BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 41(1), 37-61
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Singularities of Anti de Sitter torus Gauss maps

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Abstract. We study timelike surfaces in Anti de Sitter 3-space as an application of singularity theory. We define two mappings associated to a timelike surface which are called *Anti de Sitter nullcone Gauss image* and *Anti de Sitter torus Gauss map.* We also define a family of functions named *Anti de Sitter null height function* on the timelike surface. We use this family of functions as a basic tool to investigate the geometric meanings of singularities of the Anti de Sitter nullcone Gauss image and the Anti de Sitter torus Gauss map.

Keywords: Anti de Sitter 3-space, timelike surface, AdS-nullcone Gauss image, AdS-torus Gauss map, Legendrian singularities.

Mathematical subject classification: 53A35, 58C25.

1 Introduction

This paper is written as one of the research projects on differential geometry of submanifolds in Anti de Sitter 3-space from the viewpoint of singularity theory. There are several articles for the study of submanifolds in Minkowski space, which is a flat Lorentzian space, and also in de Sitter space, which is a Lorentzian space with positive curvature [9, 11, 13, 14, 15]. The Lorentzian space form with negative curvature is called Anti de Sitter space which is one of the vacuum solutions of the Einstein equation in the theory of relativity. Singularity theory tools, as illustrated by several papers which appeared so far ([2, 4, 5, 6, 7, 10, 11, 12, 16, 20, 21, 22, 23, 25, 28, 29, 30]), have proven to be useful in the description of geometrical properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint.

Received 17 April 2008.

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†Partially supported by Science Foundation for Young Teachers of Northeast Normal University No. 20070105 and the State Scholarship Fund of CSC No. 20073021, China.

‡Partially supported by Grant-in Aid for Scientific Research No. 18654007 and No. 18340013.

The natural connection between Geometry and Singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Lagrangian and/or Legendrian maps ([1, 24, 26]). However, there are not many results on submanifolds immersed in the Anti de Sitter space, in particular from the view point of singularity theory. In [8] we have studied the spacelike surfaces in Anti de Sitter 3-space as an application of Legendrian singularity theory. We construct a basic framework for the study of timelike surfaces in Anti de Sitter 3-space here. As it was to be expected, the situation presents certain peculiarities when compared with the Minkowski case and the de Sitter case. For instance, in our case it is always possible to choose two lightlike normal directions along the timelike surface in the frame of its normal bundle. This is similar to the de Sitter case, but the normalized image is located in the Lorentzian torus T_1^2. For the de Sitter case, the normalized image of the lightlike normal is located in the spacelike sphere S_+^2. Moreover, there are no closed timelike surfaces in de Sitter space but there are such surfaces in Anti de Sitter space.

In §2 we prepare the basic notions on timelike surfaces in Anti de Sitter 3-space. We define the Anti de Sitter nullcone Gauss image (briefly, AdS-nullcone Gauss image) and the Anti de Sitter torus Gauss map (briefly, AdS-torus Gauss map). We will find the AdS-nullcone Gauss image is more computable than the AdS-torus Gauss map. We also define the Anti de Sitter null Gauss-Kronecker curvature and the Anti de Sitter torus Gauss-Kronecker curvature. We investigate their relations. We can prove that Anti de Sitter torus Gauss-Kronecker curvature is not a Lorentz invariant but it is an $SO(2) \times SO(2)$-invariant. Moreover, these two Gauss-Kronecker curvature functions have the same zero sets. In §3 we introduce the notion of height functions on timelike surfaces, named the AdS-null height function, which is useful to show that the AdS-nullcone Gauss image has a singular point if and only if the Anti de Sitter null Gauss-Kronecker curvature vanished at such point. we also apply the Legendrian singularity theory to interpret the AdS-nullcone Gauss image as a Legendrian map. In §4 we define a surface, named Anti de Sitter torus cylindrical pedal, as a tool to study the relationship between the AdS-nullcone Gauss image and the AdS-torus Gauss map. We also study the contact of timelike surfaces with some model surfaces (i.e., AdS-horospheres) in §5. In §6 we give a generic classification of singularities of AdS-nullcone Gauss image and AdS-torus Gauss map. In the last part, §7, we introduce the notion of the AdS-null Monge form of a timelike surface in Anti de Sitter 3-space and as an application of this notion we give two examples.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 The local differential geometry of timelike surfaces

In this section we introduce the local differential geometry of timelike surfaces in Anti de Sitter 3-space. For details of Lorentzian geometry, see [27].

Let $\mathbb{R}^4 = \{(x_1, \cdots, x_4) | x_i \in \mathbb{R}\ (i = 1, \cdots, 4)\}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \cdots, x_4)$ and $\boldsymbol{y} = (y_1, \cdots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *semi-Euclidean* 4*-space with index* 2 and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We denote the signature of a vector $\boldsymbol{x}$ by

$$\operatorname{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \text{ is spacelike} \\ 0 & \boldsymbol{x} \text{ is null} \\ -1 & \boldsymbol{x} \text{ is timelike} \end{cases}$$

For a vector $\boldsymbol{n} \in \mathbb{R}^4_2$ and a real number c, we define the *hyperplane with pseudo-normal* $\boldsymbol{n}$ by

$$HP(\boldsymbol{n}, c) = \left\{ \boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c \right\}.$$

We call $HP(\boldsymbol{n}, c)$ a *Lorentz hyperplane*, a *semi-Euclidean hyperplane with index 2* or a *null hyperplane* if $\boldsymbol{n}$ is *timelike, spacelike or null* respectively.

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*) by

$$H^3_1 = \left\{ \boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \right\},$$

a *unit pseudo 3-sphere with index 2* by

$$S^3_2 = \left\{ \boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\},$$

and a *closed nullcone* with vertex $\boldsymbol{a}$ by

$$\Lambda_a = \left\{ \boldsymbol{x} \in \mathbb{R}^4_2 | \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0 \right\}.$$

In particular we call Λ_0 the *nullcone* at the origin. We also define the *Lorentz torus* by

$$T^2_1 = \left\{ \boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda_0 | x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1 \right\}.$$

If a non-zero vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \Lambda_0$, we have

$$\tilde{\boldsymbol{x}} = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}}(x_1, x_2, x_3, x_4) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}}\boldsymbol{x} \in T_1^2.$$

For any $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \in \mathbb{R}_2^4$, we define a vector $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ by

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}_2^4$ and $\boldsymbol{X}_i = \left(x_i^1, x_i^2, x_i^3, x_i^4\right)$. We can easily check that $\langle \boldsymbol{X}, \boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 \rangle = \det(\boldsymbol{X}, \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3)$, so that $\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3$ is pseudo-orthogonal to any $\boldsymbol{X}_i$ (for $i = 1, 2, 3$).

We now study the extrinsic differential geometry of timelike surfaces in Anti de Sitter 3-space. Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a regular surface (i.e., an embedding), where $U \subset \mathbb{R}^2$ is an open subset. We denote $M = \boldsymbol{X}(U)$ and identify M with U through the embedding $\boldsymbol{X}$. The embedding $\boldsymbol{X}$ is said to be timelike if the induced metric $\mathbf{I}$ of M is Lorentzian. Throughout the remainder in this paper we assume that M is a timelike surface in H_1^3. We define a vector $\boldsymbol{N}(u)$ by

$$\boldsymbol{N}(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|}.$$

By definition, we have

$$\langle \boldsymbol{N}(u), \boldsymbol{X}(u) \rangle \equiv \langle \boldsymbol{N}(u), \boldsymbol{X}_{u_i}(u) \rangle \equiv 0 \ \text{ and } \ \langle \boldsymbol{X}(u), \boldsymbol{X}_{u_i}(u) \rangle \equiv 0 \ \text{ (for } i = 1, 2).$$

This means that $\boldsymbol{X}(u), \boldsymbol{N}(u) \in N_pM$, where $u = (u_1, u_2) \in U$ and $p = \boldsymbol{X}(u) \in M$. Since the embedding is timelike and $\boldsymbol{X}(u) \in H_1^3$, $\boldsymbol{N}$ is spacelike. Therefore $\langle \boldsymbol{N}(u), \boldsymbol{N}(u) \rangle \equiv 1$. It follows that

$$\boldsymbol{X}(u) \pm \boldsymbol{N}(u) \in \Lambda_0 \cap N_pM \quad \text{and} \quad \widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}(u)} \in T_1^2 \cap N_pM.$$

Thus we can define a map $\mathbb{G}_n^{\pm} : U \longrightarrow \Lambda_0$ by $\mathbb{G}_n^{\pm}(u) = \boldsymbol{X}(u) \pm \boldsymbol{N}(u)$. This map is analogous to the hyperbolic Gauss indicatrix of hypersurfaces in $H_+^n(-1)$ which was defined in [12]. Here, we call it the *Anti de Sitter nullcone*

Gauss image (briefly, *AdS-nullcone Gauss image*) of $\boldsymbol{X}$(or M). We also define a map

$$\widetilde{\mathbb{G}_n^{\pm}} : U \longrightarrow T_1^2 \quad \text{by} \quad \widetilde{\mathbb{G}_n^{\pm}}(u) = \widetilde{\boldsymbol{X}(u) \pm \boldsymbol{N}}(u) = \frac{1}{\xi(u)} \mathbb{G}_n^{\pm}(u),$$

where

$$\xi(u) \;=\; \pm\sqrt{(x_1(u) \pm n_1(u))^2 + (x_2(u) \pm n_2(u))^2}\,.$$

We call it the *Anti de Sitter torus Gauss map* (or, *AdS-torus Gauss map*) of $\boldsymbol{X}$.

We remak that the map $\mathbb{G}_n^{\pm}(u)$ was used by S. Lee [17] to study the timelike sufaces of constant mean curvature ± 1 in Anti de Sitter 3-space. He called $\mathbb{G}_n^{\pm}(u)$ the *hyperbolic Gauss map*. By a direct calculation we know that $\mathbb{G}_n^{\pm}$ is constant if and only if $\widetilde{\mathbb{G}_n^{\pm}}$ is constant.

It is easy to show that $\boldsymbol{N}_{u_i}$ $(i = 1, 2)$ are tangent vectors of M. Therefore we have a linear transformation $S_p^{\pm} = -d\mathbb{G}_n^{\pm}(u) = -(d\boldsymbol{X}(u) \pm d\boldsymbol{N}(u)) \colon T_pM \longrightarrow T_pM$ which is called the *Anti de Sitter null shape operator* (briefly, *AdS-null shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Under the identification of U and M, the derivation $d\boldsymbol{X}(u)$ can be identified with the identity mapping id_{T_pM}, this means that $S_p^{\pm} = -d\mathbb{G}_n^{\pm}(u) = -(\mathrm{id}_{T_pM} \pm d\boldsymbol{N}(u))$. We have another linear mapping

$$d\widetilde{\mathbb{G}_n^{\pm}}(u) : T_pM \longrightarrow T_p\mathbb{R}_2^4 = T_pM \oplus N_pM.$$

If we consider the orthogonal projection $\pi^T : T_pM \oplus N_pM \longrightarrow T_pM$, then we have

$$\widetilde{S_p^{\pm}} = -(d\widetilde{\mathbb{G}_n^{\pm}}(u))^T = -\pi^T \circ d\widetilde{\mathbb{G}_n^{\pm}}(u) : T_pM \longrightarrow T_pM$$

and call it the *Anti de Sitter torus shape operator* (briefly, *AdS-torus shape operator*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We remark that $S_p^{\pm}$ (resp., $\widetilde{S_p^{\pm}}$) does not always have real eigenvalues. If the eigenvalues are real numbers, we denote it by $k_i^{\pm}$ (resp., $\widetilde{k_i^{\pm}}$) (for $i = 1, 2$).

We define

$$K_{AdSn}^{\pm}(u) = \det S_p^{\pm} = k_1^{\pm} \cdot k_2^{\pm} \quad \text{and} \quad \widetilde{K_{AdSt}^{\pm}}(u) = \det \widetilde{S_p^{\pm}} = \widetilde{k_1^{\pm}} \cdot \widetilde{k_2^{\pm}}\,.$$

We respectively call $K_{AdSn}^{\pm}(u)$ the *Anti de Sitter null Gauss-Kronecker curvature* (briefly, *AdS-null G-K curvature*) and $\widetilde{K_{AdSt}^{\pm}}(u)$ the *Anti de Sitter torus Gauss-Kronecker curvature* (briefly, *AdS-torus G-K curvature*) of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. We say that a point $p = \boldsymbol{X}(u)$ is a (*positive or negative*) *Anti de Sitter horospherical parabolic point* (briefly, *AdSh$^{\pm}$-parabolic point*) (resp. *positive* *or negative Anti de Sitter torus parabolic point, briefly, AdSt$^{\pm}$-parabolic point*) of $M = \boldsymbol{X}(U)$ if $K^{\pm}_{AdSn}(u) = 0$(resp. $\widetilde{K^{\pm}_{AdSt}}(u) = 0$). By a straightforward calculation we have the relation $S^{\pm}_p = \xi(u)\widetilde{S^{\pm}_p}$, so that we have $k^{\pm}_i(p) = \xi(u)\widetilde{k^{\pm}_i}(p)$ and $K^{\pm}_{AdSn}(u) = \xi^2(u)\widetilde{K^{\pm}_{AdSt}}(u)$. Then we have the following relations:

$$\begin{cases} k^{\pm}_i(p) = 0 \Longleftrightarrow \widetilde{k^{\pm}_i}(u) = 0 \\ K^{\pm}_{AdSn}(u) = 0 \Longleftrightarrow \widetilde{K^{\pm}_{AdSt}}(u) = 0. \end{cases}$$

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is an *umbilic point* if $S^{\pm}_p = k^{\pm}(p)id_{T_pM}$. We also say that $M = \boldsymbol{X}(U)$ is *totally umbilic* if all points on M are umbilic.

We now consider the geometric meaning of the AdS-nullcone Gauss image of a timelike surface. First, we consider a surface given by the intersection of H^3_1 with the hyperplane $HP(\boldsymbol{n}, c)$. We denote it by $AH(\boldsymbol{n}, c) = H^3_1 \cap HP(\boldsymbol{n}, c)$ and call it a *Anti de Sitter pseudohyperboloid with index* 1 (briefly, *AdS-pseudohyperboloid*), a *Anti de Sitter pseudosphere with index* 1 (briefly, *AdS-pseudosphere*) or a *Anti de Sitter horosphere* (briefly, *AdS-horosphere*) if $\boldsymbol{n}$ is spacelike, timelike and $\|\boldsymbol{n}\| < |c|$ or null respectively. Especially, we call $AH(\boldsymbol{n}, 0)$ the *Anti de Sitter small pseudohyperboloid with index* 1 (briefly, *AdS-small pseudohyperboloid*) if $\boldsymbol{n}$ is spacelike and $c = 0$. Then we have the following proposition.

Proposition 2.1. *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface in Anti de Sitter 3-space. If the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ is constant, then the timelike surface $\boldsymbol{X}(U) = M$ is a part of a AdS-horosphere.*

Proof. We consider the set $V = \{\boldsymbol{y} \in \mathbb{R}^4_2 | \langle \boldsymbol{y}, \boldsymbol{X} \pm \boldsymbol{N}\rangle = -1\}$. Since $\mathbb{G}^{\pm}_n = \boldsymbol{X} \pm \boldsymbol{N}$ is constant, the set $V = HP(\mathbb{G}^{\pm}_n, -1)$ is a null hyperplane. We also have $\langle \boldsymbol{X}, \mathbb{G}^{\pm}_n\rangle \equiv -1$, so $\boldsymbol{X}(U) = M \subset V \cap H^3_1$. $\square$

We also have the following classification theorem on umbilic points.

Proposition 2.2. *Suppose that $M = \boldsymbol{X}(U)$ is totally umbilic. Then $k^{\pm}(p)$ is constant $k^{\pm}$. Under this condition, we have the following classification.*

(1) *Suppose $k^{\pm} \neq 0$.*

 (a) *If $0 < |k^{\pm} + 1| < 1$, then M is a part of an AdS-pseudohyperboloid;*

(b) *If* $|k^{\pm}+1|>1$*, then* M *is a part of an AdS-pseudosphere;*

(c) *If* $k^{\pm}=-1$*, then* M *is a part of an AdS-small pseudohyperboloid.*

(2) *Suppose* $k^{\pm}=0$ *then* M *is a part of an AdS-horosphere.*

The proof is almost the same as that of Proposition 2.3 in [12], so that we omit it. We also call a point $p\in M$ the *Anti de Sitter horospherical point* (briefly, *AdS-horospherical point*) if $k_i^{\pm}(p)=0$ $(i=1,\ 2)$.

We now introduce the pseudo-Riemannian metric $ds^2=\sum_{i,j=1}^2 g_{ij}du_idu_j$ on $M=\boldsymbol{X}(U)$, where $g_{ij}(u)=\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. We also define the *Anti de Sitter null second fundamental invariant* by $h_{ij}^{\pm}(u)=\langle -(\mathbb{G}_n^{\pm})_{u_i}(u),$ $\boldsymbol{X}_{u_j}(u)\rangle$, the *Anti de Sitter torus second fundamental invariant* by

$$\widetilde{h_{ij}^{\pm}}(u)=\langle -(\widetilde{\mathbb{G}_n^{\pm}})_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle=\frac{1}{\xi(u)}h_{ij}^{\pm}(u)$$

for any $u\in U$. We can also show the following Weingarten formulas by exactly the same arguments as those of [8, 12, 15].

Proposition 2.3. *With the above notations the following hold*

(1) *The Anti de Sitter null Weingarten formula:*

$$\left(\mathbb{G}_n^{\pm}\right)_{u_i}=-\sum_{j=1}^{2}(h^{\pm})_i^j\boldsymbol{X}_{u_j},$$

where $\left((h^{\pm})_i^j\right)=\left(h_{ik}^{\pm}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right)=\left(g_{kj}\right)^{-1}$.

(2) *The Anti de Sitter torus Weingarten formula:*

$$\left((\widetilde{\mathbb{G}_n^{\pm}})_{u_i}\right)^T=\pi^T\circ\left(\widetilde{\mathbb{G}_n^{\pm}}\right)_{u_i}=-\sum_{j=1}^{2}(\widetilde{h^{\pm}})_i^j\boldsymbol{X}_{u_j}=-\frac{1}{\xi(u)}\sum_{j=1}^{2}(h^{\pm})_i^j\boldsymbol{X}_{u_j},$$

where $\left((\widetilde{h^{\pm}})_i^j\right)=\left(\widetilde{h_{ik}^{\pm}}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right)=\left(g_{kj}\right)^{-1}$. □

As a corollary of the above proposition, we have the following expression of the AdS-null G-K curvature and AdS-torus G-K curvature.

Corollary 2.4. *With the same notations as in the above Proposition, we have:*

$$K_{AdSn}^{\pm}=\frac{\det(h_{ij}^{\pm})}{\det(g_{ij})}=\xi^2\frac{\det(\widetilde{h_{ij}^{\pm}})}{\det(g_{ij})}=\xi^2\widetilde{K_{AdSt}^{\pm}}.$$ □

3 Height functions on timelike surfaces

In this section we define two families of functions on a timelike surface in Anti de Sitter 3-space which are useful for the study of singularities of the AdS-nullcone Gauss image and the AdS-torus Gauss map.

Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface. We define a family of functions $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ by $H(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle + 1$. We call H an *Anti de Sitter null height function* (or, *AdS-null height function*) on $M = \boldsymbol{X}(U)$. We denote the *Hessian matrix* of the AdS-null height function $h_{v_0}(u) = H(u, \boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}(h_{v_0})(u_0)$. Then we have the following proposition.

Proposition 3.1. *Let $M = \boldsymbol{X}(U)$ be a timelike surface in H^3_1 and $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ be an AdS-null height function. Then we have the following:*

(1) $H(u_0, \boldsymbol{v}) = \frac{\partial H}{\partial u_i}(u_0, \boldsymbol{v}) = 0$ *(for $i = 1, 2$) if and only if $\boldsymbol{v} = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0) = \mathbb{G}^{\pm}_n(u_0)$;*

(2) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is an AdSh$^{\pm}-$parabolic point if and only if* $\det \mathrm{Hess}(h_{v^{\pm}_0})(u_0) = 0$;

(3) *Let $\boldsymbol{v}^{\pm}_0 = \boldsymbol{X}(u_0) \pm \boldsymbol{N}(u_0)$, then $p = \boldsymbol{X}(u_0)$ is an AdS-horospherical point if and only if* $\mathrm{rank}\, \mathrm{Hess}(h_{v^{\pm}_0})(u_0) = 0$.

Proof.

(1) Since $\{\boldsymbol{X}, \boldsymbol{N}, \boldsymbol{X}_{u_1}, \boldsymbol{X}_{u_2}\}$ is a basis of the vector space $T_p\mathbb{R}^4_2$ where $p = \boldsymbol{X}(u)$, there exist real numbers $\lambda, \eta, \alpha_1, \alpha_2$ such that $\boldsymbol{v} = \lambda\boldsymbol{X} + \eta\boldsymbol{N} + \alpha_1\boldsymbol{X}_{u_1} + \alpha_2\boldsymbol{X}_{u_2}$. Therefore $H(u, \boldsymbol{v}) = 0$ if and only if $\lambda = -\langle \boldsymbol{X}(u), \boldsymbol{v}\rangle = 1$. Since $0 = \frac{\partial H}{\partial u_i}(u, \boldsymbol{v}) = \langle \boldsymbol{X}_{u_i}, \boldsymbol{v}\rangle = \sum_{j=1}^{2} g_{ij}\alpha_i$ and (g_{ij}) is non-degenerate, we have $\alpha_i = 0$ (for $i = 1, 2$). Therefore, $\boldsymbol{v} = \boldsymbol{X} + \eta\boldsymbol{N}$. From the fact that $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$, we have $\eta = \pm 1$.

(2) By definition, we have

$$\mathrm{Hess}(h_{v^{\pm}_0})(u_0) = (\langle \boldsymbol{X}_{u_iu_j}(u_0), \mathbb{G}^{\pm}_n(u_0)\rangle) = (-\langle \boldsymbol{X}_{u_i}(u_0), \mathbb{G}^{\pm}_{n\,u_j}(u_0)\rangle).$$

From the AdS-null Weingarten formula, we have

$$-\langle \boldsymbol{X}_{u_i}, (\mathbb{G}^{\pm}_n)_{u_j}\rangle = \sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_i\langle \boldsymbol{X}_{u_\alpha}, \boldsymbol{X}_{u_j}\rangle = \sum_{\alpha=1}^{2}(h^{\pm})^{\alpha}_i g_{\alpha j} = h^{\pm}_{ij}.$$

Therefore we have

$$K^{\pm}_{AdSn}(u_0) = \frac{\det(h^{\pm}_{i,j}(u_0))}{\det(g_{ij}(u_0))} = \frac{\det \operatorname{Hess}(h_{v_0^{\pm}})(u_0)}{\det(g_{ij}(u_0))}.$$

Then assertion (2) is satisfied.

(3) By the AdS-null Weingarten formula, p is an umbilic point if and only if there exists an orthogonal matrix A such that $A^t((h^{\pm})^l_i)A = k^{\pm}I$. Therefore, we have $((h^{\pm})^l_i) = Ak^{\pm}IA^t = k^{\pm}I$. Then we have

$$\operatorname{Hess}(h_{v_0^{\pm}})(u_0) = (h^{\pm}_{ij}(u_0)) = ((h^{\pm})^l_i(u_0))(g_{lj}(u_0)) = k^{\pm}(g_{ij}(u_0)).$$

Thus, $p = \boldsymbol{X}(u_0)$ is a AdS-horospherical point if and only if $\operatorname{rank}\operatorname{Hess}(h_{v_0^{\pm}})(u_0) = 0$. $\square$

As an application of the above proposition, we have the following direct corollary.

Corollary 3.2. *Let $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$, with $H(u, \boldsymbol{v}) = h_v(u)$ be an AdS-null height function on a timelike surface $M = \boldsymbol{X}(U)$ and $\mathbb{G}^{\pm}_n$ be the AdS-nullcone Gauss image, $p = \boldsymbol{X}(u)$. Suppose $\boldsymbol{v}^{\pm} = G^{\pm}_n(u)$, then the following conditions are equivalent:*

(1) *$p \in M$ is a degenerate singular point of the AdS-null height function $h_{v^{\pm}}$*

(2) *$p \in M$ is a singular point of the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$;*

(3) $K^{\pm}_{AdSn}(u) = 0$. $\square$

We can also define another family of functions $\widetilde{H} : U \times T^2_1 \longrightarrow \mathbb{R}$ by $\widetilde{H}(u, \boldsymbol{v}) = \langle \boldsymbol{X}(u), \boldsymbol{v}\rangle$. We call $\widetilde{H}$ an *Anti de Sitter torus height function* (briefly, *AdS-torus height function*) on $\boldsymbol{X}$. We denote the *Hessian matrix* of the AdS-torus height function $\widetilde{h_{v_0}}(u) = \widetilde{H}(u, \boldsymbol{v}_0)$ at u_0 by $\operatorname{Hess}(\widetilde{h_{v_0}})(u_0)$. We remark that this family satisfies the same properties as those stated in Proposition 3.1 and Corollary 3.2.

On the other hand, we can naturally interpret the AdS-nullcone Gauss image $\mathbb{G}^{\pm}_n$ of M as a Legendrian map from the viewpoint of generating family. For notations and some basic results on generating family, please refer to Arnold and Zakalyukin [1, 32]. Then, we have the following fundamental property with respect to the AdS-null height function H.

Proposition 3.3. *The AdS-null height function $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces $h_v^{-1}(0)_{v\in\Lambda_0}$.*

Proof. For any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \Lambda_0$, we have $\boldsymbol{v} \neq \boldsymbol{0}$. Without loss of generality, we might assume that $v_1 > 0$, then $v_1 = \sqrt{v_3^2 + v_4^2 - v_2^2}$. So that

$$H(u, \boldsymbol{v}) = -x_1(u)\sqrt{1 + v_3^2 + v_4^2 - v_2^2} - x_2(u)v_2 + x_3(u)v_3 + x_4(u)v_4 + 1,$$

where $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$. We have to prove the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2}\right)$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}_{u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2}, \boldsymbol{v}\rangle & x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ \langle \boldsymbol{X}_{u_1u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_1u_2}, \boldsymbol{v}\rangle & x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ \langle \boldsymbol{X}_{u_2u_1}, \boldsymbol{v}\rangle & \langle \boldsymbol{X}_{u_2u_2}, \boldsymbol{v}\rangle & x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix}.$$

We claim that it will suffice to show that the determinant of the matrix

$$A = \begin{pmatrix} x_1\frac{v_2}{v_1} - x_2 & -x_1\frac{v_3}{v_1} + x_3 & -x_1\frac{v_4}{v_1} + x_4 \\ x_{1u_1}\frac{v_2}{v_1} - x_{2u_1} & -x_{1u_1}\frac{v_3}{v_1} + x_{3u_1} & -x_{1u_1}\frac{v_4}{v_1} + x_{4u_1} \\ x_{1u_2}\frac{v_2}{v_1} - x_{2u_2} & -x_{1u_2}\frac{v_3}{v_1} + x_{3u_2} & -x_{1u_2}\frac{v_4}{v_1} + x_{4u_2} \end{pmatrix},$$

does not vanish at $(u, \boldsymbol{v}) \in \Delta^* H^{-1}(\boldsymbol{0})$. In this case, $\boldsymbol{v} = \mathbb{G}_n^{\pm}(u)$ and we denote

$$\boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ x_{1u_2} \end{pmatrix}, \ \boldsymbol{b}_2 = \begin{pmatrix} x_2 \\ x_{2u_1} \\ x_{2u_2} \end{pmatrix}, \ \boldsymbol{b}_3 = \begin{pmatrix} x_3 \\ x_{3u_1} \\ x_{3u_2} \end{pmatrix}, \ \boldsymbol{b}_4 = \begin{pmatrix} x_4 \\ x_{4u_1} \\ x_{4u_2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \det A &= -\frac{v_1}{v_1}\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) + \frac{v_2}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4) \\ &\quad -\frac{v_3}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4) + \frac{v_4}{v_1}\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} &= \big(-\det(\boldsymbol{b}_2,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4),\ \det(\boldsymbol{b}_1,\ \boldsymbol{b}_3,\ \boldsymbol{b}_4), \\ &\qquad \det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_4),\ -\det(\boldsymbol{b}_1,\ \boldsymbol{b}_2,\ \boldsymbol{b}_3)\big). \end{aligned}$$

Therefore we have

$$\begin{aligned}\det A &= \left\langle \left(-\frac{v_1}{v_1}, -\frac{v_2}{v_1}, -\frac{v_3}{v_1}, -\frac{v_4}{v_1} \right), \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \right\rangle \\ &= -\frac{1}{v_1} \langle \mathbb{G}_n^{\pm}, \| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \| \boldsymbol{N} \rangle \\ &= \mp \frac{\| \boldsymbol{X} \wedge \boldsymbol{X}_{u_1} \wedge \boldsymbol{X}_{u_2} \|}{v_1} \neq 0. \end{aligned}$$ □

Let $\boldsymbol{X} : U \longrightarrow H_1^3$ be a timelike surface in H_1^3 and $\mathbb{G}_n^{\pm}$ be the AdS-nullcone Gauss image on $M = \boldsymbol{X}(U)$. We denote $\boldsymbol{X}(u) = (x_1(u), x_2(u), x_3(u), x_4(u))$ and $\mathbb{G}_n^{\pm}(u) = (v_1(u), v_2(u), v_3(u), v_4(u))$ as coordinate representations. We define a smooth mapping

$$\mathcal{G}^{\pm} : U \longrightarrow PT^*(\Lambda_0)$$

by $\mathcal{G}^{\pm}(u) = (\mathbb{G}_n^{\pm}(u), [(x_1 v_2 - x_2 v_1) : (-x_1 v_3 + x_3 v_1) : (-x_1 v_4 + x_4 v_1)])$. Then by the above proposition we have the following corollary.

Corollary 3.4. *For any timelike surface $\boldsymbol{X} : U \longrightarrow H_1^3$, the AdS-null height function $H : U \times \Lambda_0 \longrightarrow \mathbb{R}$ on $\boldsymbol{X}$ is a generating family of the Legendrian embedding $\mathcal{G}^{\pm}$.* □

Therefore we conclude that the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ can be regarded as a Legendrian map and $\mathbb{G}_n^{\pm}(U)$ can be regarded as a wave front set of $\mathcal{G}^{\pm}$.

4 The AdS-torus cylindrical pedals of timelike surfaces

In this section we consider a surface associated to $M = \boldsymbol{X}(U)$, whose singular set is diffeomorphic to that of the AdS-nullcone Gauss image. We can use this surface to investigate the relationship between the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ and the AdS-torus Gauss map $\widetilde{\mathbb{G}}_n^{\pm}$ of a timelike surface in the Anti de Sitter 3−space. For any timelike surface $\boldsymbol{X} : U \longrightarrow H_1^3$, we define a smooth mapping $ACP_M : U \longrightarrow T_1^2 \times \mathbb{R}^*$ by

$$ACP_M(u) = \left(\widetilde{\mathbb{G}}_n^{\pm}(u), -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}}_n^{\pm}(u) \rangle \right) = \left(\widetilde{\mathbb{G}}_n^{\pm}(u), \frac{1}{\xi(u)} \right).$$

We call it the *AdS-torus cylindrical pedal* of $M = \boldsymbol{X}(U)$, where $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$. We define a diffeomorphism $\phi : T_1^2 \times \mathbb{R}^* \longrightarrow \Lambda_0$ by $\phi(\boldsymbol{v}, \lambda) = \lambda^{-1} \boldsymbol{v}$. It is easy to check that $\phi(ACP_M(u)) = \mathbb{G}_n^{\pm}(u)$, this means that the singular sets of $\mathbb{G}_n^{\pm}$ and ACP_M are diffeomorphic.

We now consider a family functions $\overline{H} : U \times T_1^2 \times \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle + \lambda = \widetilde{H}(u, \boldsymbol{v}) + \lambda,$$

we call it the *extended AdS-torus height function* on $M = \boldsymbol{X}(U)$. By similar calculations to the proof of Proposition 3.1(1), we have

$$\mathcal{D}_{\overline{H}} = \left\{ \left(\widetilde{\mathbb{G}_n^{\pm}}(u), \frac{1}{\xi(u)} \right) | u \in U \right\} = \{ ACP_M(u) | u \in U \}.$$

On the other hand, we consider the canonical projection $\pi_1 : T_1^2 \times \mathbb{R}^* \longrightarrow T_1^2$. Then we have $\pi_1 | \mathcal{D}_{\overline{H}}$ can be identified with the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ of X. Since

$$\mathbb{G}_n^{\pm}(u) = -\frac{1}{\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u) \rangle} \widetilde{\mathbb{G}_n^{\pm}}(u) = \xi(u) \widetilde{\mathbb{G}_n^{\pm}}(u),$$

we have $\phi(\mathcal{D}_{\overline{H}}) = \{ \mathbb{G}_n^{\pm}(u) | u \in U \} = \mathcal{D}_H$. Therefore, we may say that the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ is a *lift* of the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$. In fact, we also have

$$\Sigma_*(\overline{H}) = \left\{ (u,\ \widetilde{\mathbb{G}_n^{\pm}}(u),\ -\langle \boldsymbol{X}(u), \widetilde{\mathbb{G}_n^{\pm}}(u) \rangle) | u \in U \right\}.$$

We remark that similar discussions apply to the extended AdS-torus height function $\overline{H}$ and AdS-torus height function $\widetilde{H}$, and it follows that $\overline{H}$ and $\widetilde{H}$ are Morse families.

On the other hand, for any $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2$, we consider a coordinate neighborhood $U_{24}^+ = \{ \boldsymbol{v} = (v_1, v_2, v_3, v_4) \in T_1^2 | v_2 > 0 \text{ and } v_4 > 0 \}$, then

$$\overline{H}(u, \boldsymbol{v}, \lambda) = \widetilde{H}(u, \boldsymbol{v}) + \lambda = -x_1 v_1 - x_2 \sqrt{1 - v_1^2} + x_3 v_3 + x_4 \sqrt{1 - v_3^2} + \lambda.$$

We now consider smooth mappings $\mathcal{L}_{\overline{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^+) \longrightarrow T^*(T_1^2) \times \mathbb{R}^*$ defined by

$$\begin{aligned} \mathcal{L}_{\overline{H}}(u) &= \left(\widetilde{\mathbb{G}_n^{\pm}}(u),\ \left[\frac{\partial \overline{H}}{\partial v_1} : \frac{\partial \overline{H}}{\partial v_3} : \frac{\partial \overline{H}}{\partial \lambda} \right],\ \frac{1}{\xi(u)} \right) \\ &= \left(\widetilde{\mathbb{G}_n^{\pm}}(u),\ \frac{\partial \overline{H}}{\partial v_1},\ \frac{\partial \overline{H}}{\partial v_3},\ \frac{1}{\xi(u)} \right) \end{aligned}$$

and $L_{\widetilde{H}} : \widetilde{\mathbb{G}_n^{\pm}}^{-1}(U_{24}^+) \longrightarrow T^*(T_1^2)$ defined by

$$L_{\widetilde{H}}(u) = \left(\widetilde{\mathbb{G}_n^{\pm}}(u), \frac{\partial \widetilde{H}}{\partial v_1}, \frac{\partial \widetilde{H}}{\partial v_3}\right) = \left(\widetilde{\mathbb{G}_n^{\pm}}(u), \frac{\partial \overline{H}}{\partial v_1}, \frac{\partial \overline{H}}{\partial v_3}\right).$$

According to these definitions, $\mathcal{L}_{\overline{H}}$ is a Legendrian embedding whose generating family is the extended AdS-torus height function $\overline{H}$ and $L_{\widetilde{H}}$ is a Lagrangian embedding whose generating family is the AdS-torus height function $\widetilde{H}$. The details on Lagrangian singularities can be found in [1, 32]. We now consider the canonical projection

$$\pi : T^*(T_1^2) \times \mathbb{R}^* \longrightarrow T^*(T_1^2), \ \pi(\boldsymbol{v}, \lambda) = \boldsymbol{v},$$

then $\pi(\mathcal{L}_{\overline{H}}) = L_{\widetilde{H}}$. We remark that if we adopt other local coordinates on T_1^2, exactly the same results hold. Therefore we have the following proposition.

Proposition 4.1. *Under the same assumptions as in the above arguments, we have the following:*

(1) *The AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$ is a Lagrangian map. The corresponding Lagrangian embedding $L_{\widetilde{H}}$ is called the Lagrangian lift of the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$;*

(2) *The Legendrian lift $\mathcal{G}^{\pm}$ of the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ is a covering of the Lagrangian lift $L_{\widetilde{H}}$ of the AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$.*

Proof. The assertion (1) follows from the above arguments.

On the other hand, for any $\boldsymbol{v} \in T_1^2$, without loss of the generality, we can assume that $v_2 > 0$ and $v_4 > 0$. Then we have

$$v_2 = \sqrt{1 - v_1^2}, \quad v_4 = \sqrt{1 - v_3^2},$$

so we can regard (v_1, v_3) as the coordinate system of T_1^2. Therefore, the homogeneous coordinates of $PT^*(T_1^2 \times \mathbb{R}^*)$ can be expressed as $(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma])$. Moreover, if $\varsigma \neq 0$, we have

$$\big(v_1, v_3, \lambda, [\varsigma_1 : \varsigma_2 : \varsigma]\big) = \left(v_1, v_3, \lambda, \left[\frac{\varsigma_1}{\varsigma} : \frac{\varsigma_2}{\varsigma} : 1\right]\right),$$

so that we can adopt the corresponding affine coordinates $(v_1, v_3, \lambda, \rho_1, \rho_2)$, where $\rho_i = \varsigma_i/\varsigma$. By the above argument we can naturally regard $T^*(T_1^2) \times \mathbb{R}^*$ as the affine part of $PT^*(T_1^2 \times \mathbb{R}^*)$. We also have the following relation:

$$H \circ (id_U \times \phi)(u, \boldsymbol{v}, \lambda) = H(u, \lambda^{-1}\boldsymbol{v}) = \lambda^{-1}\overline{H}(u, \boldsymbol{v}, \lambda).$$

This means that $H \circ (id_U \times \phi)$ and $\overline{H}$ are C-equivalent in the sense of Mather [18]. So that these generating families correspond to the same Legendrian submanifold (cf., [1, 32]. Then we have a unique contact diffeomorphism $\Phi : PT^*(T_1^2 \times \mathbb{R}^*) \longrightarrow PT^*\Lambda_0$ covering $\phi : T_1^2 \times \mathbb{R}^* \longrightarrow \Lambda_0$ such that $\Phi \circ \mathcal{L}_{\overline{H}} = \mathcal{G}^{\pm}$. Therefore, $\mathcal{G}^{\pm}$ is a covering of $L_{\widetilde{H}}$. $\square$

5 Contact with AdS-horospheres

In this section we consider the geometric meaning of the singularities of the AdS-nullcone Gauss image of a timelike surface $M = \boldsymbol{X}(U)$ in H_1^3. We consider the contact of timelike surfaces with AdS-horospheres in the sense of Montaldi [24]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact* of X_1 and Y_1 at y_1 is the same type as the *contact* of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [24], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 5.1. *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with*

$$\dim X_1 = \dim X_2 \quad \textit{and} \quad \dim Y_1 = \dim Y_2.$$

Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, \boldsymbol{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(\boldsymbol{0}), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

For the definition of the $\mathcal{K}$-equivalence, see Martinet [19]. We now consider a function $\mathcal{H} : H_1^3 \times \Lambda_0 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + 1$. For any $\boldsymbol{v}_0 \in \Lambda_0$, we denote $\mathfrak{h}_{v_0}(\boldsymbol{u}) = \mathcal{H}(\boldsymbol{u}, \boldsymbol{v}_0)$ and we define the AdS-horosphere by $\mathfrak{h}_{v_0}^{-1}(0) = H_1^3 \cap HP(\boldsymbol{v}_0, -1)$. We write $AH(\boldsymbol{v}_0, -1) = H_1^3 \cap HP(\boldsymbol{v}_0, -1)$. For any $u_0 \in U$, we consider the null vector $\boldsymbol{v}_0^{\pm} = \mathbb{G}_n^{\pm}(u_0)$. Then we have

$$\mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{X}(u_0) = \mathcal{H} \circ (\boldsymbol{X} \times id_{\Lambda_0})(u_0, \boldsymbol{v}_0) = H(u_0, \mathbb{G}_n^{\pm}(u_0)) = 0.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0^{\pm}} \circ \boldsymbol{X}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbb{G}_n^{\pm}(u_0)) = 0,$$

for $i = 1, 2$. This means that the AdS-horosphere $AH(\boldsymbol{v}_0^{\pm}, -1)$ is tangent to $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$. In this case, we call $AH(\boldsymbol{v}_0^{\pm}, -1)$ the *tangent AdS-horosphere* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u_0)$ (or, u_0), which we write

$AH^{\pm}(\boldsymbol{X}, u_0)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be null vectors. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent, then $HP(\boldsymbol{v}_1, -1)$ and $HP(\boldsymbol{v}_2, -1)$ are parallel. Therefore, we say that AdS-horospheres $AH(\boldsymbol{v}_1, -1)$ and $AH(\boldsymbol{v}_2, -1)$ are *parallel*, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly dependent. Then we have the following lemma.

Lemma 5.2. *Let $\boldsymbol{X} : U \longrightarrow H^3_1$ be a timelike surface. Consider two points $u_1, u_2 \in U$. Then we have the following assertions:*

(1) $\mathbb{G}^{\pm}_n(u_1) = \mathbb{G}^{\pm}_n(u_2)$ *if and only if* $AH^{\pm}(\boldsymbol{X}, u_1) = AH^{\pm}(\boldsymbol{X}, u_2)$.

(2) $\widetilde{\mathbb{G}}^{\pm}_n(u_1) = \widetilde{\mathbb{G}}^{\pm}_n(u_2)$ *if and only if* $AH^{\pm}(\boldsymbol{X}, u_1)$ *and* $AH^{\pm}(\boldsymbol{X}, u_2)$ *are parallel.* □

We now consider the contact of M with the tangent AdS-horosphere at $p \in M$ as an application of Legendrian singularity theory. The main result in the theory of Legendrian singularities [1, 32] is the following:

Theorem 5.3. *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \boldsymbol{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*

(1) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are P-$\mathcal{K}$-equivalent;*

(2) *$\mathcal{L}_F$ is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\boldsymbol{0}\}$.*

For definitions of the Legendrian equivalence, Legendrian stability, P-$\mathcal{K}$-equivalence and $\mathcal{K}$-versal deformation, see [1, 19, 32].

Let
$$\mathbb{G}^{\pm}_{n\,i} : (U, u_i) \longrightarrow (\Lambda_0, \boldsymbol{v}^{\pm}_i) \quad (\text{for } i = 1, 2)$$
be AdS-nullcone Gauss image germs of timelike surface germs $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H^3_1, \boldsymbol{X}_i(u_i))$. We say that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (H^3_1, \boldsymbol{v}^{\pm}_1) \longrightarrow (H^3_1, \boldsymbol{v}^{\pm}_2)$ such that $\Phi \circ \mathbb{G}^{\pm}_{n\,1} = \mathbb{G}^{\pm}_{n\,2} \circ \phi$. Suppose the regular set of $\mathbb{G}^{\pm}_{n\,i}$ is dense in (U, u_i) for each $i = 1, 2$. It follows from Proposition A.2 in the appendix of [12] (See also [33]) that $\mathbb{G}^{\pm}_{n\,1}$ and $\mathbb{G}^{\pm}_{n\,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian embedding germs $\mathcal{G}^{\pm}_1 : (U, u_1) \longrightarrow (\Delta_1, \mathbf{z}_1)$ and $\mathcal{G}^{\pm}_2 : (U, u_2) \longrightarrow (\Delta_1, \mathbf{z}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P-$\mathcal{K}$-equivalent by Theorem 5.3. Here, $H_i : (U \times \Lambda_0, (u_i, \boldsymbol{v}^{\pm}_i)) \longrightarrow \mathbb{R}$ is the corresponding AdS-null height function germ of $\boldsymbol{X}_i$.

On the other hand, we denote $h_{i,v_i^\pm} = H_i(u, \boldsymbol{v}_i^\pm)$; then we have $h_{i,v_i^\pm}(u) = \mathfrak{h}_{v_i^\pm} \circ \boldsymbol{X}_i(u)$. By Theorem 5.1,

$$K\big(\boldsymbol{X}_1(U), \mathrm{AH}^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm\big) = K\big(\boldsymbol{X}_2(U), \mathrm{AH}^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm\big)$$

if and only if $h_{1,v_1^\pm}$ and $h_{2,v_2^\pm}$ are $\mathcal{K}$-equivalent. Therefore, we can apply the above arguments to our situation. We denote by $Q^\pm(\boldsymbol{X}, u_0)$ the local ring of the function germ $h_{v_0^\pm} : (U, u_0) \longrightarrow \mathbb{R}$, where $\boldsymbol{v}_0^\pm = \mathbb{G}_n^\pm(u_0)$. We remark that we can write the local ring explicitly as follows:

$$Q^\pm(\boldsymbol{X}, u_0) = \frac{C_{u_0}^\infty(U)}{\langle\langle \boldsymbol{X}(u), \mathbb{G}_n^\pm(u_0)\rangle + 1\rangle_{C_{u_0}^\infty(U)}},$$

where $C_{u_0}^\infty(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathcal{M}_{u_0}(U)$.

Theorem 5.4. *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H_1^3, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be timelike surface germs such that the corresponding Legendrian embedding germs $\mathcal{G}_i^\pm : (U, u_i) \longrightarrow (\Delta_1, z_i)$ are Legendrian stable. Then the following conditions are equivalent:*

(1) *AdS-nullcone Gauss image germs $\mathbb{G}_{n\,1}^\pm$ and $\mathbb{G}_{n\,2}^\pm$ are $\mathcal{A}$-equivalent;*

(2) *H_1 and H_2 are P-$\mathcal{K}$-equivalent;*

(3) *$h_{1,v_1^\pm}$ and $h_{2,v_2^\pm}$ are $\mathcal{K}$-equivalent;*

(4) *$K(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm) = K(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm)$;*

(5) *$Q^\pm(\boldsymbol{X}_1, u_1)$ and $Q^\pm(\boldsymbol{X}_2, u_2)$ are isomorphic as $\mathbb{R}$-algebras.* □

For a timelike surface germ $\boldsymbol{X} : (U, u_0) \longrightarrow (H_1^3, \boldsymbol{X}(u_0))$, we call

$$\big(\boldsymbol{X}^{-1}(AH(\,G_n^\pm(u_0), -1)), u_0\big)$$

the *tangent Anti de Sitter horospherical indicatrix germ* (briefly, *tangent AdS-horospherical indicatrix germ*) of $\boldsymbol{X}$. In general we have the following proposition:

Proposition 5.5. *Let $\boldsymbol{X}_i : (U, u_i) \longrightarrow (H_1^3, \boldsymbol{X}_i(u_i))$ (for $i = 1, 2$) be timelike surface germs such that their AdSh$^\pm$-parabolic sets have no interior points as subspaces of U. If the AdS-nullcone Gauss image germs $\mathbb{G}_{n\,1}^\pm$ and $\mathbb{G}_{n\,2}^\pm$ are $\mathcal{A}$-equivalent, then*

$$K\big(\boldsymbol{X}_1(U), AH^\pm(\boldsymbol{X}_1, u_1), \boldsymbol{v}_1^\pm\big) = K\big(\boldsymbol{X}_2(U), AH^\pm(\boldsymbol{X}_2, u_2), \boldsymbol{v}_2^\pm\big).$$

In this case, $(\boldsymbol{X}_1^{-1}(AH(\mathbb{G}_{n\,1}^{\pm}(u_1), -1)), u_1)$ *and* $(\boldsymbol{X}_2^{-1}(AH(\mathbb{G}_{n\,2}^{\pm}(u_2), -1)), u_2)$ *are diffeomorphic as set germs.* $\square$

From the above proposition, the diffeomorphism type of the tangent AdS-horospherical indicatrix germ is an invariant of $\mathcal{A}$-classification of the AdS-nullcone Gauss image germ of $\boldsymbol{X}$. Moreover, we need some numerical $\mathcal{K}$-invariants for a function germ. We denote

$$\mathrm{Ah}^{\pm} - \mathrm{ord}(\boldsymbol{X}, u_0) = \dim \frac{C^{\infty}_{u_0}(U)}{\langle h_{v_0^{\pm}}(u_0), \partial h_{v_0^{\pm}}(u_0)/\partial u_i\rangle_{C^{\infty}_{u_0}(U)}},$$

where $\boldsymbol{v}_0^{\pm} = \mathbb{G}_n^{\pm}(u_0)$. Usually $\mathrm{Ah}^{\pm}$-ord$(\boldsymbol{X}, u_0)$ is called the $\mathcal{K}$-codimension of $h_{v_0^{\pm}}$. However, We call it the *order of contact with tangent AdS-horosphere* at $\boldsymbol{X}(u_0)$. We also have the notion of *corank* of function germs:

$$Ah^{\pm} - \mathrm{corank}(\boldsymbol{X}, u_0) = 2 - \mathrm{rank}\,\mathrm{Hess}(h_{v_0^{\pm}})(u_0),$$

By Proposition 3.1, $\boldsymbol{X}(u_0)$ is an $\mathrm{AdSh}^{\pm}$-parabolic point if and only if $\mathrm{Ah}^{\pm}$-corank $(\boldsymbol{X}, u_0) \geq 1$ and $\boldsymbol{X}(u_0)$ is an AdS-horospherical point if and only if $\mathrm{Ah}^{\pm}$-corank $(\boldsymbol{X}, u_0) = 2$. On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \boldsymbol{a}) \longrightarrow \mathbb{R}$ has the A_k-singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\mathrm{Ah}^{\pm}$-corank$(\boldsymbol{X}, u_0) = 1$, the AdS-null height function $h_{v_0^{\pm}}$ has the A_k-singularity at u_0 and is generic. In this case we have $\mathrm{Ah}^{\pm}$-ord$(\boldsymbol{X}, u_0) = k$. This number is equal to the order of contact in the classical sense (cf. [3]). This is the reason why we call $\mathrm{Ah}^{\pm}$-ord$(\boldsymbol{X}, u_0)$ the order of contact with the AdS-horosphere at $\boldsymbol{X}(u_0)$.

6 Classification of singularities of AdS-nullcone Gauss images

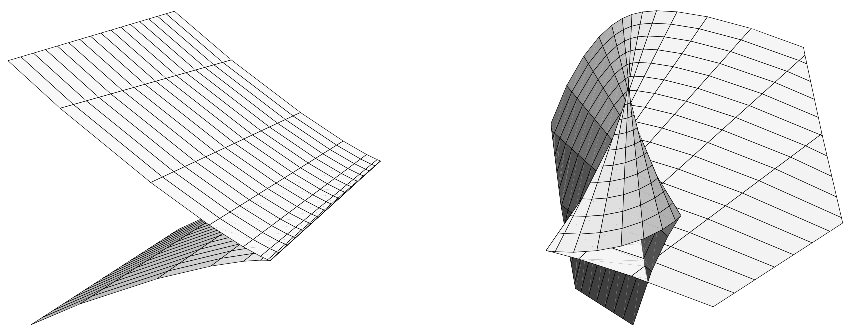
In this section we give the generic classification of singularities of the AdS-nullcone Gauss images. The arguments are almost the same as those of [12], so that we omit the details. We consider the space of timelike embeddings $\mathrm{Emb}_T(U, H_1^3)$ with the Whitney C^{∞}-topology. By the classification of stable Legendrian singularities of $n = 3$ and the transversality theorem of [12] (Proposition 7.1), we have the following theorem.

Theorem 6.1. *There exists an open dense subset* $\mathcal{O} \subset \mathrm{Emb}_T(U, H_1^3)$ *such that for any* $\boldsymbol{X} \in \mathcal{O}$ *the following conditions hold.*

(1) *The* $\mathrm{AdSh}^{\pm}$*-parabolic set* ${K^{\pm}_{AdSn}}^{-1}(0)$ *is a regular curve. We call such a curve the* $\mathrm{AdSh}^{\pm}$*-parabolic curve.*

(2) *The AdS-nullcone Gauss image* $\mathbb{G}_n^{\pm}$ *along the AdSh*$^{\pm}$*-parabolic curve is a cuspidal edge except at isolated points. At such the point* $\mathbb{G}_n^{\pm}$ *is the swallowtail.*

(3) *The cuspidal edge points (swallowtail points) of the AdS-nullcone Gauss image* $\mathbb{G}_n^{\pm}$ *correspond to fold points (cusp points) of the AdS-torus Gauss map. (cf., Figure* 1*).*



Cuspital edge Swallowtail

Figure 1

Following the terminology of Whitney [31], we say that a timelike surface $\boldsymbol{X} \colon U \longrightarrow H_1^3$ has an *excellent AdS-nullcone Gauss image* $\mathbb{G}_n^{\pm}$, the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ has only cuspidal edges and swallowtails as singularities.

We now consider the geometric meanings of cuspidal edges and swallowtails of the AdS-nullcone Gauss image. We have the following results analogous to the results of [12].

Theorem 6.2. *Let* $\mathbb{G}_n^{\pm} : (U, u_0) \longrightarrow (\Lambda_0, \boldsymbol{v}_0^{\pm})$ *be the excellent AdS-nullcone Gauss image germ of a timelike surface* $\boldsymbol{X}$ *and* $h_{v_0} \colon (U, u_0) \longrightarrow \mathbb{R}$ *the AdS-null height function germ at* u_0*, where* $\boldsymbol{v}_0^{\pm} = \mathbb{G}_n^{\pm}(u_0)$*. Then we have the following.*

(1) *The point* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$ *if and only if* Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0) = 1$.

(2) *If* u_0 *is an AdSh*$^{\pm}$*-parabolic point of* $\boldsymbol{X}$*, then* $h_{v_0^{\pm}}$ *has an* A_k*-singularity for* $k = 2, 3$.

(3) *Suppose that u_0 is an AdSh$^{\pm}$-parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent:*

(a) $\mathbb{G}_n^{\pm}$ *has a cuspidal edge at u_0;*

(b) $h_{v_0^{\pm}}$ *has an A_2−singularity;*

(c) *Ah$^{\pm}$-order* $(\boldsymbol{X}, u_0) = 2$*;*

(d) *the tangent AdS-horospherical indicatrix germ is an ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^3 = 0\}$.

(e) *for each $\varepsilon > 0$, there exist two points $u_1, u_2 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2$, neither u_1 nor u_2 is an AdSh$^{\pm}$-parabolic point and the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are parallel.*

(4) *Suppose that u_0 is an AdSh$^{\pm}$-parabolic point of $\boldsymbol{X}$. Then the following conditions are equivalent:*

(a) $\mathbb{G}_n^{\pm}$ *has a swallowtail at u_0;*

(b) $h_{v_0^{\pm}}$ *an the A_3-singularity;*

(c) *Ah$^{\pm}$-order* $(\boldsymbol{X}, u_0) = 3$*;*

(d) *the tangent AdS-horospherical indicatrix germ is a point or a tacnodal, where a curve $C \subset \mathbb{R}^2$ is called a tacnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2)|u_1^2 - u_2^4 = 0\}$.

(e) *for each $\varepsilon > 0$, there exist three points $u_1, u_2, u_3 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2, 3$, none of which is an AdSh$^{\pm}$-parabolic point and the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1, u_2 and u_3 are parallel.*

(f) *for each $\varepsilon > 0$, there exist two points $u_1, u_2 \in U$ such that $|u_0 - u_i| < \varepsilon$ for $i = 1, 2$, neither u_1 nor u_2 is an AdS-parabolic point and the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal.*

Proof. By the Proposition 3.1, we have shown that u_0 is an AdSh$^{\pm}$-parabolic point if and only if Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0) \geq 1$. Since $n = 3$, we have Ah$^{\pm}$-corank$(\boldsymbol{X}, u_0) \leq 2$. Since the AdS-null height function germ $H\colon (U \times \Lambda_0, (u_0, \boldsymbol{v}_0^{\pm})) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian embedding germ $\mathcal{G}^{\pm}$, $h_{v_0^{\pm}}$ has only A_k-singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of $h_{v_0^{\pm}}$ at an AdSh$^{\pm}$-parabolic point is 1. The assertion (2) also follows. For the same reason, the conditions (3)$\{(a), (b), (c)\}$ (respectively, (4)$\{(a), (b), (c)\}$) are equivalent.

On the other hand, if the AdS-null height function germ $h_{v_0^{\pm}}$ has an A_2-singularity, it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since $\mathcal{K}$-equivalence preserves the zero level sets, the tangent AdS-horospherical indicatrix germ is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3-singularity is given by $\pm u_1^2 + u_2^4$, so the tangent AdS-horospherical indicatrix germ is diffeomorphic to the curve given by $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3)$\{(d)\}$ (respectively, (4)$\{(d)\}$) is also equivalent to the other conditions.

Suppose that u_0 is an AdSh$^{\pm}$-parabolic point, by Proposition 4.1, the AdS-torus Gauss map has only folds or cusps. If the point u_0 is a fold point, there is a neighborhood of u_0 on which the AdS-torus Gauss map is 2 to 1 except at the AdSh$^{\pm}$-parabolic line (i.e, fold curve). By Lemma 5.2, the condition (3)(e) holds. If the point u_0 is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the AdS-torus Gauss map is 3 to 1 inside region of the critical value. Moreover, the point u_0 is in the closure of the region. This means that the condition (4)(e) is satisfied. We can also observe that nearby the cusp point, there are 2 to 1 points which approach to u_0. However, one of those points is always AdSh$^{\pm}$-parabolic point. Since other singularities do not appear in this case, so that the condition (3)(e) (respectively, (4)(e)) characterizes a fold (respectively, a cusp).

For the swallowtail point u_0, there is a self-intersection curve approaching u_0. On this curve, there are two distinct points u_1 and u_2 such that $\mathbb{G}_n^{\pm}(u_1) = \mathbb{G}_n^{\pm}(u_2)$. By Lemma 5.2, this means that the tangent AdS-horospheres to $M = \boldsymbol{X}(U)$ at u_1 and u_2 are equal. Since there are no other singularities in this case, the condition (4)$\{(f)\}$ characterizes a swallowtail point of $\mathbb{G}_n^{\pm}$. This completes the proof. $\square$

7 AdS-null Monge form

The notion of the Monge form of a surface in Euclidean 3-space is one of the powerful tools for the study of local properties of the surface from the view point of differential geometry. In this section we consider the analogous notion for a timelike surface in H_1^3.

We now consider a function $f(u_1, u_2)$ with $f(0) = f_{u_i}(0) = 0,\ i = 1, 2$. Then we have a timelike surface in H^3_1 defined by

$$\begin{aligned}\boldsymbol{X}_f(u_1, u_2) \ = \ &\Big(\sqrt{1 + \varepsilon_1 u_1^2 + \varepsilon_2 u_2^2 + f^2(u_1, u_2)},\ \frac{(1 - \varepsilon_1)u_1 + (1 - \varepsilon_2)u_2}{2},\\ &f(u_1, u_2),\ \frac{(1 + \varepsilon_1)u_1 + (1 + \varepsilon_2)u_2}{2}\Big),\end{aligned}$$

where $\varepsilon_i = \mathrm{sign}(\boldsymbol{X}_i)$ $(i = 1, 2)$. We can easily calculate

$$\boldsymbol{N}(0) = \Big(0, 0, \frac{\varepsilon_2 - \varepsilon_1}{2}, 0\Big),$$

therefore $\mathbb{G}^{\pm}_n(0) = (1, 0, \pm 1, 0)$. We call $\boldsymbol{X}_f$ an *Anti de Sitter null Monge form* (briefly, *AdS-null Monge form*). Then we have the following proposition.

Proposition 7.1. *Any timelike surface in H^3_1 is locally given by the AdS-null Monge form.*

Proof. Let $\boldsymbol{X}\colon U \longrightarrow H^3_1$ be a timelike surface. We consider a Lorentzian motion of H^3_1 which is a transitive action. Therefore, without loss of the generality, we assume that $p = \boldsymbol{X}(0) = (1, 0, 0, 0)$. We denote $M = \boldsymbol{X}(U)$, we have a basis $\{\boldsymbol{X}(0), \boldsymbol{N}(0), \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM = \langle \boldsymbol{X}_{u_1}(0), \boldsymbol{X}_{u_2}(0)\rangle_{\mathbb{R}}$. Applying the Gram-Schmidt procedure we have a pseudo-orthonormal basis $\{\boldsymbol{X}(0), \boldsymbol{N}(0), \boldsymbol{e}_1, \boldsymbol{e}_2\}$ of $T_p\mathbb{R}^4_2$ such that $T_pM = \langle \boldsymbol{e}_1, \boldsymbol{e}_2\rangle_{\mathbb{R}}$. In particular, $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is an pseudo-orthonormal basis of T_pM. Since $p = (1, 0, 0, 0)$, T_pM is considered to be a subspace of ${}_0\mathbb{R}^3_1 = \{(0, x_1, x_2, x_3) | x_i \in \mathbb{R}\}$. By a rotation of the space ${}_0\mathbb{R}^3_1$, we might assume that

$$T_pM = \Big\{\Big(0, \frac{(1 - \varepsilon_1)u_1 + (1 - \varepsilon_2)u_2}{2}, 0, \frac{(1 + \varepsilon_1)u_1 + (1 + \varepsilon_2)u_2}{2}\Big) | u_i \in \mathbb{R}\Big\} \subset \mathbb{R}^4_2.$$

Then the germ (M, p) might be written in the form

$$\Big(f_0(u_1, u_2),\ \frac{(1 - \varepsilon_1)u_1 + (1 - \varepsilon_2)u_2}{2},\ f(u_1, u_2),\ \frac{(1 + \varepsilon_1)u_1 + (1 + \varepsilon_2)u_2}{2}\Big)$$

with function germs $f_0(u_1, u_2)$, $f(u_1, u_2)$. Since $M \subset H^3_1$, we have the relation

$$f_0(u_1, u_2) = \sqrt{1 + \varepsilon_1 u_1^2 + \varepsilon_2 u_2^2 + f^2(u_1, u_2)}.$$

Since we have

$$T_pM = \Big\{\Big(0, \frac{(1 - \varepsilon_1)u_1 + (1 - \varepsilon_2)u_2}{2}, 0, \frac{(1 + \varepsilon_1)u_1 + (1 + \varepsilon_2)u_2}{2}\Big) | u_i \in \mathbb{R}\Big\},$$

the condition $f(0) = 0$, $f_{u_i}(0) = 0$ are automatically satisfied. □

For the null vector $\boldsymbol{v}_0^{\pm} = (1, 0, \pm 1, 0)$, we consider the AdS-horosphere $\mathrm{AH}(\boldsymbol{v}_0^{\pm}, -1)$. Then we have the AdS-null Monge form of $\mathrm{AH}(\boldsymbol{v}_0^{\pm}, -1)$:

$$\begin{aligned} \boldsymbol{a}^{\pm}(u_1, u_2) &= \Big(\frac{\varepsilon_1 u_1^2 + \varepsilon_2 u_2^2}{2} + 1,\ \frac{(1-\varepsilon_1)u_1 + (1-\varepsilon_2)u_2}{2}, \\ &\qquad \pm \frac{\varepsilon_1 u_1^2 + \varepsilon_2 u_2^2}{2},\ \frac{(1+\varepsilon_1)u_1 + (1+\varepsilon_2)u_2}{2}\Big). \end{aligned}$$

Here, we can easily check the relation $\langle \boldsymbol{a}(u), \boldsymbol{v}_0^{\pm}\rangle = -1$.

On the other hand, $\boldsymbol{a}^{\pm}(0) = (1, 0, 0, 0) = p$ and $\boldsymbol{a}_{u_i}^{\pm}(0)$ is equal to the $x_{3+\varepsilon_i}$-axis for $i = 1, 2$. This means that $T_pM = T_p(\boldsymbol{a}^{\pm}(U))$. Therefore $\boldsymbol{a}^{\pm}(U) = \mathrm{AH}(\boldsymbol{v}_0^{\pm}, -1)$ is the tangent AdS-horosphere of $M = \boldsymbol{X}_f(U)$ at $p = \boldsymbol{X}_f(0)$. It follows from this fact that the tangent AdS-horospherical indicatrix of the AdS-null Monge form germ $(\boldsymbol{X}_f, 0)$ is given as follows:

$$\boldsymbol{X}_f^{-1}\big(AH(\boldsymbol{v}_0^{\pm}, 0)\big) = \big\{(u_1, u_2) \mid \pm 2f(u_1, u_2) = \varepsilon_1 u_1^2 + \varepsilon_2 u_2^2\big\}.$$

On the other hand, since $f(0) = f_{u_i}(0) = 0$, we may write

$$f(u_1, u_2) = \frac{1}{2}\bar{k}_1 u_1^2 + \frac{1}{2}\bar{k}_2 u_2^2 + g(u_1, u_2)$$

where $g \in \mathcal{M}_2^3$ and $\bar{k}_1, \bar{k}_2$ are eigenvalues of $(f_{u_iu_j}(0))$. Under this representation, we can easily calculate $(\boldsymbol{X}_f)_{u_iu_j}(0) = (\varepsilon_i\delta_{ij}, 0, \bar{k}_i, \delta_{ij}, 0)$. It follows from this fact that

$$h_{ij}^{\pm}(0) = \langle \mathbb{G}_n^{\pm}(0), (\boldsymbol{X}_f)_{u_iu_j}(0)\rangle = \varepsilon_i\delta_{ij}(-1 \pm \bar{k}_i),$$

and

$$g_{ij}(0) = \langle (\boldsymbol{X}_f)_{u_i}(0), (\boldsymbol{X}_f)_{u_j}(0)\rangle = \varepsilon_i\delta_{ij}.$$

Therefore, we have $k_i^{\pm}(0) = -1 \pm \varepsilon_i\bar{k}_i$ and

$$K_{AdSn}^{\pm}(0) = k_1^{\pm}(0)k_2^{\pm}(0) = (-1 \pm \varepsilon_1\bar{k}_1)(-1 \pm \varepsilon_2\bar{k}_2).$$

The tangent AdS-horospherical indicatrix is given by

$$\begin{aligned} \boldsymbol{X}_f^{-1}(AH(\boldsymbol{v}_0^{\pm}, -1)) &= \big\{(u_1, u_2) \mid \pm\bar{k}_1 u_1^2 \pm \bar{k}_2 u_2^2 \pm 2g(u_1, u_2) - \varepsilon_1 u_1^2 - \varepsilon_2 u_2^2 = 0\big\} \\ &= \big\{(u_1, u_2) \mid \varepsilon_1 k_1^{\pm}(0)u_1^2 + \varepsilon_2 k_2^{\pm}(0)u_2^2 \pm 2g(u_1, u_2) = 0\big\}. \end{aligned}$$

If we try to draw the picture of the AdS-nullcone Gauss image, it might be very hard to give a parameterization. However, by the AdS-null Monge form of the tangent AdS-horospherical indicatrix, we can easily detect the type of singularities of the AdS-nullcone Gauss image $\mathbb{G}_n^{\pm}$ (or, AdS-torus Gauss map $\widetilde{\mathbb{G}_n^{\pm}}$).

Example 7.1. Consider the function given by

$$f(u_1, u_2) = \frac{1}{2}u_1^2 + u_2^2 + \frac{1}{2}u_1^3.$$

Suppose that $\varepsilon_1 = -1, \varepsilon_2 = 1$ Then $\bar{k_1} = 1$, $\bar{k_2} = 2$. We have $k_1^+(0) = -2$, $k_2^+(0) = 1$, $k_1^-(0) = 0, k_2^-(0) = -3$. So that the origin is an AdSh$^-$-parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the ordinary cusp $u_2^2 = -\frac{1}{3}u_1^3$. By Theorem 6.2, $\mathbb{G}_n^-$ $(\widetilde{\mathbb{G}_n^-})$ *is the cuspidal edge* (*fold*) *at the origin.*

Example 7.2. Consider the function given by

$$f(u_1, u_2) = \frac{1}{2}u_1^2 + \frac{1}{4}u_2^2 + \frac{3}{2}u_1^4.$$

Suppose that $\varepsilon_1 = 1, \varepsilon_2 = -1$ Then $\bar{k_1} = 1$, $\bar{k_2} = 1/2$. We have $k_1^+(0) = 0$, $k_2^+(0) = -3/2$, $k_1^-(0) = -2, k_2^-(0) = -1/2$. So that the origin is an AdSh$^+$- parabolic point. The tangent AdS-horospherical indicatrix germ at the origin is the tacnodal $u_2^2 = u_1^4$. By Theorem 6.2, $\mathbb{G}_n^- \widetilde{\mathbb{G}_n^-})$ is the swallowtail (cusp) at the origin.

Acknowledgments. The work was completed when the first author was a joint PhD candidate at the Hokkaido University. He would like to thank the peoples in department of mathematics for their hospitality.

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