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On differentiable area–preserving maps of the plane

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Abstract. $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an *almost–area–preserving* map if: (a) F is a topological embedding, not necessarily surjective; and (b) there exists a constant $s > 0$ such that for every measurable set B, $\mu(F(B)) = s\mu(B)$ where μ is the Lebesgue measure. We study when a differentiable map whose Jacobian determinant is nonzero constant to be an almost–area–preserving map. In particular, if for all z, the eigenvalues of the Jacobian matrix DF_z are constant, F is an almost–area–preserving map with convex image.

Keywords: area-preserving maps, Jacobian conjecture.

Mathematical subject classification: 14R15, 26B10, 14E07, 14E25.

1 Introduction and statement of the results

We say that $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an **almost–area–preserving** map if it satisfies the following two conditions:

(a) This F is a topological embedding; that is, a globally injective local homeomorphism.

(b) There exists a constant $s > 0$ such that for every measurable set $B \subset \mathbb{R}^2$ we have
$$\mu(F(B)) = s\mu(B),$$
where μ is the Lebesgue measure.

This topological embedding does not have to be surjective, because its image $F(\mathbb{R}^2)$ might be a proper subset of $\mathbb{R}^2$ not necessarily convex.

The standard area–preserving diffeomorphisms of class C^1 are examples of *almost–area–preserving* maps, see [OU41, KH].

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A differentiable planar map F is called **unipotent**, if its spectrum $\mathrm{Spc}(F) = \{\text{eigenvalues of } DF_z \colon z \in \mathbb{R}^2\}$[1] is the one point set $\{1\}$. In [Ch00], Chamberland proves that a real–analytic map of $\mathbb{R}^2$ into itself has an inverse as long as it is unipotent (see also [Che99] and [VH97]). So, real–analytic–unipotent maps are almost–area–preserving. A proof for C^1-maps of the plane appears in [Ca00], where Campbell gives a normal form; this impressive result may be paraphrased as follows: A C^1-planar map $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ is unipotent, if and only if, it has the form

$$F(x, y) = \big(x + b\phi(ax + by) + c,\ y - a\phi(ax + by) + d\big) \quad \forall x, y \in \mathbb{R}, \quad (1)$$

where a, b, c and d are real constants, and ϕ is a C^1-function on a single variable. Thus, any C^1-unipotent map of the plane is bijective. Therefore, *the C^1-unipotent maps of the plane are also almost–area–preserving.*

In [Ch03], Chamberland research the C^1-maps $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ with **constant eigenvalues**; that is, its spectrum has at most two elements. These maps are completely characterized in two cases: **(a)** F is a C^1-unipotent map, then F has the form (1) and **(b)** F is a polynomial map, then F takes the form

$$F(x, y) = \big(ax + by + \beta\varphi(\alpha x + \beta y) + e,\ cx + dy - \alpha\varphi(\alpha x + \beta y) + f\big) \quad (2)$$

for some real constant a, b, c, d, e, f, α and β, and some polynomial φ of one variable. Furthermore, if F is a polynomial map on the plane and $\mathrm{Spc}(F)$ is bounded, Lemma 2.1 of [CGM01] implies that $\mathrm{Spc}(F)$ has at most two elements. Therefore, *polynomial planar maps whose bounded* $\mathrm{Spc}(F)$ *misses the zero are almost–area–preserving.*

Theorem 1. *Let $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map with* $\det(DF_z) \neq 0$*, for all $z \in \mathbb{R}^2$. If* $\mathrm{Spc}(F)$ *has at most two elements then F is an almost–area–preserving map with convex image.*

The almost–area–preserving maps of Theorem 1 extends the last examples, described in [Ca00] and [Ch03], to maps not necessarily C^1.

A differentiable map $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ shall be called **Jacobian map**, if for all $z \in \mathbb{R}^2$, its Jacobian determinant $\det(DF_z)$ is nonzero constant; in this way, a Jacobian planar map $F = (f, g)$ with f and g polynomials and $\det(DF_z) = 1$, shall be refereed as a **Keller map**. In this context, the injective Jacobian maps of class C^1 are almost–area–preserving. However, it is well know that there are Jacobian maps which are not injective as shown the map

$$F(x, y) = \sqrt{2}\big(e^{\frac{x}{2}} \cos(ye^{-x}),\ e^{\frac{x}{2}} \sin(ye^{-x})\big)$$

[1] The spectrum is also denoted by $\mathrm{Spec}(F)$, but we prefer $\mathrm{Spc}(F)$ in order to avoid some confusion with the algebraic notation $\mathrm{Spec}(\mathfrak{F})$ associated to a ring $\mathfrak{F}$.

mentioned in [ChM98]. Consequently, the goal is to give sufficient conditions on differentiable maps to insure that it is an almost–area–preserving map. To this end, for each $\theta \in \mathbb{R}$ we denote by R_θ the linear rotation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and define the map $F_\theta = R_\theta \circ F \circ R_{-\theta}$.

Definition 1. *We say that the differentiable map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition if for each $\theta \in \mathbb{R}$, there does not exist a sequence $(x_k, y_k) \in \mathbb{R}^2$ with $x_k \to +\infty$ such that $F_\theta((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DF_\theta(x_k, y_k)$ has a real eigenvalue λ_k satisfying $x_k\lambda_k \to 0$.*

Definition 1 was motivated by the condition (*) of [GN07] which claims: "For each $\theta \in \mathbb{R}$, there does not exist a sequence $z_k \in \mathbb{R}^2$ with $z_k \to \infty$ such that $F_\theta(z_k) \to p \in \mathbb{R}^2$ and $DF_\theta(z_k)$ has a real eigenvalue $\lambda_k \to 0$."

If $g(x, y)$ is a C^1 function such that $g(x, y) = \frac{y}{x}$ as long as $x \geq 3$, and the map $F(x, y) = (e^{-x}, g(x, y))$ satisfies $\det(DF_z) \neq 0$, for all $z \in \mathbb{R}^2$. Then, for any unbounded sequence $3 \leq x_k \to +\infty$ there exist $p = (0, 0)$ such that $F(x_k, 0) \to p$ and $DF(x_k, 0)$ has a real eigenvalue

$$\frac{1}{x_k} = \lambda_k \to 0.$$

However, the limit of the product $x_k\lambda_k$ is different from zero.

Theorem 2. *If the differentiable Jacobian map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition then F is an almost–area–preserving map with convex image.*

Theorem 2 improves the main result of [GN07] (see also [Ra02, R05]). Moreover, if Theorem 2 is valid for maps F, it remains true for $-F$. In fact, if in such theorem we change the pair $\{F, \mathrm{Spc}(F)\}$ by $\{-F, \mathrm{Spc}(-F)\}$ we may see that its conclusion remains valid. Also, for each $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ any arbitrary invertible linear map, we have that if F is as in Theorem 2 then $A \circ F \circ A^{-1}$ is also an almost–area–preserving map with convex image.

The maps in Theorem 2 are injective, so we obtain the following corollary whose proof is presented at the end of Section 3.

Corollary 1. *If F is as in Theorem 2 and* $\mathrm{Spc}(F) \subset \{z \in \mathbb{C}\colon ||z|| < 1\}$, *then F has at most one fixed point.*

Another interesting property of the Keller maps as in Corollary 1 (i.e with $\mathrm{Spc}(F) \subset \{z \in \mathbb{C}\colon ||z|| < 1\}$) is obtained from Theorem B of [CGM99]. This theorem proves the existence of a unique fixed point of F which is a global attractor for the discrete dynamical system generated by F (i.e for each $p \in \mathbb{R}^2$ the ω-limit set of the orbit $(F^k(p))_{k\geq 0}$[2] is the one point set $\{z \in \mathbb{R}^2 \colon F(z) = z\}$). This dynamical property is false for all the maps of class C^1 as shown the global diffeomorphisms given in Theorem E of [CGM99]. If a is small enough, this map G_a given by

$$G_a(x, y) = \left(-\frac{ky^3}{1 + x^2 + y^2} - ax, \frac{ky^3}{1 + x^2 + y^2} - ay\right), \quad 1 < k < \frac{2}{\sqrt{3}}$$

satisfies $G_a(0) = 0$ and $\mathrm{Spc}(G_a) \subset \{z \in \mathbb{C} \colon ||z|| < 1\}$, but it has a periodic orbit. However, the Jacobian determinant of G_a is not constant.

Problem 1. Let $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a Jacobian map of class C^1 with $F(0) = 0$ and $\mathrm{Spc}(F) \subset \{z \in \mathbb{C} \colon ||z|| < 1\}$. Does it follow that 0 is a global attractor for F?

Observe that Theorem 1 follows directly of Theorem 2.

This paper is organized as follows: In Section 2, we show that the maps of Theorem 2 are injective maps whose image is convex. In Section 3, we conclude the proof of Theorem 2.

2 A topological embedding on the plane

The present section is related to the Keller Jacobian Conjecture, more precisely its real version which claims: *Every polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with a constant nonzero Jacobian determinant is invertible.* (See [ChM98, SX96, NX02, FGR03]). This conjecture remains open even when $n = 2$ but one knows that injectivity implies the surjectivity in this context, [BA62, Pa04]. There is quite a lot literature in problem of finding sufficient conditions for a C^1 map from $\mathbb{R}^2$ to $\mathbb{R}^2$ to be injective. For example, Campbell in [Ca00] proved in the C^1-case that the fact that 1 is the only eigenvalue of all the Jacobian matrices is a sufficient condition. Then, Chamberland conjectured in [Ch00] that the fact that the spectrum is bounded away from 0 is sufficient to get the injectivity.

In [FGR04], we gave a proof of Chamberland Conjecture in dimension 2 (without assuming the C^1-hypothesis). More precisely, a differentiable map $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ is invertible as long as its spectrum, $\mathrm{Spc}(F)$ misses a set $[0, \varepsilon)$ for some $\varepsilon > 0$. The idea of the proof is to study the foliation $\mathcal{F}(f)$ given by the level curves $\{f = constant\}$ of the first coordinate f of $F = (f, g)$, [RR89].

[2]The orbit $(F^k(p))_{k\geq 0}$ is the planar set given by the iterations of the map: $\{F^k(p) \colon k \in \mathbb{N} \cup \{0\}\}$, where F^0 denotes the identity map.

The non–injectivity of F implies that the foliation admits a Reeb component as in [HR57]. With a nice argument a sequence $((x_k, y_k))_{k\geq 0}$ is found such that one of the eigenvalue λ_k of the Jacobian matrix $DF_{(x_k,y_k)}$ is positive and satisfies $\lim_{k\to\infty}\lambda_k = 0$. A remark can be found in [GN07], where it is said that one can suppose that the sequence $(F(x_k, y_k))_{k\geq 0}$ has a limit in $\mathbb{R}^2$. This motives to present Definition 1 and the following

Theorem 3. *Suppose that the differentiable map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the B-condition and* $\det(DF_z) \neq 0$, *for all $z \in \mathbb{R}^2$ then F is a topological embedding.*

Proof. From [FGR04, Proposition 1.4], we shall have established that $F = (f, g)$ will be injective if we prove that $\mathcal{F}(f)$ has no half–Reeb component. In order to obtain this result we consider the map $(f_\theta, g_\theta) = R_\theta \circ F \circ R_{-\theta}$.

Suppose by way of contradiction that $\mathcal{F}(f)$ has a half–Reeb component. From [FGR04, Proposition 1.5] we can select some $\theta \in \mathbb{R}$ for which $\mathcal{F}(f_\theta)$ has a half–Reeb component whose projection is an unbounded interval; thus there are $a_0 > 0$ and a half–Reeb component $\mathcal{A}$ of $\mathcal{F}(f_\theta)$ such that $[a_0, +\infty) \subset \Pi(\mathcal{A})$, where $\Pi(x, y) = x$. Then, *for some $a > a_0$ large enough* the line $\Pi^{-1}(a)$ intersects the two non–compact edges of $\mathcal{A}$ but it is disjoint of the compact edge. Since $\mathcal{A}$ is the union of an increasing sequence of compact sets bounded by the compact edge and a compact segment of leaf. Then, from our selection of $a > a_0$ we have that for any $x \geq a$, the line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\mathcal{F}(f_\theta)|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap [x, +\infty) = \{x\}$. If $x \geq a$, the intersection $\alpha_x \cap \Pi^{-1}(x)$ is a compact subset of $\mathcal{A}$. Therefore, we can define two functions

$$H\colon (a, +\infty) \to \mathbb{R} \quad \text{by} \quad H(x) = \sup\left\{y\colon (x, y) \in \alpha_x \cap \Pi^{-1}(x)\right\},$$

and

$$\varphi\colon (a, +\infty) \to \mathcal{A} \quad \text{by} \quad \varphi(x) = f_\theta(x, H(x)).$$

For every interval $[a, b) \subset \mathbb{R}$ the function H is bounded, because the graph of the restriction $H|_{[a,b)}$ is contained in the compact subset of $\mathcal{A}$ whose boundary is $\Pi^{-1}(a) \cup \alpha_b$. Moreover, when $x \geq a$ is kept fixed, the point $(x, H(x)) \in \alpha_x \cap \Pi^{-1}(x)\}$ is a local extremal of the differentiable function $(x, y) \mapsto f_\theta(x, y)$. Thus,

(a) if $x \geq a$ every partial derivative satisfies $(f_\theta)_y(x, H(x)) = 0$.

The function φ is bounded because its image is contained in $f_\theta(\Gamma)$ where Γ is the compact edge of $\mathcal{A}$. This φ is continuous because the leaves α_x depends continuously of x (i.e $\mathcal{F}(f_\theta)$ is a C^0-foliation). And, φ is strictly monotone because $\mathcal{F}(f_\theta)$ is topological transversal to $\Gamma \setminus \{\text{some point}\}$. Therefore:

(b) This $\varphi(x) = f_\theta(x, H(x))$ is bounded, continuous and strictly monotone; in particular φ is differentiable a.e.

(c) We claim that the image $F_\theta(\mathcal{A})$ is bounded, where $F_\theta = (f_\theta, g_\theta)$.

Consider Γ the compact edge of $\mathcal{A}$ which is homeomorphic to an interval, and denote by I the compact interval $f_\theta(\Gamma)$. From our construction of $\mathcal{F}(f_\theta)$, the compact continuous curve $F_\theta(\Gamma)$ is the union of exactly two graphs of continuous functions $I \mapsto \mathbb{R}$. Moreover, the intersection of this two graphs is a point in $F_\theta(\Gamma)$ where the vertical foliation has a topological tangency. Both functions $I \mapsto \mathbb{R}$ are bounded, and every vertical interval whose end points belong these graphs is the image under F_θ of a compact segment of leaf contained in $\mathcal{A}$. Since the adherence of a bounded set is also bounded and $\mathcal{A} \setminus \{\text{non-compact edges}\}$ is the union of such type of compact segment of leafs we obtain (c).

Since f_θ is differentiable at $(x, H(x))$ and H is upper semicontinuous, we can proceed as in [FGR04] and obtain that φ has the following property:

(d) For some full measure subset $M \subset (a, +\infty)$ such φ is differentiable on M and for all $x \in M$ the Jacobian matrix of F_θ at $(x, H(x))$ is

$$DF_\theta(x, H(x)) = \begin{pmatrix} \varphi'(x) & 0 \\ (g_\theta)_x(x, H(x)) & (g_\theta)_y(x, H(x)) \end{pmatrix}.$$

In other words, if $x \in M$, then $\varphi'(x) = (f_\theta)_x(x, H(x)) \in \mathrm{Spc}(F)$.

To proceed we shall only consider the case in which $\varphi'(x) \geq 0$, because in the other case we can use $\limsup_{x\to\infty} x\varphi'(x)$.

If $\liminf_{x\to\infty} x\varphi'(x) = 0$, there is a sequence $(x_k, H(x_k)) \to \infty$ such that $DF_\theta(x_k, H(x_k))$ has a real eigenvalue $\lambda_k = \varphi'(x_k)$ for which $\lim x_k\lambda_k = 0$ and $F_\theta(x_k, H(x_k))$ tends to a finite value in the closure $\overline{F_\theta(\mathcal{A})}$ (which is compact by (c)). This contradicts the B-condition.

If $\liminf_{x\to\infty} x\varphi'(x) \neq 0$, then $\liminf_{x\to\infty} x\varphi'(x) > 0$. This implies that there are constants $\alpha_0 \geq a$ and $\ell > 0$ such that $\ell \leq x\varphi'(x)$ if $x \geq \alpha_0$. From (b) there is a constant $K > 0$ such that for all $x > \alpha_0$, $0 \leq \varphi(x) - \varphi(\alpha_0) \leq K$. Take $c_0 > \alpha_0$ so that $K < \int_{\alpha_0}^{c_0} \frac{\ell}{x} dx$. Then

$$K < \int_{\alpha_0}^{c_0} \frac{\ell}{x} dx \leq \int_{\alpha_0}^{c_0} \varphi'(x) dx \leq \varphi(c_0) - \varphi(\alpha_0) \leq K.$$

This contradiction proves the theorem. $\square$

Lemma 1. *If some level curve $\{f = c\}$ is disconnected, then $\mathcal{F}(f)$ has a half-Reeb component.*

Proof. We refeer the reader to the proof of Theorem 2.1 of [GR06]. $\square$

Corollary 2. *If $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is as in Theorem 2, then F is injective and $F(\mathbb{R}^2)$ is convex.*

Proof. Let $p, q \in F(\mathbb{R}^2)$ and let $[p, q] = \{(1-t)p + tq : 0 \le t \le 1\}$. Take $\theta \in \mathbb{R}$ so that $R_\theta([p, q])$ is contained in the vertical line $x = c$. Lemma 1 implies that the level curve $\{f_\theta = c\}$ is a connected subset of the straight line $x = c$ connecting $R_\theta(p)$ with $R_\theta(q)$; that is $R_\theta([p, q]) \subset F_\theta(\mathbb{R}^2)$ which implies that $[p, q] \subset F(\mathbb{R}^2)$ and concludes the proof. $\square$

Let us finish with an example of a map $F = (f, g)$ whose image is not convex and the foliation $\mathcal{F}(f)$ has no Reeb component in the sense of [HR57].

Example. If we consider

$$f(x, y) = \exp(y) \cos\left(2 \arctan(x) - \frac{\pi}{2}\right)$$

and

$$g(x, y) = \exp(y) \sin\left(2 \arctan(x) - \frac{\pi}{2}\right),$$

the map $F = (f, g)$ is injective because of the foliation $\mathcal{F}(f)$ has no half-Reeb component ([FGR04, Proposition 1.4]). More precisely, for every $c \neq 0$ the level curve $\{f = c\}$ is the graph of a function and $\{f = 0\}$ is the vertical axis. Therefore, F is a smooth diffeomorphism between $\mathbb{R}^2$ and the open set $\mathbb{R}^2 \setminus (\{0\} \times (-\infty, 0])$ which in particular is a *non convex image.*

3 Differentiable almost–area–preserving maps

In this section we conclude the proof of Theorem 2 which implies Theorem 1. This proof shall be completed at the end of this section. By [RR89, Theorem 3.4] if $\det(DF_z) \neq 0$ for all $z \in \mathbb{R}^2$, the non–connected function $z \mapsto \det(DF_z)$ has constant sign, so in the rest of this section we may assume that $\det(DF_z) > 0$.

Lemma 2. *Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an injective differentiable (not necessarily C^1) map and let $s > 0$ a constant such that for all $z \in \mathbb{R}^2$, $|\det(DG_z)| = s$. If $B \subset \mathbb{R}^2$ is a measurable set and $G(B)$ has finite Lebesgue measure then*

$$\mu(G(B)) = s\mu(B)$$

where μ is the Lebesgue-measure in $\mathbb{R}^2$.

Proof. If follows directly of applying [P, Corollary 10.6.10] (a change of variables); because $\mu(G(B)) < \infty$, G is a differentiable injective map and the Jacobian determinant $\det(DG_z)$ is constant. $\square$

Proof of Theorem 2. By Corollary 2, F is an injective map with convex image. Then, it is sufficient to prove that Statement (b) in the definition of almost–area–preserving maps is true.

Let $B \subset \mathbb{R}^2$ be any measurable set.

(a.1) We claim that if $\mu(F(B)) < \infty$ or $\mu(B) < \infty$, then $\mu(F(B)) = s\mu(B)$ where $s = |\det(DF_z)|$.

By Lemma 2 (a.1) holds when $\mu(F(B)) < \infty$. In the other case, $\mu(B) < \infty$, so we consider $A = F(B)$ and $G = F^{-1}$. As $\mu(G(A)) = \mu(B) < \infty$ we may apply Lemma 2 and obtain that $\mu(G(A)) = \frac{1}{s}\mu(A)$ because $|\det(DG_z)| = \frac{1}{s}$. From this, (a.1) holds.

(a.2) We claim that if B is a measurable set of the plane $\mu(F(B)) = s\mu(B)$.

If we does not have the condition of (a.1) $\mu(F(B))$ and $\mu(B)$ have no finite measure, so (a.2) is true.

Therefore, F is an almost–area–preserving map. $\square$

Corollary 3. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in Theorem 2. If $(B_n)_n$ is an infinite sequence of measurable sets for which there exist a constant $\delta > 0$, such that for all $N \in \mathbb{N}$, $\mu(B_{N+1} \setminus \cup_{n=1}^{N} B_n) > \delta$. Then, $F(\cup_n B_n)$ has no finite measure.*

Proof. Suppose, by contradiction, that $\mu(F(\cup_n B_n)) < \infty$. Consider, $A_1 = B_1$, and for every N natural greater than one set $A_{N+1} = B_{N+1} \setminus \cup_{n=1}^{N} B_n$. Given $K \in \mathbb{N}$, by using the definition of almost–area–preserving, we have that

$$\mu\left(F\left(\bigcup_n B_n\right)\right) > \mu\left(\bigcup_{N=2}^{K+1} F(A_N)\right) = \sum_{N=2}^{K+1} s\mu(A_N) = s\delta K.$$

Then, the natural numbers $\mathbb{N}$ is bounded. This contradiction conclude the proof. $\square$

Remark 1. Corollary 3 remains true if take F as any almost–area–preserving map whose image may be non–convex. For instance, take the map $F(x, y) = (\exp(x), y\exp(-x))$ for which $F(\mathbb{R}^2) \neq \mathbb{R}^2$.

Proof of Corollary 1. Consider $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $G(z) = F(z) - z$ for all $z \in \mathbb{R}^2$. This map has no positive eigenvalue because $\mathrm{Spc}(G)$ is contained in $\{z \in \mathbb{R}^2 \colon \Re(z) < 0\}$, so it satisfies Theorem 2 because $\mathrm{Spc}(G)$ avoid an open real neighborhood of the origin. Since G is injective we conclude the proof of Corollary 1.

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