BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 41(1), 63-72

Maximal divisible subgroups in modular group rings of p-mixed abelian groups

Peter Danchev

Abstract. The isomorphism structure of the maximal divisible subgroup of the subgroup $V_p(R(G); H)\,\mathrm{Id}\,R(G)$ of the normalized unit group $VR(G)$ in a commutative group ring $R(G)$ is completely described only in terms of R, G and H whenever R is a commutative unital ring of prime characteristic p and G is a p-mixed abelian group. In particular, the maximal divisible subgroup of $VR(G)$ is characterized. This extends a result due to Nachev (Commun. Algebra, 1995) as well as a result due to the author (Commun. Algebra, 2010).

Keywords: abelian groups, divisible subgroups, commutative rings, idempotents, nilpotents, normalized units, cardinalities.

Mathematical subject classification: 16S34, 16U60, 20K10, 20K21.

1 Introduction

Throughout the rest of the present paper, let it be agreed that $R(G)$ is the group ring of a multiplicative abelian group G over a commutative unital ring R, with normalized unit group $VR(G)$ and its p-component of torsion $V_pR(G)$. As usual, G_t denotes the maximal torsion subgroup of G with p-primary part G_p, $G^{(p)}$ denotes the maximal p-divisible subgroup of G and dG denotes the maximal divisible subgroup of G; note that the inclusion $dG \subseteq G^{(p)}$ always holds. Moreover, $\mathrm{id}(R) = \{e \in R : e^2 = e\}$ designates the set of all idempotents in R, $N(R)$ designates the nil-radical of R, $R^{(p)}$ designates the maximal (p)-divisible subring of R when $\mathrm{char}(R) = p$ is a prime, and $\mathrm{Id}\,R(G)$ designates the idempotent subgroup of $VR(G)$ generated by (and hence consisting of) all elements of the form $\sum_{g\in G} e_g g$, where the sum is finite, for which $e_g \in \mathrm{id}(R)$, $\sum_{g\in G} e_g = 1$ and $e_g.e_h = 0$ whenever $g \neq h$ are elements of the sum. Furthermore, for a subgroup H of G and a subring L of R containing the same identity,

Received 9 September 2008.

we let $I(L(G); H)$ denote the relative augmentation ideal of $L(G)$ with respect to H, and let $I_p(L(G); H)$ denote its nil-radical. To facilitate the exposition, denote $1 + I_p(L(G); H) = V_p(L(G); H)$.

All other notions and notations are standard and follow essentially those from [6], [8] and [9], respectively.

In [1] we have established the isomorphism classification of $dVR(G)$ only in terms associated with G and R where G is a p-mixed group (i.e., $G_t = G_p$) and R is a field of characteristic p (see also [2], [3] and [4] for related type results). Note that this was slightly extended in [11] to an indecomposable ring R of prime characteristic p; however notice that the used approach is the same as that in [1]. Nevertheless, the results from [1] and [11] were superseded in [5] to an arbitrary ring R of prime $\mathrm{char}(R) = p$. Likewise, in ([4], Theorem 2.1) was described $V_p(R(G); H)$ when $1 \neq H \leq G_p$, $\mathrm{char}(R) = p$ and either $|R^{(p)}| \geq \aleph_0$ or $|G^{(p)}| \geq \aleph_0$ The purpose of this article is to generalize both situations from [4] and [5] in the converse case when $G_p \subseteq H$, H is isotype in G and $\mathrm{char}(R) = p$.

2 Main Results

Before stating and proving our chief theorem, we start with a series of preparatory technicalities.

Lemma 1 ([5]). *The commutative unital ring P is a direct sum of exactly n indecomposable subrings if and only if* $\mathrm{id}(P)$ *has exactly* 2^n *elements.*

Proof. "$\Rightarrow$". The set B of all idempotents in P is a Boolean algebra, with infima given by $e \wedge f = ef$, suprema by $e \vee f = e + f - ef$, and complements by $e' = 1 - e$. If B is finite, let $e_1, \cdots, e_n$ be its atoms (i.e., the primitive idempotents of P). Then $P = Pe_1 \oplus \cdots \oplus Pe_n$ where the direct summands Pe_i's are indecomposable rings for each $1 \leq i \leq n$. On the other hand, a minor manipulation shows that the elements of B are precisely the sums $\sum_{i \in I} e_i$ for subsets $I \subseteq \{1, \cdots, n\}$, and these are all distinct. Thus, B has exactly 2^n elements as claimed.

"$\Leftarrow$". Since $\mathrm{id}(P)$ is finite, P can be decomposed into a direct sum of finitely many indecomposable rings, say m. By what we have just shown in the necessity, $|\mathrm{id}(P)| = 2^m$. Thus, $2^m = 2^n$ and consequently $m = n$. $\square$

Proposition 2 ([5]). *Suppose that A is an abelian group and P is a commutative unital ring. Then*

$$\mathrm{Id}\, P(A) \cong \coprod_{\mu} A$$

where $\mu = \log_2 |\operatorname{id}(P)|$ *if* $|\operatorname{id}(P)| < \aleph_0$ *or* $\mu = |\operatorname{id}(P)| \geq \aleph_0$.

Proof. Utilizing [10], the isomorphism

$$\operatorname{Id} P(A) \cong \coprod_{\mu} A$$

holds. If $\operatorname{id}(P) = \{0, 1\}$, it is well known that $\operatorname{Id} P(A) = A$ and hence $\mu = 1$. Let us now $\operatorname{id}(P)$ contain a non-trivial idempotent, say e. Consider the elements $ea+(1-e)$ where $1 \neq a \in A$. Observe that $ea+(1-e) = fb+(1-f)$ for some $f \in \operatorname{id}(P) \setminus \{0, 1\}$ and $1 \neq b \in A$ only when $f - e + ea - fb = 0$, i.e., when $e = f$ and $a = b$. Therefore, $\mu = |\operatorname{id}(P)|$ if $\operatorname{id}(P)$ is infinite. Otherwise, if $\operatorname{id}(P)$ is finite, say with 2^n elements for some natural n, the number of primitive orthogonal idempotents is precisely n owing to Lemma 1. Since all elements of $\operatorname{Id} P(A)$ are finite sums of members of A with coefficients from $\operatorname{id}(P)$ which are orthogonal with sum 1, we elementarily observe that $\mu = n = \log_2 |\operatorname{id}(P)|$. □

Proposition 3 ([5]). *Suppose* $G_t = G_p$ *and* $\operatorname{char}(R) = p$. *Then the following decomposition holds:*

$$VR(G) = V_p R(G) \operatorname{Id} R(G).$$

Proof. Since the left hand-side obviously contains the right one, it suffices to show only the converse. To this aim, letting $x = r_1 g_1 + \cdots + r_t g_t \in VR(G)$, and consider the natural map $\psi \colon G \to G/G_p$. It can be linearly extended to the surjective homomorphism $\Psi : R(G) \to R(G/G_p)$, which restriction on $VR(G)$ gives an epimorphism from $VR(G)$ to $VR(G/G_p)$ with kernel $\ker(\Psi) = 1 + I(R(G); G_p)$. But $\Psi(x) \in VR(G/G_p)$ and since $(G/G_p)_t = G_t/G_p = 1$, by virtue of [10] we have that

$$\Psi(x) = \big[G_p + v_1(g_1 G_p - G_p) + \cdots + v_t(g_t G_p - G_p)\big] \cdot \big[e_1 g_1 G_p + \cdots + e_s g_s G_p\big],$$

where $v_1, \cdots, v_t \in N(R)$ and $e_1, \cdots, e_s$ are orthogonal idempotents from R with sum 1. Set

$$y = \big[1 + v_1(g_1 - 1) + \cdots + v_t(g_t - 1)\big] \cdot \big[e_1 g_1 + \cdots + e_s g_s\big].$$

Clearly, $y \in V_p R(G) \operatorname{Id}(R(G)) \subseteq VR(G)$ because $1 + v_1(g_1 - 1) + \cdots + v_t(g_t - 1) \in V_p R(G)$ and $e_1 g_1 + \cdots + e_s g_s \in \operatorname{Id}(R(G)) \leq VR(G)$ with the inverse $e_1g_1^{-1} + \cdots + e_sg_s^{-1}$. But we observe that $\Psi(x) = \Psi(y)$ and hence $\Psi(x)\Psi(y)^{-1} = \Psi(x)\Psi(y^{-1}) = \Psi(xy^{-1}) = 1$ that is $xy^{-1} \in \ker(\Psi) = 1 + I(R(G); G_p) \subseteq V_pR(G)$, i.e., $x \in yV_pR(G)$ and $x \in V_pR(G)\,\mathrm{Id}(R(G))$, as required. □

The importance of the above special decomposition of $VR(G)$ stems from the truthfulness of the following statement, which is pivotal.

Lemma 4. *Let* $\mathrm{char}(R) = p$ *be a prime. Then*

$$\mathrm{Id}_p\, R(G) = \mathrm{Id}\, R(G_p).$$

Proof. Given $x \in \mathrm{Id}_p\, R(G)$, hence $x = e_1g_1 + \cdots + e_sg_s$ with $e_1 + \cdots + e_s = 1$, $x^{p^n} = 1 = e_1g_1^{p^n} + \cdots + e_sg_s^{p^n}$ for some natural n. Thus we write

$$1 = g_1^{p^n} = \cdots = g_k^{p^n} \neq g_{k+1}^{p^n} = \cdots = g_m^{p^n} \neq g_{m+1}^{p^n} \neq \cdots \neq g_s^{p^n}$$

with

$$e_1 + \cdots + e_k = 1, e_{k+1} + \cdots + e_m = 0, e_{m+1} = \cdots = e_s = 0.$$

Since $\{e_{k+1}, \cdots, e_m\}$ is a system of orthogonal idempotents, we easily obtain that $e_{k+1} = \cdots = e_m = 0$. Finally, we write $x = e_1g_1 + \cdots + e_kg_k$, where $g_1, \cdots, g_k \in G_p$, and we are done.

The converse is obvious. □

Lemma 5. *Suppose* $1 \in L \leq R$ *and* $A, H \leq G$. *Then*

$$V_p(R(G); H) \cap V_pL(A) = V_p(L(A); H \cap A).$$

Proof. The inclusion "$\supseteq$" is evident.

As for the inclusion "$\subseteq$", take x to belong in the left hand-side. Write $x = \sum_{g\in G} r_gg$ where $r_g \in G$ and for each element $b \in G$ of this sum we have $\sum_{g\in bH} r_g = 0$ if $b \notin H$ and $\sum_{g\in bH} r_g = 1$ if $b \in H$, and $x = \sum_{a\in A} f_aa$ where $f_a \in L$. Thus the canonical records $\sum_{g\in G} r_gg = \sum_{a\in A} f_aa$ imply that $r_g = f_a$ and $g = a$. Furthermore,

$$x = \sum_{a\in bH\cap A} f_a = \sum_{a\in b(H\cap A)} f_a = 0$$

when $b \notin H \cap A$ and $x = \sum_{a\in bH\cap A} f_a = \sum_{a\in b(H\cap A)} f_a = 1$ when $b \in H \cap A$, because $b \in A$, as required. □

Proposition 6. *Suppose $G_t = G_p \subseteq H$ where H is isotype in G and* $\mathrm{char}(R) = p$ *is a prime. Then*

$$\begin{aligned}\left[V_p(R(G); H)\,\mathrm{Id}\,R(G)\right]^{p^\alpha} &= V_p^{p^\alpha}\big(R(G); H\big)\,\mathrm{Id}^{p^\alpha} R(G)\\ &= V_p\big(R^{p^\alpha}(G^{p^\alpha}); H^{p^\alpha}\big)\,\mathrm{Id}\,R(G^{p^\alpha}).\end{aligned}$$

Proof. Clearly the inclusion "$\supseteq$" is true.

Conversely, in view of Proposition 3, $A = [V_p(R(G); H)\,\mathrm{Id}\,R(G)]^{p^\alpha} \subseteq V^{p^\alpha} R(G) = V R^{p^\alpha}(G^{p^\alpha}) = V_p R^{p^\alpha}(G^{p^\alpha})\,\mathrm{Id}\,R(G^{p^\alpha})$, whence by the modular law we have

$$\begin{aligned}A &\subseteq \big(V_p R^{p^\alpha}(G^{p^\alpha})\,\mathrm{Id}\,R(G^{p^\alpha})\big) \cap (V_p(R(G); H)\,\mathrm{Id}\,R(G))\\ &= \mathrm{Id}\,R(G^{p^\alpha})\big[V_p R^{p^\alpha}(G^{p^\alpha}) \cap (V_p(R(G); H)\,\mathrm{Id}\,R(G))\big]\\ &= \mathrm{Id}\,R(G^{p^\alpha})\big[V_p R^{p^\alpha}(G^{p^\alpha}) \cap (V_p(R(G); H)\,\mathrm{Id}_p\,R(G))\big].\end{aligned}$$

Using Lemma 4 we write $\mathrm{Id}_p\,R(G) = \mathrm{Id}\,R(G_p)$. On the other hand, it follows that $\mathrm{Id}\,R(G_p) \subseteq V_p(R(G_p); G_p) \subseteq V_p(R(G); H)$; in fact, each element of $\mathrm{Id}\,R(G_p)$ can be written as $x = e_1 g_{1p} + \cdots + e_s g_{sp}$ where $e_1, \cdots, e_s$ are orthogonal idempotents of R with $e_1 + \cdots + e_s = 1$. Thus $x = 1 + e_1(g_{1p} - 1) + \cdots + e_s(g_{sp} - 1) \in 1 + I(R(G_p); G_p) = V_p(R(G_p); G_p)$, and the wanted relation holds as claimed.

Furthermore, employing Lemma 5,

$$\begin{aligned}A &\subseteq \mathrm{Id}\,R(G^{p^\alpha})[V_p R^{p^\alpha}(G^{p^\alpha}) \cap V_p(R(G); H)]\\ &= \mathrm{Id}\,R(G^{p^\alpha}) V_p(R^{p^\alpha}(G^{p^\alpha}); G^{p^\alpha} \cap H)\\ &= \mathrm{Id}\,R(G^{p^\alpha}) V_p(R^{p^\alpha}(G^{p^\alpha}); H^{p^\alpha})\\ &= \mathrm{Id}^{p^\alpha} R(G) V_p^{p^\alpha}(R(G); H),\end{aligned}$$

as required. □

We now have all the machinery needed to prove the following chief statement.

Theorem 7. *Suppose $G_t = G_p$, H is an isotype subgroup of G such that $H \supseteq G_p$ and suppose char$(R) = p$ is a prime. Then*

$$\begin{aligned}d[V_p(R(G); H)\,\mathrm{Id}\,R(G)] &= dV_p(R(G); H) d\,\mathrm{Id}\,R(G)\\ &= V_p\big(R^{(p)}(G^{(p)}); H^{(p)}\big)\,\mathrm{Id}\,R(G^{(p)}).\end{aligned}$$

Proof. Appealing to [6], for an arbitrary multiplicative group A we may write that

$$dA = \cap_p A^{p^{\omega\tau}} = A^\tau = A^{\tau+1}$$

where τ is the minimal (i.e., the first) ordinal with this property.

We now pause to note that the following five formulas are true for any ordinal number α.

(a) $V^{p^\alpha} R(G) = V R^{p^\alpha}(G^{p^\alpha})$;

(b) $V_p^{p^\alpha} R(G) = V_p R^{p^\alpha}(G^{p^\alpha})$;

(c) $\mathrm{Id}^{p^\alpha} R(G) = \mathrm{Id}\, R(G^{p^\alpha})$;

(d) $\mathrm{Id}^{q^\alpha} R(G) = \mathrm{Id}\, R(G^{q^\alpha})$;

(e) $V_p^{p^\alpha}(R(G); H) = V_p(R^{p^\alpha}(G^{p^\alpha}); H^{p^\alpha})$.

Point (a) is well-known, and (b) is its direct consequence.

As for (c), it is obviously true for $\alpha = 0$ and suppose it is valid for all ordinals strictly less than α. Observe that if $x \in \mathrm{Id}\, R(G)$, then $x = e_1 g_1 + \cdots + e_k g_k$ for some orthogonal idempotents $e_1, \cdots, e_k$ with sum 1 and some $g_1, \cdots, g_k \in G$. Thus $x^p = e_1 g_1^p + \cdots + e_k g_k^p \in \mathrm{Id}\, R(G^p)$, so that the formula follows for $\alpha = 1$. If α is isolated, then in view of the induction hypothesis

$$\begin{aligned}\mathrm{Id}^{p^\alpha} R(G) &= (\mathrm{Id}^{p^{\alpha-1}} R(G))^p = (\mathrm{Id}\, R(G^{p^{\alpha-1}}))^p \\ &= \mathrm{Id}^p R(G^{p^{\alpha-1}}) = \mathrm{Id}\, R((G^{p^{\alpha-1}})^p) = \mathrm{Id}\, R(G^{p^\alpha}).\end{aligned}$$

If now α is limit, then by the induction hypothesis we have

$$\begin{aligned}\mathrm{Id}^{p^\alpha} R(G) &= \cap_{\beta<\alpha} \mathrm{Id}^{p^\beta} R(G) = \cap_{\beta<\alpha} \mathrm{Id}\, R(G^{p^\beta}) \\ &= \mathrm{Id}\, R\big(\cap_{\beta<\alpha} G^{p^\beta}\big) = \mathrm{Id}\, R(G^{p^\alpha}),\end{aligned}$$

where the identity $\cap_{\beta<\alpha} \mathrm{Id}\, R(G^{p^\beta}) = \mathrm{Id}\, R(\cap_{\beta<\alpha} G^{p^\beta})$ follows easily by comparison of the canonical records of elements from the left hand-side. The obtained sequence of equalities is tantamount to the expected equality.

Point (d) follows in the same manner because referring to the Newton's binomial formula and to the orthogonality of the system $\{e_1, \cdots, e_k\}$ we derive that $(e_1 g_1 + \cdots + e_k g_k)^q = e_1 g_1^q + \cdots + e_k g_k^q$.

Point (e) follows in the same manner as ([4], Lemma 1.1) even when H is not isotype in G.

Next, in accordance with Proposition 6 we write

$$\big[V_p(R(G);H)\,\mathrm{Id}\,R(G)\big]^{p^\alpha} = V_p^{p^\alpha}(R(G);H)\,\mathrm{Id}^{p^\alpha} R(G). \tag{1}$$

Further, we shall show that for every prime $q \neq p$ the following holds:

$$\big[V_p(R(G);H)\,\mathrm{Id}\,R(G)\big]^{q^\alpha} = V_p(R(G);H)\,\mathrm{Id}^{q^\alpha} R(G). \tag{2}$$

For $\alpha = 0$ and $\alpha = 1$ we are done. As above the case when α is isolated is plain. If α is limit, we have by induction that

$$\begin{aligned}\big[V_p(R(G);H)\,\mathrm{Id}\,R(G)\big]^{q^\alpha} &= \cap_{\beta<\alpha}[V_p(R(G);H)\,\mathrm{Id}\,R(G)]^{q^\beta}\\ &= \cap_{\beta<\alpha}\big[V_p(R(G);H)\,\mathrm{Id}^{q^\beta} R(G)\big]\\ &= \cap_{\beta<\alpha}\big[V_p(R(G);H)\,\mathrm{Id}\,R(G^{q^\beta})\big],\end{aligned}$$

where the last equality follows from (d). We claim that the last intersection is equal to $V_p(R(G);H)[\cap_{\beta<\alpha}\mathrm{Id}\,R(G^{q^\beta})] = V_p(R(G);H)\,\mathrm{Id}\,R(G^{q^\alpha})$. In fact, letting $x \in \cap_{\beta<\alpha}[V_p(R(G);H)\,\mathrm{Id}\,R(G^{q^\beta})]$ whence $x = (r_1g_1 + \cdots + r_kg_k)(e_1a_{1\beta} + \cdots + e_sa_{s\beta}) \in V_p(R(G);H)\,\mathrm{Id}\,R(G^{q^\gamma})$ for each γ with $\beta < \gamma < \alpha$, where

$$r_1g_1 + \cdots + r_kg_k \in V_p(R(G);H) \quad \text{and} \quad e_1a_{1\beta} + \cdots + e_sa_{s\beta} \in \mathrm{Id}\,R(G^{q^\beta}).$$

Thus, $e_1a_{1\beta} + \cdots + e_sa_{s\beta} \in V_p(R(G);H)\,\mathrm{Id}\,R(G^{q^\gamma})$ and hence there exists a natural t with the property $a_{1\beta}^{p^t} \in (G^{q^\gamma})^{p^t}, \cdots, a_{s\beta}^{p^t} \in (G^{q^\gamma})^{p^t}$; note that $e_i + \cdots + e_j \neq 0$ for all $1 \leq i,j \leq s$ since otherwise by multiplying with $e_i, \cdots, e_j$ both sides of $e_i + \cdots + e_j = 0$ we deduce $e_i = \cdots = e_j = 0$ which is false. Therefore, $a_{1\beta} \in G_pG^{q^\gamma} \subseteq G^{q^\gamma}, \cdots, a_{s\beta} \in G_pG^{q^\gamma} \subseteq G^{q^\gamma}$ because $G_p = G_p^q$. Finally, we conclude that $e_1a_{1\beta} + \cdots + e_sa_{s\beta} \in \mathrm{Id}\,R(G^{q^\gamma})$, that is, $e_1a_{1\beta} + \cdots + e_sa_{s\beta} \in \cap_{\beta<\alpha}\mathrm{Id}\,R(G^{q^\beta}) = \mathrm{Id}\,R(G^{q^\alpha})$ as promised.

As a final third step, with the aid of (1) and (2) plus (c), (d) and (e), we shall prove in addition that

$$\begin{aligned}\cap_l[V_p(R(G);H)\,\mathrm{Id}\,R(G)]^{l^\alpha} &= \cap_l\big[V_p^{l^\alpha}(R(G);H)\,\mathrm{Id}^{l^\alpha} R(G)\big]\\ &= \big[\cap_l V_p^{l^\alpha}(R(G);H)\big]\big[\cap_l \mathrm{Id}^{l^\alpha} R(G)\big]\\ &= V_p^{p^\alpha}(R(G);H)\big[\cap_l \mathrm{Id}\,R(G^{l^\alpha})\big]\\ &= V_p^{p^\alpha}(R(G);H)\,\mathrm{Id}\,R(\cap_l G^{l^\alpha}),\end{aligned} \tag{3}$$

where the intersection is taken over all primes l.

Indeed, we shall use the idea as in the above second step. Writing $x \in V_pR^{p^\alpha}((G^{p^\alpha}); H^{p^\alpha}) \operatorname{Id} R(G^{p^\alpha})$ and $x \in V_p(R(G); H) \operatorname{Id} R(G^{q^\alpha})$ as

$$\begin{aligned} x &= \big(r_1g_1 + \cdots + r_kg_k\big)\big(e_1a_{1p} + \cdots + e_sa_{sp}\big) \\ &= \big(f_1h_1 + \cdots + f_kh_k\big)\big(e'_1b_{1q} + \cdots + e'_sb_{sq}\big) = \ldots, \end{aligned}$$

where it is apparent in what algebraic structure the symbols belong, we find that there is a positive integer t such that $a_{1p}^{p^t} \in (G^{q^\alpha})^{p^t}, \cdots, a_{sp}^{p^t} \in (G^{q^\alpha})^{p^t}$; notice that $e_i + \cdots + e_j \neq 0$ for all $1 \leq i, j \leq s$ since otherwise by multiplying with $e_i, \cdots, e_j$ both sides of $e_i + \cdots + e_j = 0$ we get $e_i = \cdots = e_j = 0$ which is impossible. Consequently, $a_{1p} \in G_pG^{q^\alpha} \subseteq G^{q^\alpha}$, i.e., $a_{1p} \in G^{p^\alpha} \cap G^{q^\alpha}$ etc. $a_{sp} \in G^{p^\alpha} \cap G^{q^\alpha}$ for each prime $q \neq p$. Finally, $e_1a_{1p} + \cdots + e_sa_{sp} \in \operatorname{Id} R(\cap_p G^{p^\alpha})$, as required.

With all of these three main equalities at hand, we immediately infer by substituting $\alpha = \omega\tau$ that the wanted formula for $d(V_p(R(G); H) \operatorname{Id} R(G))$ holds. □

As a direct consequence, we yield the following.

Corollary 8 ([5]). *Let $G_t = G_p$ and let $\operatorname{char}(R) = p$ be a prime. Then*

$$dVR(G) = dV_pR(G)d \operatorname{Id} R(G) = V_pR^{(p)}(G^{(p)}) \operatorname{Id} R(G^{(p)}).$$

Proof. Choose $H = G$ and so $V_p(R(G); H) = V_p(R(G); G) = V_pR(G)$. Henceforth, we apply Proposition 3 and Theorem 7 to infer the desired equalities. □

Remark. However, Theorem 7 gives a more general strategy than Corollary 8 via the various choices of H; in fact we may also take $H = G_p$ which leads us to another interesting situations.

So, we are now in a position to deduce the isomorphism classification of $dVR(G)$.

Theorem 9 ([5]). *Let G be a p-mixed abelian group and R a commutative unital ring of prime characteristic p. Then the following isomorphism holds:*

$$dVR(G) \cong \coprod_\lambda \mathbf{Z}(p^\infty) \times \coprod_\mu \left(\frac{dG}{dG_p}\right)$$

where $\lambda = \max(|R^{(p)}|, |G^{(p)}|)$ if $dG_p \neq 1$, or $\lambda = \max(|N(R^{(p)})|, |G^{(p)}|)$ if $dG_p = 1$, $G^{(p)} \neq 1$ and $N(R^{(p)}) \neq 0$, or $\lambda = 0$ if $G^{(p)} = 1$ and $N(R^{(p)}) = 0$, and $\mu = |\operatorname{id}(R)| \geq \aleph_0$ or $\mu = \log_2 |\operatorname{id}(R)|$ if $|\operatorname{id}(R)| < \aleph_0$.

Proof. According to Corollary 8, we write $dVR(G) = dV_pR(G)d\,\mathrm{Id}\,R(G)$, and so with the help of [5] we deduce that

$$\begin{aligned} dVR(G) &\cong dV_pR(G) \times \big(dVR(G)/dV_pR(G)\big) \\ &\cong dV_pR(G) \times \big(d\,\mathrm{Id}\,R(G)/d\,\mathrm{Id}_p\,R(G)\big). \end{aligned}$$

For the classification of the first factor, $dV_pR(G)$, we apply either [12] or [13]. As for the second one, we employ Proposition 2 to infer that $d\,\mathrm{Id}\,R(G) \cong \coprod_\mu dG$, where μ is calculated in the same manner as above and thus, under the validity of this isomorphism, we obtain that $d\,\mathrm{Id}_p\,R(G) \cong \coprod_\mu (dG)_p = \coprod_\mu dG_p$. Finally, via the canonical isomorphism, we conclude that

$$\frac{d\,\mathrm{Id}\,R(G)}{d\,\mathrm{Id}_p\,R(G)} \cong \frac{\coprod_\mu dG}{\coprod_\mu dG_p} \cong \coprod_\mu \left(\frac{dG}{dG_p}\right).$$

So, the isomorphism relation for $dVR(G)$ is really true, as desired. □

3 Left-open problems

There are a few questions that remain unanswered. The first of them asks whether the main formula used in the proof of Theorem 7 is valid for an arbitrary subgroup H.

Problem 1. Let $H \leq G$ where $G_t = G_p$ and $\mathrm{char}(R) = p$. Does it follow that

$$\left[V_p(R(G); H)\,\mathrm{Id}\,R(G)\right]^{p^\alpha} = V_p^{p^\alpha}(R(G); H)\,\mathrm{Id}^{p^\alpha}\,R(G)?$$

Let $\mathrm{supp}(G) = \{p | G_p \neq 1\}$, $\mathrm{inv}(R) = \{p | p \cdot 1_R \in R^*\}$ where R^* is the multiplicative group (i.e., the group of units) of R, and $\mathrm{zd}(R) = \{p | \exists r \in R \setminus \{0\} \colon pr = 0\}$.

Problem 2. If $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{zd}(R)) = \emptyset$, does it follow that

$$VR(G) = \mathrm{Id}\,R(G)V(R(G_t) + N(R(G)))?$$

In order $dVR(G)$ to be comprehensively described up to isomorphism only in terms associated with R and G, we also state the following

Problem 3. Suppose $\mathrm{supp}(G) \cap (\mathrm{inv}(R) \cup \mathrm{zd}(R)) = \emptyset$. Is the following equality true

$$dVR(G) = d\,\mathrm{Id}\,R(G)dV(R(G_t) + N(R(G)))?$$

The solution of these three questions will be the theme of a later research paper.

References

[1] P.V. Danchev. *Maximal divisible subgroups in modular group algebras of p-mixed and p-splitting abelian groups.* Rad. Mat., **13** (2004), 23–32.

[2] P.V. Danchev. *Isomorphism classifications of maximal divisible subgroups in modular group rings.* Bull. Georg. Natl. Acad. Sci. (Math.), **174** (2006), 238–242.

[3] P.V. Danchev. *Maximal divisible subgroups in modular group algebras of torsion abelian groups over certain commutative rings.* An. St. Univ. "Al. I. Cuza" Iasi – Math., **53** (2007), 325–329.

[4] P.V. Danchev. *Isomorphism characterization of divisible groups in modular abelian group rings.* Georgian Math. J., **16** (2009), 49–54.

[5] P.V. Danchev. *Maximal divisible subgroups in p-mixed modular abelian group rings.* Commun. Algebra, **38** (2010).

[6] L. Fuchs. Infinite Abelian Groups, volumes I and II, Mir, Moskva 1974 and 1977 (translated in Russian).

[7] G. Karpilovsky. *On finite generation of unit groups of commutative group rings.* Arch. Math. (Basel), **40** (1983), 503–508.

[8] G. Karpilovsky. Commutative Group Algebras, Marcel Dekker, New York (1983).

[9] G. Karpilovsky. Unit Groups of Group Rings, Longman Sci. and Techn., Harlow (1989).

[10] G. Karpilovsky. *Units of commutative group algebras.* Expo. Math., **8** (1990), 247–287.

[11] V. Kuneva. *Maximal divisible subgroups in modular group algebras of p-mixed abelian groups.* Compt. rend. Acad. bulg. Sci., **59** (2006), 899–902.

[12] N.A. Nachev. *Invariants of the Sylow p-subgroup of the unit group of commutative group ring of characteristic p.* Compt. rend. Acad. bulg. Sci., **47** (1) (1994), 9–12.

[13] N.A. Nachev. *Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p.* Commun. Algebra, **23** (1995), 2469–2489.

Peter Danchev
Department of Mathematics
University of Plovdiv
4000 Plovdiv
BULGARIA

E-mail: pvdanchev@yahoo.com