XX}(k) = A(x)$$

exists, for all $x \in \mathbb{R}_+$. *Then* $d^{(N)}(x)$ *is asymptotically normal* $\mathcal{N}(0, NB(x))$, *where* $B(x) = E\{X_0^2\} + 2A(x)$.

PROOF. We have that $E\{d^{(N)}(x)\} = 0$ and

$$E\{d^{(N)}(x)^2\} = N\, E\{X_0^2\} + 2 \sum_{t=1}^{N} \sum_{s=1}^{N} \psi_{t \dot{+} s}(x)\, E\{X_t X_s\};$$

by stationarity this equals to

$$N\, E\{X_0^2\} + 2 \sum_{k=1}^{N-1} \sum_{s-1}^{N-k} \psi_{s \dot{+} (s+k)}(x)\, R_{XX}(k),$$

and therefore $N^{-1} \cdot E\{d^{(N)}(x)^2\} \longrightarrow B(x)$, by (6). For the higher order cumulant,

$$cum\{d^{(N)}(x_1), \ldots, d^{(N)}(x_r)\} = \sum_{n_1=0}^{N-1} \cdots \sum_{n_r=0}^{N-1} \psi_{n_1}(x_1) \ldots \psi_{n_r}(x_r).$$

$$\cdot\, cum\{X_{n_1}, \ldots, X_{n_r}\} = O(N),$$

hence $N^{-r/2} \cdot cum\{d^{(N)}(x_1), \ldots, d^{(N)}(x_r)\} \longrightarrow 0$ as $N \longrightarrow \infty$, if $r > 2$, and the theorem is proved by a basic lemma of Chapter 4 of [1].

3. COMMENTS. The theorem holds true for an m-dependent stationary process. Here, $R_{XX}(k) = 0$, for $|k| > m$. For this case, and $x = 0$, we have that

$$d^{(N)}(0) = \sum_{n=0}^{N-1} X_n \longrightarrow \mathcal{N}(0, N \int_{-\pi}^{\pi} D_m(\alpha)\, g_{XX}(\alpha) d\alpha),$$

where $D_m(\alpha) = \sum_{|j|<m} e^{ij\alpha}$ is the Dirichlet kernel. Here, $B(0) = \sum_{u=-m}^{m} R_{XX}(u)$ and use (2). Note that the distribution of $d^{(N)}(0)$ may be approximated by $\mathcal{N}(0, N g_{XX}(0))$, since $D_m(\alpha)$ is concentrated near 0. In particular, for $m = 0$, that is, a 0-dependent stationary sequence, $d^{(N)}(x)$ is asymptotically

$$\mathcal{N}\,(0, N \int_{-\pi}^{\pi} g_{XX}(\alpha)\, d\alpha).$$

REFERENCES

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