Asymptotic Integration of Nonlinear Systems of Ordinary Differential Equations*

Dedicado à memória de Ayrton Badelucci

A. F. IZÉ**

1. The asymptotic behavior of the solutions of the nonlinear system of differential equations

$$\dot{x} = A(t)x + f(t, x) + h_i(t), \qquad A(t) = (a_{ij}(t)), \tag{1}$$

$$i, j = 1, 2, \ldots, n \qquad (\dot{\cdot}) = \frac{d}{dt}$$

will be considered, where $x = (x_1, x_2, \ldots, x_n)$. We assume that $h_i(t)$, $a_{ij}(t)$, $i, j = 1, 2, \ldots, n$ are complex continuous functions in $[t_0, \infty)$, $f_i(t_1 x)$ are continuous in

$$\Omega = \{(t, x) \in E^{n+1} \mid t_0 < t < \infty, |x| < R \leq \infty\},$$

$E = \mathbb{R}$ or $\mathbb{C}$ where $t_0 \geq 1$ can be chosen sufficiently large. (The results stated here are also true if we assume that $h_i(t)$ is L-measurable and $f_i(t, x)$ is Lebesgue measurable for each x and continuous in x for each t.) We assume also the unicity of the solutions of (1) in the points of Ω.

We will show that all the solutions of (1) are "close" respectively to the solutions of the linear homogeneous system associated to (1):

$$\dot{z} = A(t)z. \tag{2}$$

In Section 3 we study the asymptotic behavior of the solutions of (1) and the results obtained there generalize results of Hille [12], Haupt[11], and Waltman [19] on the existence of nonoscillating solutions of a second order differential equation, Hallam [8] and [9], Faedo [5], Ghizzetti [6] (see also Sobol [18]) for an n^{th} order linear system. Cesari [4, p. 42 and 83] and Hartman [10, p. 321, Section 17] give a survey of early results on this sucject.

*Recebido pela SBM em 3 de outubro de 1973.

**This research was completed under partial support of "Fundação de Amparo à Pesquisa do Estado de São Paulo", (FAPESP), "Coordenação de Aperfeiçoamento de Pessoal de Nivel Superior", (CAPES), and "Conselho Nacional de Pesquisas", (CNPq).

The results of Wilkins [22], Bellman [2] and Waltman [20] from a different point of view are also particular cases of the results of Section 3 (see also Cesari [4, p. 42] for further references). Theorem 1 of Section 3 contains, in particular, the Dini-Hukuhara theorem [14] and its extensions made by Bellman [1], Caligo [3] and Weyl [21] and the results on boundedness and asymptotic stability in the large of Golomb [7].

2. We will need the following lemmas:

Lemma 1. *If*

$$v(t) \leq \rho + \int_T^s \rho(s, v(s))$$

where $f(t, v)$ is continuous and monotonic nondecreasing in v in the region defined by $|t - T| < a$, $|v - \rho| < b$, where a and b are positive real numbers, then $v(t) \leq z(t)$ where $z(t)$ is the maximal solution of the differential equation $\dot{z} = f(t, z)$ through (T, ρ) for $t \leq T$.

For a proof of Lemma 1, see Hartman [Corollary 4.4, p. 29] and for a more general form see Nohel [16, p. 326].

Lemma 2. *Let $a_i \geq 0$, $b_i \geq 0$, $r_i \geq 0$ and $r = \max r_i$, $i = 1, 2, \ldots, n$. If $b_i > 1$ for some i then*

$$\sum_{i=1}^n a_i b_i^{r_i} \leq \left[\sum_{i=1}^n a_i\right]\left[\sum_{i=1}^n b_i\right]^r.$$

Proof.

$$\sum_{i=1}^n a_i b_i^{r_i} \leq \sum_{i=1}^n a_i \sum_{j=1}^n b_j^{r_j} \leq \sum_{i=1}^n a_i \left(\sum_{j=1}^n b_j\right)^r.$$

Lemma 3. *Let $\Omega \subset E^n = \mathbb{R}^n$ or $\mathbb{C}^n$ a measurable set of points and let $f(t, s)$ be summable in Ω for values of t in $[t_0, \infty)$. Assume that there exists a summable non-negative function $\phi(s)$ such that $|f(t, s)| \leq \phi(s)$ for almost all values of s in Ω and all values of t in $[t_0, \infty)$, Then if $\lim_{t\to\infty} f(t, s)$ exists for all (or almost all) values of s in Ω we have*

$$\lim_{t\to\infty} \int_\Omega f(t, s)\, ds = \int_\Omega \lim_{t\to\infty} f(t, s)\, ds.$$

Lemma 3 was proved in [13, p. 322].

We need also the simple lemma on matrices [15].

Lemma 4. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be real or complex square matrices $n \times n$ with B non singular. If $C = AB^{-1}$ where $C = (c_{ij})$ then*

$$c_{ij} = \frac{\det C'_{ij}}{\det B}$$

where C' is the matrix obtained by substitution of the i^{th} row of A in the j^{th} row of B.

3 Let $U(t) = (z_{ij}(t))$, $i, j = 1, 2, \ldots, n$ be a fundamental matrix of (2) and suppose that there exists a diagonal matrix $\rho(t) = (\rho_i(t))$, $i = 1, 2, \ldots, n$ satisfying the condition $z_{ij}(t) = (a_i + o(1))\rho_i(t)$ ($(a_i(t) = a_i + o(1)$ means $\lim_{t\to\infty} a_i(t) = a_i$)) where $\rho_i(t)$ are defined and continuous in $[t_0, \infty)$.

Assume with respect to system (1) the hypothesis

$H_1)$
$$|f_i(t, x)| \leq \sum_{j=1}^n \varepsilon_{ij}(t)\, |x_j|^{r_j}$$

where $\varepsilon_{ij}(t)$ is continuous and

$$\int^\infty \varepsilon_{ij}(t)\, |\rho_i(t)|^{r_j}\, |\det \rho(t)| \left[\exp - \int_T^t Tr \cdot A(s)\, ds\right] dt < \infty$$

$r_j > 0$, $j = 1, 2, \ldots, n$.

In hypothesis H_1), $Tr \cdot A(t) = \sum_{i=1}^n a_{ii}(t)$.

$H_2)$
$$\int^\infty |h_i(t)\, \rho_i(t)|^{-1}\, |\det \rho_k(t)| \left[\exp - \int_T^t Tr \cdot A(s)\, ds\right] dt < \infty.$$

Theorem 1. *Let hypotheses H_1) and H_2) be satisfied with respect to system (1) and let $r = \max r_i$. Then, for any $r > 0$, for every solution $z(t) = (z_1(t), z_2(t), \ldots, z_n(t)$ of (2) with $|z(t_0)|$ sufficiently small there exists a solution $x(t)$*

of (1) *with* $x(t_0) = z(t_0)$ *such that* $x_i(t) = z_i(t) + (a_i + \text{o}(1))\, \rho_i(t)$. *In particular, if there exists a solution* $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ *of* (2) *such that* $z_i(t) = (a_i + o(1))\, \rho_i(t)$ *with* $(a_1, a_2, \ldots, a_n) \neq 0$ *then there exists at least a solution* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *of* (1) *with* $x(t_0) = z(t_0)$ *such that* $x_i(t) = z_i(t) + (a_i + o(1))\, \rho_i(t)$ *with* $(a_i, a_2, \ldots, a_n) \neq 0$.

If $0 < r \leq 1$, *then every solution of* (1) *satisfies the following condition*

H) For every solution $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ *of* (2) *there exists a solution* $x(t) = (x_1(t), \ldots, x_n(t))$ *of* (1) *with* $x(t_0) = z(t_0)$, *such that* $x_i(t) = z_i(t) + (a_i + 0(1))\, \rho_i(t)$ *and conversely, for every solution* $x(t)$ *of* (1) *there exists a solution* $z(t)$ *of* (2) *such that* $x_i(t) = z_i(t) + (a_i + o(1))\, \rho_i(t)$.

PROOF. A general solution of (1) can be written in the form

$$x(t) = z(t) + \int_{t_0}^{t} U(t)\, U^{-1}(s)\, f(s, x(s))\, ds + \int_{t_0}^{t} U(t)\, U^{-1}(s)\, h(s)\, ds.$$

By Lemma 4, $U(t)\, U^{-1}(s) = C_{ir}(t, s)$ where

$$C_{ir}(t, s) = det\, \frac{C'_{ir}(t, s)}{det\, U(s)}$$

and $(C'_{ir}(t, s))$ is a matrix which we obtain by substitution of the row of order i of matrix $U(t)$ in the row of order r in the matrix $U(s)$. Then

$$\begin{aligned} C'_{ir}(t, s) &= \sum_{l=1}^{n} z_{il}(t)\, G_{rl}(s) \\ &= \sum_{l=1}^{n} z_{il}(t)\, \rho_i(t)\, \rho_i(t)^{-1}\, \rho_r(s)^{-1} \prod_{k=1}^{n} \rho_k(s)\, U_{rl}(s) \\ &= \rho_i(t)\, \rho_r(s)^{-1}\, det\, \rho(s) \sum_{l=1}^{n} D_{rl}(t, s) \end{aligned}$$

where $D_{rl}(t, s)$ is a bounded function and $\lim_{t \to \infty} D_{rl}(t, s)$ exists, since $\lim z_{il}(t)\, \rho_i(t)^{-1}$ exists and is bounded by hypothesis.

By Jacobi-Liouville's formula we have

$$det\, U(s) = K \int_{T}^{s} Tr \cdot A(v)\, dv, \qquad T \geq t_0$$

and

$$C_{ir}(t, s) = \frac{1}{k} \left[exp - \int_{T}^{s} Tr \cdot A(v)\, dv \right] \rho_i(t)\, \rho_r(s)^{-1}\, det\, \rho(s)\, D_{rl}(t, s).$$

Then

$$\begin{aligned} \frac{x_i(t)}{\rho_i(t)} &= \frac{z_i(t)}{\rho_i(t)} + \frac{1}{\rho_i(t)} \int_{t_0}^{t} \sum_{r=1}^{n} C_{ir}(t, s)\, \rho_r(s, x(s))\, ds + \\ &+ \frac{1}{\rho_i(t)} \int_{t_0}^{t} \sum_{j=1}^{n} C_{ij}(t, s)\, h_j(s) ds = \frac{z_i(t)}{\rho_i(t)} + \\ &+ \frac{1}{K} \int_{t_0}^{t} det\, \rho(s) \left[exp - \int_{T}^{s} Tr \cdot A(v)\, dv \right] \sum_{r=1}^{n} \rho_r^{-1}(s) \sum_{l=1}^{n} D_{rl}(t, s)\, f_r(s, x(s))\, ds + \\ &+ \frac{1}{K} \int_{t_0}^{t} det\, \rho(s) \left[exp - \int_{T}^{s} Tr \cdot A(v)\, dv \right] \sum_{j=1}^{n} \rho_j^{-1}(s) \sum_{l=1}^{n} D_{jl}(t, s)\, h_j(s)\, ds. \end{aligned}$$

By hypothesis H_1) and H_2) we have

$$\begin{aligned} \frac{|x_i(t)|}{|\rho_i(t)|} &\leq \frac{|z_i(t)|}{|\rho_i(t)|} + \frac{1}{K} \int_{t_0}^{t} |det\, \rho(s)| \left[exp - \int_{T}^{s} Tr \cdot A(v)\, dv \right] \cdot \\ &\cdot \sum_{r=1}^{n} |\rho_r^{-1}(s)| \sum_{l=1}^{n} |D_{rl}(t, s)| \sum_{j=1}^{n} \varepsilon_{rj}(s) \frac{|x_j(s)|^{r_j}}{|\rho_j(s)|^{r_j}} |\rho_j(s)|^{r_j}\, ds + \\ &+ \frac{1}{K} \int_{t_0}^{t} |det\, \rho(s)| \left[exp - \int_{t_0}^{t} Tr \cdot A(v)\, dv \right] \sum_{j=1}^{n} \sum_{l=1}^{n} |D_{jl}(t, s)| \frac{|h_j(s)|}{|\rho_j(s)|}\, ds. \end{aligned}$$

By hypothesis $\dfrac{|z_i(t)|}{|\rho_i(t)|} \leq k_i$

$$\begin{aligned} &\frac{1}{K} \int_{t_0}^{t} |det\, \rho(s)| \left[exp - \int_{t_0}^{t} T_r\ A(v)\, dv \right] \sum_{j=1}^{n} \sum_{l=1}^{n} |D_{jl}(t, s)|\, |h_j(s)\, \|\, \rho_j(s)^{-1}|\, ds \leq \\ &\leq C \int_{t_0}^{\infty} |det\, \rho(s)| \left[exp - \int_{t_0}^{s} Tr \cdot A(v)\, dv \right] \sum_{j=1}^{n} |h_j(s)|^{-1} |\rho_j(s)|^{-1}\, ds = C_i\,. \end{aligned}$$

Then we have

$$\frac{|x_i(t)|}{|\rho_i(t)|} \leq C_i + k_i + K_1 \int_{t_0}^{t} |det\, \rho(s)| \left[exp - \int_{t_0}^{t} Tr \cdot A(v)\, dv \right] \cdot$$

$$\cdot\left[\sum_{r,j=1}^{n} \varepsilon_{rj}(s)\,|\rho_r(s)|^{-1}\,|\rho_j(s)|^{r_j} \sum_{k=1}^{n} \frac{|x_k(s)|^{r_j}}{|\rho_k(s)|^{r_j}}\right] ds.$$

If $\sum_{i=1}^{n} (k_i + C_i) = M$,

$$|det\,\rho(s)|\left[exp - \int_T^s Tr \cdot A(v)\,dv\right] \sum_{r,j=1}^{n} \varepsilon_{rj}(s)\,|\rho_r(s)|^{-1}\,|\rho_j(s)|^{r_j} = b(s)$$

$$\sum_{i=1}^{n} \frac{|x_i(t)|}{|\rho_i(t)|} \leq M + nK_1 \int_{t_0}^{t} b(s) \sum_{i=1}^{n} \frac{|x_i(s)|^{r}}{|\rho_i(s)|^{i}}\, ds$$

and by Lemma 2

$$\sum_{i=1}^{n} \frac{|x_i(t)|}{|\rho_i(t)|} \leq M + n^2 k_1 \int_{t_0}^{t} b(s) \sum_{i=1}^{n} \left(\frac{|x_i(s)|}{|\rho_i(s)|}\right)^{r} ds$$

where $r = \max_i r_i$ and by Lemma 1,

$$\sum_{i=1}^{n} \frac{|x_i(t)|}{|\rho_i(t)|} \leq z(t)$$

and $z(t)$ is the maximal solution of the equation

$$(3) \qquad \dot{z} = n^2 k_1 b(s)\, z(s)^r, \qquad \int^{\infty} b(t)\,dt < \infty.$$

The solutions of (3) are

$$z(t) = M\, exp\, n^2 k_1 \int_{t_0}^{t} b(s)\,ds \qquad \text{if} \qquad r = 1$$

$$z(t)^{1-r} = M^{1-r} + (1-r)\, n^2 k_1 \int_{t_0}^{t} b(s)\,ds \qquad \text{if} \qquad r \neq 1.$$

If $0 < r \leq 1$ then $z(t)$ is bounded independently of initial conditions because the coefficients of z^r in the right side of (3) has a convergent integral. If $r > 1$ any solution $z(t)$ of (3) will be bounded provided

$$M^{1-r} > (r-1)\,c_2 \int_{t_0}^{t} b(s)\,ds.$$

Since $z(t_0) = M$ this corresponds to choosing the initial conditions of (1) in such a way that M be sufficiently small. For such a choice of initial con-

ditions any solution $x(t)$ of (1) may be continued to all $t \geq t_0$. We observe that when we use Lemma 2 we must have

$$\frac{|x_i(t)|}{|\rho_i(t)|} > 1$$

for some i but if this is not true $\sum_{i=1}^{n} \left|\frac{x_i(t)}{\rho_i(t)}\right|$ is bounded and this is what is ultimately desired.

Let us show that $\dfrac{x_i(t) - z_i(t)}{\rho_i(t)}$ has a finite limit when $t \to \infty$. From the proof of the theorem

$$\left|\frac{1}{\rho_i(t)} \int_{t_0}^{t} \sum_{r=1}^{n} C_{ir}(t, s)\, f_r(s, x(s))\, ds\right| \leq$$

$$\leq C \int_{t_0}^{t} |det\,\rho(s)| \left[exp - \int_{t_0}^{t} Tr \cdot A(v)\,dv\right] \sum_{r,j=1}^{n} \varepsilon_{rj}(s)\,|\rho_r(s)|^{-1}\,|\rho_j(s)|^{r_j}$$

where C does not depend on t_0, then by Lemma 3

$$\lim_{t\to\infty} \frac{1}{\rho_i(t)} \int_{t_0} \sum_{r=1}^{n} C_{ir}(t, s)\, f_r(s, x(s))\, ds =$$

$$= \lim_{t\to\infty} \frac{1}{K} \int_{t_0}^{t} |det\,\rho(s)| \left[exp - \int_T^s Tr \cdot A(v)\,dv\right] \sum_{r=1}^{n} \rho_r(s)^{-1} \sum_{l=1}^{n} D_{rl}(t, s)\, f_r(s, x(s))\, ds$$

$$= A_{i1} < \infty.$$

Since $\lim_{t\to\infty} D_{rl}(t, s)$ exists and is bounded, and in an analogous manner

$$\lim_{t\to\infty} \int_{t_0}^{t} \sum_{r=1} C_{ir}(t, s)\, h(s)\, ds = A_{i2} < \infty.$$

It remains to show that if there exists a solution of (2) such that $z_i(t) = a_i + o(1))\,\rho_i(t)$, with $(a_1, a_2, \ldots, a_n) \neq 0$ then there exists a least a solution $x(t)$ of (1) with $x(t_0) = z(t_0)$ for which $x_i(t) = z_i(t) + (a_i + o(1))$. Then for some k, $a_k \neq 0$, if we take t_0 large enough, A_{i1} and A_{i2} can be made arbitrarily small in such a way that $|a_{i1}| > |A_{i1} + A_{i2}|$, then

$$\lim_{t\to\infty} \frac{x_k(t)}{\rho_i(t)} = a_{k1} + A_{k1} + B_{k1} \neq 0.$$

We have to prove now that if $r \leq 1$, for every solution $x(t)$ of (1) there exists a solution $z(t)$ of (2) such that $x_i(t) = z_i(t) + (a_i + o(1))\, \rho_i(t)$.

For any $x(t)$ choose $\bar{z}(t)$ given by the integral equation

$$x(t) = \bar{z}(t) + \int_{t_0}^{t} U(t)U^{-1}(s)\,[f(s, x(s)) + h(s)]\, ds.$$

$\bar{z}(t)$ is a solution of (2) which satisfies $\bar{z}(t_0) = x(t_0)$ and by the reasoning made in the proof of the theorem we conclude that the solution $x(t)$ can be written in the form $x_i(t) = \bar{z}_i(t) + (a_i + o(1))\, \rho_i(t)$ and the proof is complete.

Theorem 1 can be used to obtain results on the asymptotic behavior of a large class of systems of differential equations. As a simple application consider the singular system of differential equations

(4) $$\dot{x}_i = \sum_{j=1}^{n} C_{ij}\, t^{-p_{ij}}\, x_j + \sum_{j=1}^{n} f_i(t, x) + h_i(t)$$

and the associate linear system

(5) $$\dot{z}_i = \sum_{j=1}^{n} C_{ij}\, t^{-p_{ij}}\, z_j \qquad i, j = 1, 2, \ldots, n$$

where $p_{ij} = \alpha_j - \alpha_i + 1$ and

(6) $$\dot{x}_i = \sum_{j=1}^{n} C_{ij} e^{-q_{ij}} x_j + g_i(t, x) + h_i(t), \qquad i, j, = 1, 2, \ldots, n$$

(7) $$\dot{z}_i = \sum_{j=1}^{n} C_{ij}\, e^{-q_{ij}}\, z_j \qquad i, j = 1, 2, \ldots, n$$

where $q_{ij} = \alpha_j - \alpha_i$.

(5) or (7) has a solution of the form $z_i(t) = a_i t^{\alpha_i}$ $(z_i(t) = a_i e^{\alpha_i t})$ with $(a_1, a_2, \ldots, a_n) \neq 0$ if and only if the column vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies the equation

(8) $$det(C - I\alpha) = 0.$$

It was shown in [15] that equation (8) admits n-vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ which are solutions of (8). Let $\lambda_j = \beta_j + i\gamma_j$, $j = 1, 2, \ldots, n$ be the characteristic roots of

(9) $$det(C - I\alpha - \lambda I) = 0.$$

By (8) at least one of the roots of (9) is equal to zero, we put $\lambda_1 = 0$. If $g_j(t) = \cos(\gamma_j \log t) + i\, sen(\gamma_j \log t)$, then a fundamental matrix of solutions $U(t) = (z_1(t), \ldots, z_n(t))$ of (5) is given by

(10) $$U(t) = (a_{ij}\, t^{\alpha_j + \beta_j}\, g_j(t)), \qquad i, j = 1, 2, \ldots, n$$

and $z_1(t)$ is the column vector $(a_1 t^{\alpha_1}, a_2 t^{\alpha_2}, \ldots, a_n t^{\alpha_n})$.

We consider equation (4) subjected to the following set of conditions

H'_1) $$|f_i(t, x)| \leq \sum_{j=1}^{n} \varepsilon'_{ij}(t)\, t^{(\alpha - \alpha_j - 1) r_j}\, |x_j|^{r_j} \qquad i, j = 1, 2, \ldots, n$$

with ε'_{ij} non-negative, $r_j > 0$, $j = 1, 2, \ldots, n$ and

$$\int^{\infty} \varepsilon'_{ij}(t) t^{\alpha_i (rj - 1) - r_j}\, dt < \infty.$$

H'_2) $$\int^{\infty} |h_i(t)\, t^{-\alpha_i}|\, dt < \infty$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector solution of (8).

We are considering for convenience in this case

$$\varepsilon'_{ij}(t)\, t^{(\alpha_i - \alpha_j - 1) r_j} = \varepsilon_{ij}(t).$$

As we know that

$$exp - \int_{T}^{t} Tr \cdot A(v)\, dv = exp \int_{T}^{t} v^{-1} (\sum_{j=1}^{n} C_{ij})\, dv = K\, t^{-(\sum_{p=1}^{n} C_{pp})}$$

From the fact the sums of the roots of equation (9) is equal to the sum of the elements of the diagonal of matrix given in (8) we have

$$\sum_{p=1}^{n} \beta_p = \sum_{p=1}^{n} (C_{pp} - \alpha_p) - \sum_{p=1}^{n} i\gamma_p$$
$$det\, \rho(t) = t^{\sum_{i=1}^{n} \alpha_i}$$

then if $\beta_p \leq 0$,

$$|det\, \rho(t)\, exp - \int_{T}^{t} Tr\, A(v)\, dv| \leq K$$

if we take $\rho_i(t) = t^{\alpha_i}$ hypotheses H'_1) and H'_2) are the same as H_1) and H_2) because

$$\int^{\infty} \left| \det \rho(t) \left[\exp - \int_T^t Tr \cdot A(s)\, ds \right] \right| |\rho_i(t)|^{-1} |\rho_j(t)^{r_j}| \varepsilon_{ij}(t)\, dt$$

$$\leq K \int^{\infty} |t^{-\alpha_i} t^{\alpha_j r_j} \varepsilon'_{ij}(t)\, t^{(\alpha - \alpha_j - 1) r_j}|\, dt < \infty$$

by H_4) and H'_2) is obiously verified.

The results bellow can be reformulated for system (4) (or (6)) even when the characteristic roots of (9) are multiple, but to avoid complicated calculations we consider only the case in which the characteristic roots of (9) are simple. For system (4) we have then the following

COROLLARY 1. *Let hypotheses H'_1) and H'_2) be satisfied with respect to system (5) with $p_{ij} = \alpha_j - \alpha_i + 1$ for every $C_{ij} \neq 0$, $i, j = 1, 2, \ldots, n$. where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a solution of (8), $r = \max r_i$ and $\beta_i \leq 0$ with $\beta_i < 0$, if $\gamma_i \neq 0$; then for any $r > 0$, for every solution $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ of (5) with $|z(t_0)|$ sufficiently small there exists a solution $x(t)$ of (4) with $x(t_0) = z(t_0)$ such that $x_i(t) = z_i(t) + (a_i + o(1))\, t^{\alpha_i}$ and there exists at least one solution $x(t) = (x_i(t), \ldots, x_n(t))$ which satisfies the condition*

$$x_i(t) = (b_i + o(1))\, t^{\alpha_i}$$

that is

$$\lim_{t \to \infty} \frac{x_i(t)}{t^{\alpha_i}} = b_i$$

constant with $(b_1, b_2, \ldots, b_n) \neq 0$.

If $0 < r \leq 1$, then every solution of (4) satisfies the following condition:

H'). For every solution $z(t) = (z_1(t), \ldots, z_n(t))$ of (5) there exists a solution $x(t) = (x_1(t), \ldots, x_n(t))$ of (4) with $x(t_0) = z(t_0)$ such that

$$x_i(t) = z_i(t) + (a_i + o(1))\, t^{\alpha_i}$$

and conversely, for every solution $x(t)$ of (4) there exists a solution $z(t)$ of (5) such that

$$x_i(t) = z_i(t) + (a_i + o(1))\, t^{\alpha_i}$$

If for some i, $\beta_i = 0$ and $\gamma_i \neq 0$ we can guarantee only the boundedness of $a_i(t)$, $i = 1, 2, \ldots, n$.

THEOREM 2. *Let hypotheses H'_1) and H'_2) be satisfied with respect to system (4) with $p_{ij} = \alpha_j - \alpha_i + 1$ and suppose that for every i there exists at least one $j = j(i)$ such that $c_{ij} \neq 0$. Suppose also that $\mathcal{R}(\alpha_i) > 0$, $i = 1, 2, \ldots, n$ and $\beta_i \leq 0$ with $\beta_i < 0$ if $\gamma_i \neq 0$. Then for any $r > 0$ a necessary and sufficient condition for the solutions of (4) to satisfy the condition*

$$x_i(t) = z_i(t) + (a_i + o(1))\, t^{\alpha_i}$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ is a solution of (5) with $|z(t_0)|$ sufficiently small with at least one solution $x(t) = (x_1(t), \ldots, x_n(t))$ such that

$$x_i(t) = (b_i + o(1)) t^{\alpha_i}$$

with $(b_1, b_2, \ldots, b_n) \neq 0$ is that $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a solution of (8).

If $0 < r \leq 1$ a necessary and sufficient condition for the solutions of (4) to satisfy condition H') is that $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a solution of (8).

PROOF. The sufficient condition is a consequence of Corollary 1. To prove the necessary condition suppose that there exists a solution $x(t)$ of (4) satisfying the condition $x_i(t) = z_i(t) + (a_i + o(1))\, t^{\alpha_i}$. Integration of equation (4) followed by multiplication by t^{α_i} gives

$$a_i(t) = a_i(t_0)\ t_0^{\alpha_i} t^{-\alpha_i} + t^{-\alpha_i} \int_{t_0}^{t} \sum_{j=1}^{n} c_{ij} s^{\alpha_i - 1} a_j s^{\alpha_j - \alpha_i - p_{ij}}\ ds + \tag{11}$$

$$+ t^{-\alpha_i} \int_{t_0}^{t} \sum_{j=1}^{n} c_{ij} s^{\alpha_i - 1} \phi_j(s) s^{\alpha_j - \alpha_i + 1 - p_{ij}}\, ds + t^{-\alpha_i} \int_{t_0}^{t} f_i(s, x(s))\, ds.$$

When $t \longrightarrow \infty$ the first term of the right side of (11) goes to zero and by L'Hospital's rule the third term also goes to zero. By hypothesis H'_1) we have

$$\left| t^{-\alpha_i} \int_{t_0}^{t} f_i(s, x(s))\, ds \right| < t^{-\mathcal{R}(\alpha_i)} \int_{t_0}^{t} \sum_{j=1}^{n} \left| \varepsilon_{ij}(s) s^{\alpha_i (r_j - 1) - r_j} \frac{x_j(s)^{r_j}}{s^{\alpha_j r_j}} s^{\alpha_i} \right| ds \leq$$

$$\leq t^{-\mathcal{R}(\alpha_i)} \int_{t_0}^{t} \sum_{j=1}^{n} \left| \varepsilon_{ij}(s)\, s^{\alpha_i (r_j - 1) - r_j} \right| \frac{|z_j(s) + (a_j + o(1))\, t^{\alpha_j}|^{r_j}}{|t^{\alpha_j}|^{r_j}} ds \leq$$

$$\leq K\, t^{-\mathcal{R}(\alpha_i)} \int_{t_0}^{\infty} \sum_{j=1}^{n} \left| \varepsilon_{ij}(s)\, s^{\alpha_i (r_j - 1) - r_j} \right| ds$$

because $\dfrac{|z_j(s)|}{|t^{\alpha_j}|}$ is bounded and by Lemma 3,

$$\lim_{t\to\infty} t^{-\alpha_i}\int_{t_0}^{t} f_i(s, x(s))\,ds = 0.$$

As we have $\alpha_j + 1 - p_{ij} = \alpha_i \neq 0$

$$\lim_{t\to\infty} t^{-\alpha_i}\int_{t_0}^{t} s^{\alpha_i-1}\, c_{ij}a_j s^{\alpha_j-\alpha_i+1-p_{ij}}\,ds = \frac{c_{ij}a_j}{\alpha_j+1-p_{ij}}.$$

Taking the limit as $t \longrightarrow \infty$ of both sides of Equation (11) we have

$$a_i\alpha_i = \sum_{j=1}^{n} c_{ij}\alpha_j$$

which has a solution $(a_1, a_2, \ldots, a_n) \neq 0$ if and only if (8) is satisfied.

Remark. In Theorem 1 we can not use a weaker hypothesis than H_1) as is shown by the simple example

$$\dot{x} = \frac{1}{t} + \frac{1}{t(\log t)}x$$

which has the solution $x = c(\log t)$ that does not satisfy the condition H'). Here $\alpha = 1$, $f(t,x) = \dfrac{1}{t\log t}x$, $\varepsilon_{11}(t) = \dfrac{1}{\log t}$ which goes to zero as $t \longrightarrow \infty$ but $\displaystyle\int^{\infty} \frac{1}{t\log t}dt$ is divergent. On the other hand, Moore and Nehari [17] proved that the condition

$$\int^{\infty} a(t)\,t^{2n+1}\,dt < \infty$$

is necessary and sufficient for the equation

$$x'' + a(t)x^{2n+1} = 0, \qquad a(t) > 0$$

to have a solution satisfying $\lim \dfrac{x(t)}{t} = \alpha > 0$. This equation written in system form is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a(t)\,x_1^{2n+1}$$

and in the context of Corollary 1, we have $\alpha_1 = 1$, $\alpha_2 = 0$, $f_2(t,x) = -a(t)x_1^{2n+1}$, $r_1 = 2n+1$, $\varepsilon'_{21}(t) = a(t)\,t^{2r_j}$, $\varepsilon'_{21}(t)\,t^{2(r_1-1)-r_j} = a(t)\,t^{2n+1}$.

4. We consider now Equation (6). We observe that Equation (7) can be reduced to Equation (5) by the change of variable $\tau = \log t$, then a fundamental matrix of (7) is obtained from the fundamental matrix of (5) by the inverse change of variables $t = e^{\tau}$. All results of Section 3 have their analogy for Equation (6) and the proofs are essentially the same. Hypothesis H'_1) and H'_2) become

H_1^*) $$|f_i(t,x)| \leq \sum_{r=1}^{n} \varepsilon'_{ij}(t)\,e^{(\alpha_i-\alpha_j)r_j t}\,|x_j|^{r_j}$$

with $\varepsilon'_{ij}(t)$ non-negative, $r_j > 0$, $j = 1, 2, \ldots, n$ and

$$\int^{\infty} \varepsilon'_{ij}(t)\,e^{\alpha_i(r_j-1)t}\,dt < \infty$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a vector solution of (8).

H_2^*) $$\int^{\infty} |h_i(t)\,e^{-\alpha_i t}|\,dt < \infty$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a vector solution of (8).

Condition H') becomes

H^*). For every solution $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))$ of (7) there exists a solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of (6) with $x_i(t_0) = z_i(t_0)$ such that

$$x_i(t) = z_i(t) + (a_i + o(1))\,e^{\alpha_i t}$$

and conversely for every solution $x(t)$ of (6) there exists a solution $z(t)$ of (7) such that

$$x_i(t) = z_i(t) + (a_i + o(1))\,e^{\alpha_i t}.$$

The analog of Corollary 1 is:

Corollary 1′. *Let hypotheses H_1^*) and H_2^* be satisfied with respect to system (6) with $q_{ij} = \alpha_j - \alpha_i$ for every $c_{ij} \neq 0$, $i, j = 1, 2, \ldots, n$ where $(\alpha_i, \alpha_2, \ldots, \alpha_n)$ is a solution of (8), $r = \max r_i$ and $\beta_i \leq 0$ with $\beta_i < 0$ if $\gamma_i \neq 0$. Then for any $r > 0$,* *for every solution* $z(t) = (z_1(t), z_2(t), \dots, z_n(t))$ *of* (7) *with* $|z(t_0)|$ *sufficiently small there exists a solution* $x(t)$ *of* (6) *with* $x(t_0) = z(t_0)$ *such that*

$$x_i(t) = z_i(t) + (a_i + o(1))\, e^{\alpha_i t}.$$

Also, there exists at least one solution $x(t)$ *of* (6) *satisfying the condition*

$$x_i(t) = (b_i + o(1))\, e^{\alpha_i t}, \qquad i = 1, 2, \dots, n$$

with $(b_1, b_2, \dots, b_n) \neq 0$.

If $0 < r \leq 1$, *then every solution of* (6) *satisfies condition* H^*).

If for some i, $\beta_i = 0$ *and* $\gamma_i \neq 0$, *we can guarantee only the boundedness of* $a_i(t)$, $i = 1, 2, \dots, n$.

The statement of a theorem analogous to Theorem 2 is obvious.

5. Consider now the n^{th}-order equations

(12) $$u^{(n)} - \sum_{i=0}^{n-1} c_i t^{-n+i} u^{(i)} = f(t, u, \dots, u^{(n-1)}) + h(t).$$

(13) $$u^{(n)} - \sum_{i=0}^{n-1} c_i u^{(i)} = f(t, u, \dots, u^{(n-1)}) + h(t).$$

Where $p_i = n-1$ and $q_i = n-i-1$, and the associated homogeneous equations

(14) $$z^{(n)} - \sum_{i=0}^{n-1} c_i t^{-n+i} z^{(i)} = 0.$$

(15) $$z^{(n)} - \sum_{i=0}^{n-1} c_i z^{(i)} = 0.$$

The hypotheses H_1') and H_2') become (as we will see later) the hypotheses

H_1^{**}) $$|f(t, x)| \leq \Sigma\, \varepsilon_i(t)\, t^{-n+i} |u^{(i)}|^{v_i}$$

with $\varepsilon_i(t)$ non-negative, $v_i > 0$, $i = 0, 1, \dots, n-1$ and

$$\int^{\infty} \varepsilon_i(t)\, t^{-n+i-1}\, dt < \infty, \qquad i = 0, 1, \dots, n-1.$$

H_2^{**}) $$\int^{\infty} |h(t)\, t^{n-\alpha_0-1}|\, dt < \infty$$

where α_0 satisfies the equation

$$\alpha(\alpha-1)\dots(\alpha-n+1) - \sum_{i=1}^{n-1} c_i \alpha(\alpha-1)\dots(\alpha-i+1) - c_0 = 0.$$

Condition H') becomes

H^{**}). For every solution $z(t)$ of (14) there exists a solution $u(t)$ of (12) such that

$$u^{(i)}(t) = z^{(i)}(t) + (a_i + o(1)) t^{\alpha_0 - i}$$

and, conversely, for every solution $u(t)$ of (12) there exists a solution $z(t)$ of (14) such that

$$u^{(i)}(t) = z^{(i)}(t) + (a_i + o(1))\, t^{\alpha_0 - i}.$$

In particular, there exists at least one solution $u(t)$ of (12) such that

$$u^{(i)}(t) = t^{\alpha_0 - i}(b_i + o(1))$$

with $b_i \neq 0$, $i = 0, 1, \dots, n-1$.

The following corollaries of Theorem 2 generalize a theorem of Faedo [5] and Ghizzetti [6] (Hartman [10, Theorem 17.1, p. 315] presents a sharper formulation of the result of [5] and [6] which is not contained in Corollary 2). The following theorems are also particular cases of Corollary 2: Theorems 2.1, 2.2 of [8], Theorem 2.1 of [9], Theorem 1 of [19], Theorem 1 of [20], Theorem 1 of [22], a Theorem of Haupt [11] and Theorem 2 of Hille [12] who gives a careful discussion of the non-oscillations of equation $x'' + a(t)x = 0$.

COROLLARY 2. *Let hypotheses* H_1^{**}) *and* H_2^{**}) *be satisfied with respect to Equation* (12) *with* $p_i = n-i$ *if* $c_i \neq 0$, $i = 0, 1, \dots, n-1$ *and let* α_0 *be a root of Equation* (16) *such that* $\beta_j \leq \mathcal{R}(\alpha_0)$ *with* $\beta_j < \mathcal{R}(\alpha_0)$ *if* $\gamma_j \neq 0$ *for every* $j = 2, \dots, n$. *Then for any* $v > 0$, *for every solution* $z(t)$ *of* (14) *with* $|z^{(i)}(t_0)|$ *sufficiently small* $i = 0, 1, \dots, n-1$ *there exists a solution* $u(t)$ *of* (12) *with* $z^{(i)}(t_0) = u^{(i)}(t_0)$ *such that*

$$u^{(i)}(t) = z^{(i)}(t) + (a_i + o(1))\, t^{\alpha_0 - i}.$$

In particular, there exists at least a solution $u(t)$ *of* (12) *satisfying*

$$u^{(i)}(t) = (b_i + o(1))\, t^{\alpha_0 - i}$$

with $b_i \neq 0$, $i = 0, 1, \dots, n-1$. *If* $0 < v \leq 1$ *then the solutions of* (12) *satisfy condition* H^{**}).

if for some i, $\beta_i = \mathcal{R}(\alpha_0)$ and $\gamma_i \neq 0$ we can guarantee only the boundedness of the $a_i(t)$, $i = 1, 2, \ldots, n$.

PROOF. Writing Equation (12) in system form

$$x_1 = u^{(0)}, x_2 = u^{(1)}, \ldots, x_n = u^{(n-1)},$$

$$\varepsilon_0 = \varepsilon_{n1}, \varepsilon_1 = \varepsilon_{n2}, \ldots, \varepsilon_{n-1} = \varepsilon_{nn},$$

$$c_{12} = 1,\ c_{23} = 1, \ldots, c_{n-1,n} = 1$$

$$\alpha_2 - \alpha_1 + 1 = 0, \ldots, \alpha_n - \alpha_{n-1} + 1 = 0.$$

Hence

$$\alpha_2 = \alpha_1 - 1, \alpha_3 = \alpha_2 - 1 = \alpha_1 - 2, \ldots, \alpha_n = \alpha_1 - n + 1.$$

As we have $p_{nj} = \alpha_j - \alpha_n + 1$, it follows that

$$p_{n1} = p_0 = n, \quad p_{n2} = p_1 = n-1, \ldots, \ p_{nn} = p_{n-1} = 1.$$

Then with

$$\alpha_1 = \alpha, \ldots, \alpha_n = \alpha - n + 1$$

the hypotheses H_1^{**}) and H_2^{**}) become hypotheses H_1') and H_2').

Equation (8) can be written

$$(16)\ det(C' - I\alpha) = \alpha(\alpha-1)\ldots(\alpha-n+1) - \sum_{i=0}^{n-1} c_i\alpha(\alpha-1)\ldots(\alpha-i+1) - c_0 = 0$$

which is Equation (8), but if λ_j' is a root of this equation this root corresponds to a root $\lambda_j = 0$ of Equation (9) which is

$$(17) \qquad det(C' - I\alpha - \lambda I) = 0.$$

Let α_0 be a fixed root of (16) and let λ_j be a root of (17) corresponding to α_0. Then $\lambda_j' = \alpha_0 + \lambda_j$ is a root of (16) and the condition $\mathcal{R}(\lambda_j') < \mathcal{R}(\alpha_0)$ corresponds to the condition $\mathcal{R}(\lambda_j) < 0$ and the hypotheses of Corollary 2 bellow are the same as Theorem 2. The following corollary generalizes also Theorem 2.3 of [9].

COROLLARY 3. *Let hypotheses H_1^{**}) and H_2^{**}) be satisfied with respect to Equation (12). Suppose that $p_i = i$ if $c_i \neq 0$ with $c_i \neq 0$ for some i, $i = 0, 1, \ldots, n-1$. Suppose also that $\mathcal{R}(\alpha_0) > n-1$ and the elementary divisors of the minimal polynomial of matrix $C' - I\alpha$ are linear and $\beta_i \leq \mathcal{R}(\alpha_0)$ with $\beta_i < \mathcal{R}(\alpha_0)$ if $\gamma_i \neq 0$. Then for any $v > 0$, a necessary and sufficient condition for the solution of (12) to satisfy the condition*

$$u^{(i)}(t) = z^{(i)} + (a_i + o(1))\, t^{\alpha_0 - i}$$

where $z(t)$ is a solution of (14), $|z^{(i)}(t_0)|$, $i = 0, 1, \ldots, n-1$, sufficiently small with at least one solution $u(t)$ such that

$$u^{(i)} = (b_i + o(1))\, t^{\alpha_0 - i},$$

$b_i \neq 0$, $i = 1, 2, \ldots, n$, is that α_0 be a root of (16).

If $0 < v \leq 1$ a necessary and sufficient condition for the solutions of (12) to satisfy condition H^{**}) is that α_0 be a root of (16).

A similar result can be obtained for equation (13).

Finally, we observe that Theorem 1 is related to the theory of singularities of differential equation. In fact, consider the system

$$(18) \qquad \dot{W} = \sum_{j=1}^{n} a_{ij}(z)\, W_j, \qquad i = 1, 2, \ldots, n$$

where a_{ij} are analytic in a punctured vicinity of the origin and have a pole of order p_{ij} where $p_{ij} = \alpha_j - \alpha_i + 1$ if $a_{ij}(z) \neq 0$ then

$$a_{ij} = z^{\alpha_i - \alpha_j - 1}\, b_{ij}(z)$$

with $b_{ij}(z)$ analytic. Then the system above can be written in the form

$$\dot{W}_i = \sum_{j=1}^{n} z^{\alpha_i - \alpha_j - 1}\, b_{ij}(z)\, W_j, \qquad i = 1, 2, \ldots, n.$$

If we put $W_j = z^{\alpha_j} v_j$ the system is transformed into the system

$$(19) \qquad \dot{v}_i = z^{-1}(b_{ii}(z) - \alpha_i)v_i + \sum_{j=1}^{n} z^{-1} b_{ij}(z)\, v_j$$

which has a pole of order 1 and by Sauvage's Theorem (see, Hartman, [10], p. 73) this system has regular solutions and then system (18) also has regular solutions.

The determination of the solutions of (18) can be carried out by well known methods, where instead of the Euler indicial equation we have Equation (8). Then we have a slight generalization of Sauvage's theorem (see Hartman, [10], Theorem 11, p. 73).

THEOREM 4. *If the system* (18) *has poles at most of order* p_{ij} *at the origin where* $p_{ij} = \alpha_j - \alpha_i + 1$ *if* $a_{ij}(z) \neq 0$ *then the origin is a regular singular point for* (18).

These results can be extended to the non-analytic case in order to contain the results of Faedo [5].

BIBLIOGRAPHY

[1] BELLMAN, R., *The Stability of Solutions of Linear Differential Equations* (3), Duke Math., J., **10**, 643-647 (1943).

[2] BELLMAN, R., *The Boundedness of Solutions of Linear Differential Equations* (3), Duke Math. J., **14**, 83-97 (1947).

[3] CALIGO, D., *Un Criterio Sufficiente di Stabilità per le Soluzioni dei Sistemi di Equazioni Integrali Lineari e sue Applicazioni* (3), Rend. Accad. Italia, 7, 497-506 (1940).

[4] CESARI, L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, 2nd Edition, Academic Press, New York (1963).

[5] FAEDO, S., *Il Teorema di Fuchs per le Equazione Differenziale Lineari a Coefficiente nonAnalitici e Proprietà Asintotiche delle Soluzioni*, Ann. Mat. Pura ed Appl. (4) **25** (1946), 111-133 (X 17).

[6] GHIZZETTI, A., *Sur Comportamento Asintotico degli Integrali delle Equazioni Differentiali Ordinarie Lineari ed Omogenee*, Rend. Mat. Univ. Roma (5), **8** (1949), 28-42.

[7] GOLOMB, M., *Bounds for Solutions of Nonlinear Differential Systems* (6), Archive Rat. Mech. Anal. **1**, 272-282 (1958).

[8] HALLAM, T. G., *Asymptotic Behavior of the Solutions of an n^{th} Order Nonhomogeneous Ordinary Differential Equation*, Trans. Amer. Math. Soc., **122** (1966), 177-194.

[9] HALLAM, T. G., *Asymptotic Behavior of the Solutions of a Nonhomogeneous Singular Equation*, J. Diff. Eqns. **3** (1967), 135-152.

[10] HARTMAN, P., Ordinary Differential Equations, Wiley, New York (1964).

[11] HAUPT, O., *Uber das asymptotische Verhalten der Losungen gewissen linearer gewohnlicher Differentialgleichungen* (3, 4, 5), Math. A., **48** (1943), 289-292.

[12] HILLE, E., *Nonoscillations Theorems* (5), Trans. Am. Math. Soc., **64** (1948), 232-252.

[13] HOBSON, E. N., The Theory of Functions of a Real Variable, Volume 2, Dover Publ., Inc. (1957).

[14] HUKUHARA, M., *Sur les Points Singuliers des Equations Differentielles Linéaires*, J. Fac. Sci. Univ. Hokkaido (1), Math. **2**, 13-81 (1934-1936).

[15] IZE, A. F., *Asymptotic Integration of a Nonhomogeneous Singular Linear System of Ordinary Differential Equations*, J. of Diff. Eq. v. **8**, n.^{0} 1, pp. 1-15 (1970).

[16] NOHEL, J. A., *Some Problems in Nonlinear Volterra Integral Equations*, Bull. Amer. Math. Soc. **68** (1962), pp. 323-329.

[17] MOORE, R. A., and NEHARI, Z., *Nonoscillation Theorems for a Class of Nonlinear Differential Equations*, Trans. Am. Math. Soc., **131** (1959), pp. 30-52.

[18] SOBOL, I. M., *On the Asymptotic Behavior of Solutions of Linear Differential Equations*, Doklady Akad. Nauk **1**, N.° 2 (1948).

[19] WALTMAN, P., *On the Asymptotic Behavior of Solutions of a Nonlinear Equation*, Proc. Amer. Math. Soc. **15** (1964), 918-923.

[20] WALTMAN, P., *On the Asymptotic Behavior of the Solutions of an n^{th} Order Equation*, Monatsch. Math. **69** (1965). 427-430.

[21] WEYL, H., *Comment on the Preceding Paper* (3), Amer. J. Math. **68**, 7-12 (1946).

[22] Wilkins, A., *On the Growth of Solutions of Linear Differential Equations* (3), Bull. Amer. Math. Soc., **50**, 388-394 (1944).

[23] Winter, A., *Linear Variations of Constants*, American J. of Math., Vol. LXVIII (1946), pp. 185-213.

Escola de Engenharia de S. Carlos
Universidade de S. Paulo
S. Carlos - SP
BRASIL