A Characterisation of Henselian Valuations via the Norm*

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Let (K, ϕ) be a non-trivial valuated field, where ϕ is a real valuation which may be archimedean or non-archimedean. It is well-known that if ϕ admits a unique prolongation Ψ to a finite extension $L|K$ of degree n, then Ψ is given by the norm, namely

$$\Psi(x) = [\phi \circ N_{L|K}(x)]^{\frac{1}{n}}$$

In general however, the norm does not define a valuation and the question when it does is not dealt with explicity in the literature. In this note, we give an answer to this question. It is interesting to note that the uniqueness of the prolongation of ϕ to L can also be settled in reference to the norm $N_{L|K}$. This immediately gives a characterisation of Henselian valuations. The same question is then treated for Krull valuations of arbitrary rank. It is also curious that the proofs in the two cases should be so different.

PROPOSITION 1. *Let L be a finite extension of K of degree n. Then ϕ admits a unique prolongation to L if and only if the function $\Psi\colon L \longrightarrow \mathbb{R}_+$ defined by*

$$\Psi(x) = [\phi \circ N_{L|K}(x)]^{\frac{1}{n}}$$

defines a valuation of L.

PROOF. Assume that Ψ is a valuation of L. $\Psi_1, \Psi_2, \ldots, \Psi_s$ be the prolongations of Ψ to L. Assume that $s > 1$, we will show that this leads to a contradiction. Sence Ψ is given to be a valuation of L and it certainly extends ϕ, it must be equal to one of the Ψ_i, say $\Psi = \Psi_1$. Relating the global norm to the local norms (See 2-5-1, Proposition of [1]), there exist integers $t_i > 0$, $i = 1, 2, \ldots, s$ such that

$$\prod_{i=1}^{s} \Psi_i^{t_i}(a) = \phi(N_{L|K}(a)), \qquad \forall\, a \in L.$$

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Taking the n^{th} root and using $\Psi = \Psi_1$, we get

$$\Psi_1^{(t_1-n)/n}(a) \prod_{i=2}^{s} \Psi_i^{t_i/n}(a) = 1, \qquad \forall\, a \in L.$$

This means a finite product formula holds in L with the exponents different from 0 and this is impossible by the Approximation Theorem (See 1-4-3, Exercise (ii) of [1]), Thus $s = 1$ and so ϕ admits a unique prolongation to L, given by Ψ. The other part of the Theorem is obvious.

The following characterisation of Henselian valuations is now immediate.

THEOREM 1. *ϕ is a Henselian valuation if and only if for every finite algebric extension L of K, the function Ψ defined by*

$$\Psi(x) = [\phi \circ N_{L|K}(x)]^{1/[L:K]}$$

is a valuation of L.

We shall now extend this result to Krull valuations of arbitrary rank. We note that the simple proof presented here also applies to rank-1 non-archimedean valuations.

(K, v) will denote a Krull-valued field with valuation ring A and maximal ideal M. $L|K$ will denote a finite algebraic extension of K of degree n. Let $\tilde{A}$ be the integral closure of A in L. If G is the value group of v, we denote by $\tilde{G}$ its divisible closure. We will use the following known results:

1) $a \in \tilde{A}$ is a unit if and only if $N_{L|K}(a)$ is a unit of A [6, §12, Proposition 12].

2) v has a unique prolongation to L if and only if $\tilde{A}$ is a local ring (Corollary (13.5) of [4]).

PROPOSITION 2. *v admits a unique prolongation to L if and only if the function w: $L \longrightarrow \tilde{G}$ defined by*

$$w(x) = \frac{1}{n}\, v(N_{L|K}(x))$$

defines a Krull valuation of L.

PROOF. It is enough to prove that if w does define a valuation of L then the prolongation is unique. For this again by the result above, it suffices to prove that $\tilde{A}$ is a local ring. Using the result above on the units of $\tilde{A}$, we note that an element x of $\tilde{A}$ is a non-unit if and only if $N_{L|K}(x)$ is a non-unit of A, that is if and only if $w(x)$ is positive in $\tilde{G}$. Since w is given to be a valuation, this means that the set of all non-units of $\tilde{A}$ is an ideal of $\tilde{A}$. This proves that $\tilde{A}$ is a local ring.

The following result is immediate now.

THEOREM 2. *v is a Henselian valuation if and only if for every finite algebraic extension L of K, the function w defined by*

$$w(x) = \frac{1}{[L:K]}\, v(N_{L|K}(x)), \qquad \forall\, x \in L$$

is a valuation of L.

REFERENCES

[1] WEISS, E., Algebraic Number Theory, McGraw-Hill Book Company, Inc., New York, 1963.

[2] MCCARTHY, P. J., Algebraic Extensions of Fields, Blaisdell Publishing Company, Waltham, 1966.

[3] RIBENBOIM, P., L'Arithmétique des Corps, Hermann, Paris, 1972.

[4] ENDLER, O., Valuation Theory, Springer-Verlag, New York, 1972.

[5] RIBENBOIM, P., Algebraic Numbers, Wiley-Interscience, New York, 1972.

[6] BOURBAKI, N., Algèbre, Chapitre 8, Hermann, Paris, 1958.

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