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Symmetries of quadratic form classes and of quadratic surd continued fractions. Part II: Classification of the periods' palindromes

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Abstract. According to a theorem by Lagrange, the continued fractions of quadratic surds are periodic. Their periods may have different types of symmetries. This work relates these types of symmetries to the symmetries of the classes of the corresponding indefinite quadratic forms. This allows classifying the periods of quadratic surds and simultaneously finding the symmetry type of the class of an arbitrary indefinite quadratic form and the number of its integer points contained in each domain of the Poincaré tiling of the de Sitter world, introduced in Part I of this paper. Moreover, we obtain the same result for every class of forms representing zero, i.e., when the quadratic surds are replaced by rational, using the finite continued fraction obtained from a special representative of that class. Finally, we show the relation between the reduction procedure for indefinite quadratic forms defined by continued fractions and the classical reduction theory, which acquires a geometric description by the results in Part I.

Keywords: continued fractions, quadratic forms, reduction theory.

Mathematical subject classification: 11A55, 11H55.

1 Definition of the palindromes

Definition 1. A finite sequence $[\alpha_0, \alpha_1, \ldots, \alpha_N]$ is said to be *palindromic*[1] iff

$$\alpha_i = \alpha_{N-i}, \quad i = 0, \ldots, \mathbb{N}.$$

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[1] The word palindrome means exactly "that which reads the same backwards and forwards," e.g., the word RADAR or entire phrases like the Latin riddle IN GIRUM IMUS NOCTE ET CONSUMIMUR IGNI (we go around at night and are consumed by fire).

Definition 2. A *period of length* P is a finite sequence of P natural numbers that cannot be written as a sequence of identical subsequences. A period is said to be *even* if P is even and *odd* otherwise.

Example. The sequence [1, 2, 3, 1, 2, 3] is not a period.

Definition 3. An infinite continued fraction $[\alpha_0, \alpha_1, \alpha_2, \dots]$ is said to be *periodic* if for some nonnegative integer N and some natural number P

$$\alpha_{N+j} = \alpha_{N+j+kP} \quad \forall j, k \in \mathbb{N}. \tag{1}$$

Definition 4. A periodic continued fraction $[\alpha_0, \alpha_1, \alpha_2, \dots]$ is denoted by

$$\big[\alpha_0, \alpha_1, \dots, \alpha_{N-1}, [\alpha_N, \alpha_{N+1}, \dots, \alpha_{N+P-1}]\big]$$

if N is the minimal nonnegative integer satisfying (1). The sequence $[a_1, a_2, \dots, a_P] := [\alpha_N, \alpha_{N+1}, \dots \alpha_{N+P-1}]$ is called the *period* of the periodic continued fraction, and the natural number P is called its *length*.

Definition 5. The *inverse* of the period of a periodic continued fraction is the period obtained by writing the period backwards.

Example. The period $[a, b, c, d]$ is the inverse of the period $[d, c, b, a]$.

Definition 6. The period of a periodic continued fraction is said to be

- *palindromic* if there exists a cyclic permutation[2] of it such that the permuted period is equal to its inverse period,
- *bipalindromic* if there is a cyclic permutation of it such that the permuted period can be subdivided into two palindromic odd sequences, and
- *nonpalindromic* if it is neither palindromic nor bipalindromic.

Examples. The periods (different letters denote different naturals)

$$[a, a, b, b], \quad [a, a, b, a, a] \quad \text{and} \quad [a, b, a, c, c]$$

are palindromic. The periods

$$\Gamma_1 = [a, b, c, b, a, d] \quad \text{and} \quad \Gamma_2 = [a, a, a, b]$$

are bipalindromic because Γ_1 can be written as $[(b, c, b)(a, d, a)]$ (or $[(b, a, d, a, b)(c)]$, etc.) and Γ_2 can be written as $[(a)(a, b, a)]$ (or $[(b)(a, a, a)]$). The periods $[a, b, c]$ and $[a, b, b, c]$ are nonpalindromic.

[2] A cyclic permutation transforms the ordered finite sequence $[a_1, a_2, \dots, a_P]$ into one of the P sequences $[a_k, a_{k+1}, \dots, a_P, a_1, a_2, \dots, a_{k-1}]$, where $k = 1, 2, \dots, P$.

Remarks. 1. Any period of length 1 is palindromic.

2. Any period of length 2 is bipalindromic according to the definition above. Hence, a nonpalindromic period contains at least three different elements.

3. If a palindromic period is even, then there exist at least two different cyclic permutations of it such that the permuted periods are equal to their inverses. *Example.* $[abccba]$ and $[cbaabc]$.

4. The palindromicity of a period of P elements (natural numbers) can be seen as an axial symmetry of a regular plane polygon whose vertices are labeled by these natural numbers. If P is odd, then a vertex must belong to the symmetry axis; if P is even, then either no vertex belongs to the symmetry axis (and every element hence has its symmetric element), or two vertices belong to the symmetry axis (and these two vertices hence have no symmetric element). The latter case corresponds to the bipalindromicity.

2 The symmetry types of the classes of quadratic forms

Let $\mathbf{f}$ denote the triple of integer coefficients of the binary quadratic form $f = mx^2 + ny^2 + kxy$. As in Part I [1], $\mathcal{T}$ denotes the group isomorphic to $PSL(2, \mathbb{Z})$ that acts on the space of the form coefficients (m, n, k) and whose action is induced by that of $SL(2, \mathbb{Z})$ on the xy plane. The class of the form $\mathbf{f} = (m, n, k)$ under $\mathcal{T}$ is denoted by $C(\mathbf{f})$ or $C(m, n, k)$.

We recall the classification of the symmetry types of the classes of indefinite binary quadratic forms (i.e., with a discriminant $\Delta = k^2 - 4mn < 0$), already introduced in Part I. We considered three commuting involutions acting in the space of forms and, with $\mathbf{f}$, defining eight forms (see Fig. 1-III). In particular, for a given $\mathbf{f} = (m, n, k)$,

1. the form $\mathbf{f}_c = (n, m, -k)$ is the *complementary* of the form $\mathbf{f}$,
2. the form $\overline{\mathbf{f}} = (m, n, -k)$ is the *conjugate* of the form $\mathbf{f}$,
3. the form $\mathbf{f}^* = (-n, -m, k)$ is the *adjoint* of the form $\mathbf{f}$,
4. the form $\overline{\mathbf{f}}^* = (-n, -m, -k)$ is the *antipodal* of the form $\mathbf{f}$ and is the adjoint of the conjugate (or the conjugate of the adjoint) of the form $\mathbf{f}$, and
5. the form $-\mathbf{f} = (-m, -n, -k)$ is the *opposite* of the form $\mathbf{f}$ and is the complementary of the adjoint of the form $\mathbf{f}$.

Moreover, a form $\mathbf{f}$ is said to be *self-conjugate* if $\overline{\mathbf{f}} = \mathbf{f}$ and *self-adjoint* if $\mathbf{f}^* = \mathbf{f}$.

Remarks. Any form $(m, n, 0)$ is *self-conjugate*. Any form $(m, -m, k)$ is *self-adjoint*.

The complementary of a form $\mathbf{f} = (m, n, k)$ belongs to the class of $\mathbf{f}$, $C(m, n, k)$, while the conjugate and/or the adjoint of $\mathbf{f}$ *may* or *may not* belong to the class of $\mathbf{f}$.

But *if a class contains a pair of forms related by some involution or a form that is invariant under some involution, then the entire class is invariant under that involution* (see Proposition 1.2 of Part I).

Hence, there are exactly five types of symmetries of the classes, according to the different symmetries considered for the forms.

Definition 8. A class of forms is said to be

1. *asymmetric* if it contains only pairs of complementary forms,
2. *k-symmetric* if, in addition to the pairs of complementary forms, it contains only pairs of conjugate or isolated self-conjugate forms,
3. $(m+n)$*-symmetric* if, in addition to the pairs of complementary forms, it contains only pairs of adjoint and isolated self-adjoint forms,
4. *antisymmetric* if, in addition to the pairs of complementary forms, it contains only pairs of antipodal forms, and
5. *supersymmetric* if it contains all pairs of complementary forms, conjugates, and adjoints (and hence antipodal) forms.

3 Results

In this section, we state our results. The proofs are provided in the following sections, mainly using the results of Part I [1].

3.1 Basic Theorems

Let $\mathbf{f} := (m, n, k)$ be a triple of integers such that $k^2 - 4mn > 0$. The ordered pair $(\xi^+(\mathbf{f}), \xi^-(\mathbf{f}))$ denotes the roots of the quadratic equation $m\xi^2 + k\xi + n = 0$:

$$\xi^\pm(\mathbf{f}) = \frac{-k \pm \sqrt{k^2 - 4mn}}{2m}, \tag{2}$$

the first with the plus sign and the second with the minus sign.

We assume that $\xi^\pm(\mathbf{f})$ are irrational.

Notation. We denote $\Gamma(m, n, k)$ the period of the continued fraction of $\xi^+(m, n, k)$ considered up to cyclic permutations.

Remark. In the sequel the word *period* will be used for the period of a continued fraction considered up to cyclic permutations of its elements.

Theorem 3.1. *The continued fractions of the roots $\xi^+(\mathbf{f})$ and $\xi^-(\mathbf{f})$ are periodic, and their periods are mutually inverse.*[3] *The ordered pair of periods of the continued fractions of $(\xi^+(\mathbf{f}), \xi^-(\mathbf{f}))$ is an invariant of the class $C(m, n, k)$.*

By Theorem 3.1, the periods of ξ^+ and ξ^- have the same palindromic type.

Definition. A form $\mathbf{f}$ is said to be *primitive* if cannot be written as $a\mathbf{f}'$ for another integer form $\mathbf{f}'$ and $a > 0$.

Remark. All forms in the same class are either primitive or nonprimitive.

A class consisting of primitive forms is said to be *primitive*.

Theorem 3.2. *For every period s there are two and only two primitive classes $C(m, n, k)$ such that $\Gamma(m, n, k) = s$. These classes, which may coincide, are sent one to the other by the antipodal involution.*

3.2 The palindromic type of the period from the symmetry type of the class

In this section we state the correspondence between the five symmetry types (in Definition 8) of the classes of forms and the five palindromic types of their corresponding periods $\Gamma(m, n, k)$.

Theorem 3.3. *Let $\Gamma(m, n, k)$ be the period of the class $C(m, n, k)$. Then*

a. $\Gamma(m, n, k)$ *is palindromic and even iff $C(m, n, k)$ is $(m+n)$-symmetric,*

b. $\Gamma(m, n, k)$ *is palindromic and odd iff $C(m, n, k)$ is supersymmetric,*

c. $\Gamma(m, n, k)$ *is bipalindromic iff $C(m, n, k)$ is k-symmetric,*

d. $\Gamma(m, n, k)$ *is nonpalindromic and odd iff $C(m, n, k)$ is antisymmetric, and*

e. $\Gamma(m, n, k)$ *is nonpalindromic and even iff $C(m, n, k)$ is asymmetric.*

In the following examples, we give all the forms for each case of the class $C(m, n, k)$ satisfying $m > 0$ and $n < 0$; the reader may verify the symmetry type of the class.

[3] This fact was probably already known to Lagrange, Galois, etc. A geometrical proof of the first part is in [4].

Example a. $m = 5$, $n = -7$, $k = 9$. $\Gamma = [1, 2, 2, 1]$. In the same class, we have the following forms:

m	n	k	period
5	-7	9	$[1, 1, 2, 2]$
7	-7	-5	$[1, 2, 2, 1]$
7	-5	9	$[2, 2, 1, 1]$
5	-5	-11	$[2, 1, 1, 2]$

We note that there is one pair of adjoints and two self-adjoint forms.

Example b. $m = 1$, $n = -2$, $k = -3$. $\Gamma = [1, 3, 1]$. In the same class, we have the following forms:

m	n	k	period
1	-2	-3	$[3, 1, 1]$
1	-2	3	$[1, 1, 3]$
2	-2	-1	$[1, 3, 1]$
2	-1	3	$[3, 1, 1]$
2	-1	-3	$[1, 1, 3]$
2	-2	1	$[1, 3, 1]$

We note that there are two self-adjoint forms, which are conjugates.

Example c. $m = 1$, $n = -2$, $k = -5$. $\Gamma = [5, 2, 1, 2]$. In the same class, we have the following forms:

m	n	k	period
1	-2	-5	$[5, 2, 1, 2]$
1	-2	5	$[2, 1, 2, 5]$
3	-2	-3	$[1, 2, 5, 2]$
3	-2	3	$[2, 5, 2, 1]$

We note that there are two pairs of conjugate forms.

Example d. $m = 5$, $n = -3$, $k = -13$. $\Gamma = [2, 1, 4]$. In the same class, we have the following forms:

m	n	k	period
5	-3	-13	$[2, 1, 4]$
5	-9	7	$[1, 4, 2]$
3	-9	-11	$[4, 2, 1]$
3	-5	13	$[2, 1, 4]$
9	-5	-7	$[1, 4, 2]$
9	-3	11	$[4, 2, 1]$

We note that the orbit contains three pairs of antipodal forms.

Example e. $m = 5,\ n = -15,\ k = 18.\ \Gamma = [1, 2, 3, 4]$. In the same class, we have the following:

m	n	k	period
5	-15	18	$[1, 2, 3, 4]$
8	-15	-12	$[2, 3, 4, 1]$
8	-7	20	$[3, 4, 1, 2]$
5	-7	-22	$[4, 1, 2, 3]$

We note that there are no pairs of symmetric forms.

Corollary 3.4. *Given a period s of length P, the two primitive classes $C(m, n, k)$ such that $\Gamma(m, n, k) = s$ coincide iff P is odd.*

Remark. The square root of a rational number $\sqrt{p/q}$ has a continued fraction whose period is either odd and palindromic or bipalindromic because it is the root of the equation $qx^2 - p = 0$, corresponding to a class of forms (m, n, k) that is either k-symmetric or supersymmetric, since contains a form with $k = 0$. This answers a question posed by Arnold in [3].

In [2], Arnold posed the question whether the roots of all quadratic equations of type $x^2 + kx + n = 0$ are palindromic. The following corollary answers this question.

Corollary 3.5. *The continued fractions of the quadratic surds corresponding to a form whose class represents* 1 *have a period that is either odd and palindromic or even and bipalindromic.*

3.3 The numbers of forms with $mn < 0$ from the periods of the surds

The theorems below, referring to some special domains of the space of forms, complete the results of Part I.

We show that the period $\Gamma(m, n, k)$ is related to the set, called *cycle* (or to half of it), composed by the forms of the class $C(m, n, k)$, satisfying $m > 0$ and $n < 0$ and hence belonging to H^0, defined in Sec. 4 of Part I.

Besides under the considered involutions, a cycle could be *a priori* invariant under a n-cyclic symmetry, i.e., it could satisfy the following: for every point $\mathbf{f}$ of the cycle there exists an operator $M \in \mathcal{T}$ such that the n points $M\mathbf{f}, M^2\mathbf{f}, \ldots, M^n\mathbf{f} = \mathbf{f}$ belong to the cycle and are distinct.

Corollary 3.6. *The cycle of all forms lying in H^0 cannot have the n-cyclic symmetry.*

Let $\Gamma(m,n,k) = [a_1, \ldots, a_P]$.

Definition. If P is odd, then the *geometric period* of the continued fraction with period Γ is

$$\Pi(m,n,k) := \Gamma^2 = [a_1, a_2, \ldots, a_P, a_{P+1}, \ldots, a_p],$$

where $p = 2P$ and $a_{P+i} = a_i$ for $i = 1, \ldots, P$; otherwise, the geometric period coincides with

$$\Gamma : \Pi(m,n,k) := \Gamma(m,n,k) \quad \text{and} \quad p = P.$$

Theorem 3.7. *Let (m,n,k) be any triple of integers such that $k^2 - 4mn > 0$ is a nonsquare number, and let $\Pi(m,n,k) = [a_1, a_2, \ldots, a_p]$. We define*

$$t_{\text{odd}} := \sum_{i \text{ odd}}^{p} a_i, \qquad t_{\text{even}} := \sum_{i \text{ even}}^{p} a_i, \qquad t := \sum_{i}^{p} a_i. \tag{3}$$

The class $C(m,n,k)$ has t points in H^0 and in H^0_R, has t_{odd} points in every domain of G_A and $G_{\bar{A}}$, and has t_{even} points in every domain of G_B and $G_{\bar{B}}$ (or vice versa*). Moreover, $t_{\text{odd}} = t_{\text{even}} = t/2$ if Γ is either odd or even palindromic, i.e., if the corresponding form is supersymmetric, antisymmetric, or $(m+n)$-symmetric.*

In Sec. 4 of Part I, we showed that each class whose discriminant is a square, has representatives on the boundaries of the domains of the tiling. In particular, Theorem 4.13 states that there are k distinct classes whose discriminant is equal to k^2. These k classes have a fixed number of representatives in the interior of each domain. The following theorems provide, for every class, the number of its forms in H^0 and its symmetry type from the finite continued fraction of a rational number related to a representative of that class.

Remark. The last element of a finite continued fraction is greater than 1.

Definition. The *odd continued fraction* of a rational number $r > 1$ is the finite continued fraction $[a_1, \ldots, a_N]$ of r if N is odd and the continued faction $[a_1, a_2, \ldots, a_N - 1, 1]$ otherwise. Similarly, the *even* continued fraction of a rational number $r > 1$ is the finite continued fraction $[a_1, \ldots, a_N]$ of r if N is even and the continued faction $[a_1, a_2, \ldots, a_N - 1, 1]$ otherwise.

Note that the odd and the even continued fraction of $r = [a_1, \ldots, a_N]$ both represent r:

$$\begin{aligned}[a_1, \ldots, (a_N - 1), 1] &= a_1 + \cfrac{1}{\cdots + \frac{1}{(a_N-1)+\frac{1}{1}}} \\ &= a_1 + \cfrac{1}{\cdots + \frac{1}{a_N}} = [a_1, \ldots, a_N].\end{aligned}$$

Theorem 3.8. *Let $k > m > 0$ and $[a_1, \ldots, a_N]$ be the even continued fraction of the rational number k/m. We define*

$$\hat{t}_{\text{odd}} := \sum_{i \text{ odd}}^{N-1} a_i - 1, \qquad \hat{t}_{\text{even}} := \sum_{i \text{ even}}^{N} a_i - 1, \qquad \hat{t} := \sum_{i}^{N} a_i - 1. \tag{4}$$

The following statements hold:

i. *The class $C(m, 0, k)$ has $\hat{t}$ points in H^0 and H^0_R, has $\hat{t}_{\text{odd}}$ points in the interior of every domain of G_A and $G_{\bar{A}}$, and has $\hat{t}_{\text{even}}$ points in the interior of every domain of G_B and $G_{\bar{B}}$.*

ii. *Moreover, $\hat{t}_{\text{odd}} = \hat{t}_{\text{even}} = (\hat{t} - 1)/2$ if $C(m, 0, k)$ is $(m+n)$-symmetric.*

Theorem 3.9. *If $0 \leq m < |k|$, the class $C(m, 0, k)$ is not antisymmetric and is*

i. *supersymmetric iff either $m = 0$ or k is even and $m = k/2$;*

ii. *$(m+n)$-symmetric iff the even continued fraction of k/m is palindromic;*

iii. *k-symmetric iff the odd continued fraction of k/m is palindromic;*

iv. *asymmetric iff both the odd and the even continued fraction of k/m are not palindromic.*

The appendix contains examples illustrating these theorems.

Section 5 is devoted to the reduction theory for indefinite forms from the geometric standpoint of our model.

The (finite or infinite) sequence of integers $(b_0, b_1, b_2, \ldots)$, $b_i \geq 2$, denotes the *minus continued fraction* of the number ξ:

$$\xi = b_0 - \cfrac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\cdots}}}.$$

We prove the following theorem on periodic minus continued fractions:

Theorem 3.10. *Let $(c_1, c_2, \ldots, c_L)$ be the period of the minus continued fraction of a quadratic surd. If the corresponding class is supersymmetric, antisymmetric, or $(m+n)$-symmetric, then*

$$\sum_{i=1}^{L} c_i = 3L$$

4 Proofs

4.1 Fundamental lemmas

In Part I, for every $\Delta > 0$ such that $\Delta \equiv 0$ or $\Delta \equiv 1 \bmod 4$, we introduced the set

$$H_\Delta = \{(m, n, k) \in \mathbb{Z}^3 : k^2 - 4mn = \Delta\}.$$

This is the space of quadratic forms

$$f = mx^2 + ny^2 + kxy$$

with real coefficients and fixed discriminant Δ (see Fig. 1-I). Moreover, we defined the projection $\mathcal{Q}$ of the hyperboloid H_Δ to the open cylinder C_H (Fig. 1-III).

Remark. Let $\xi^+(\mathbf{f})$ and $\xi^-(\mathbf{f})$ be the roots of the equation $f = 0$ for the variable $\xi = x/y \in \mathbb{R}P^1$. Then the cylinder C_H and the space

$$\Xi = \left\{(\xi^+(\mathbf{f}), \xi^-(\mathbf{f})) \in \mathbb{R}P^1 \times \mathbb{R}P^1 \setminus \{(\xi, \xi) | \xi \in \mathbb{R}P^1\},\ \mathbf{f} \in H_\Delta\right\} \tag{5}$$

are homeomorphic. Indeed, the map $\mathcal{Q} : H_\Delta \to C_H$ (see eq. (17) of Part I) is a homeomorphism. Also, eq. (2) defines a homeomorphism between H_Δ and Ξ.

We show explicitly how the domains characterizing the tiling of C_H are mapped by this homeomorphism to Ξ.

In Fig. 1-II, the cylinder C_H is depicted with the curved segments bounding some of its domains replaced with straight line segments. Note that the cylinder (5) is obtained from the torus $\mathbb{R}P^1 \times \mathbb{R}P^1$ minus its diagonal (Fig. 1-IV). The circles c_1 and c_2, the boundary of C_H, represent the points of the cone $\Delta = 0$, to which the points at infinity of the hyperboloid approach. For such limit values of the coefficients, the roots of the corresponding quadratic equations tend to a same value. Hence, the two circles correspond to the diagonal $\xi^+ = \xi^-$. The root ξ^+ vanishes when $n = 0$ and $k > 0$, and ξ^- vanishes when $n = 0$ and $k < 0$. The roots ξ^+ and ξ^- attain $\pm\infty$ when $m = 0$, and they also change sign when m changes sign. Note that the lines $m = 0$ and $n = 0$

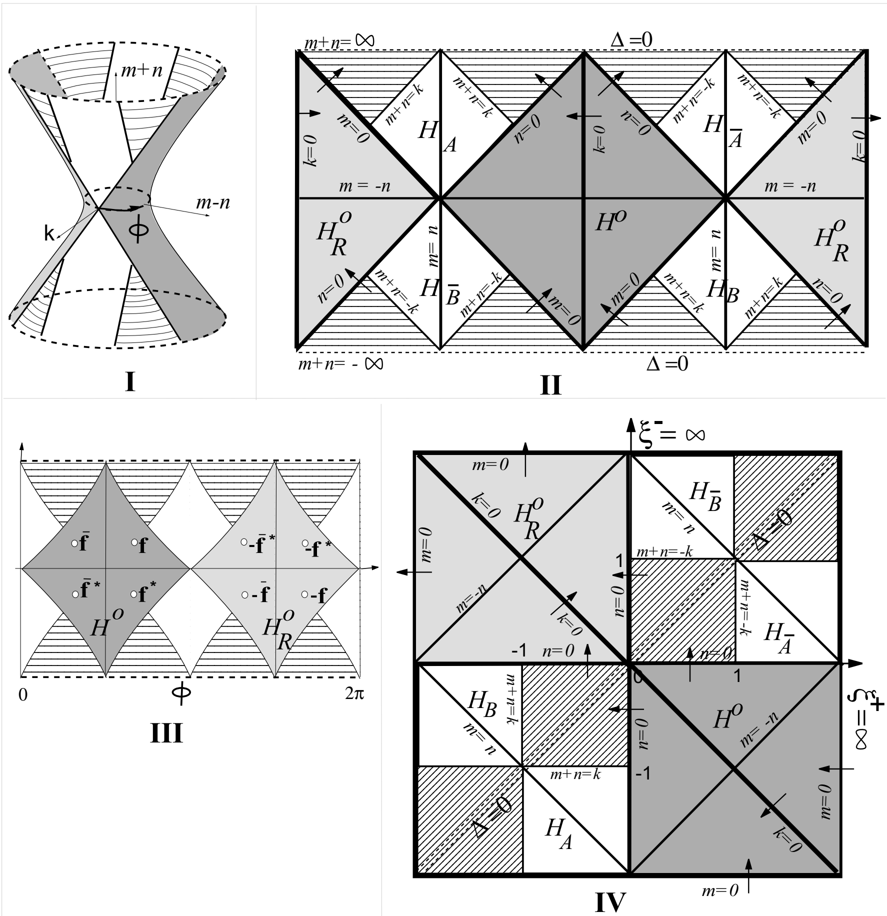

Figure 1: The arrows coorienting a line where a coefficient vanishes point towards the region where the value of that parameter is positive.

are the boundaries of the domains H^0 and H^0_R. The rhomboidal regions H^0 and H^0_R in C_H (Fig. 1-III and Figs. 8 and 10 in Part I) are represented by true rhombi in Fig. 1-II. These regions are thus represented in Ξ by the square regions $\xi^+ \cdot \xi^- < 0$, also denoted by H^0 and H^0_R in Fig. 1-IV. Outside H^0 and H^0_R, the coefficients m and n have the same sign, and there are four special domains: H_A and $H_{\bar{A}}$, where m and n are positive, $m + n < k$, $k > 0$ (H_A) and $m + n < -k$, $k < 0$ ($H_{\bar{A}}$); H_B and $H_{\bar{B}}$, where m and n are negative, $m + n > k$, $k < 0$ (H_B) and $m + n > -k$, $k < 0$ ($H_{\bar{B}}$). These domains are mapped to the respective domains

$$\begin{aligned} H_A &= \big\{(\xi^+,\xi^-) : -1 < \xi^+ < 0,\ \xi^- < -1\big\},\\ H_{\bar{A}} &= \big\{(\xi^+,\xi^-) : \xi^+ > 1,\ 0 < \xi^- < 1\big\},\\ H_B &= \big\{(\xi^+,\xi^-) : \xi^+ < -1,\ -1 < \xi^- < 0\big\},\\ H_{\bar{B}} &= \big\{(\xi^+,\xi^-) : 0 < \xi^+ < 1,\ \xi^- > 1\big\}. \end{aligned} \tag{6}$$

Figure 2 shows how to obtain the square representing Ξ from the rectangle representing C_H of Fig. 1-IV and II respectively: we cut the rectangle along the lines $m = 0$, thus obtaining two triangles: one containing H^0 (with the circle c_1 as base) and the other containing H^0_R (with the circle c_2 as base; see Fig. 2). We then place the triangle containing H^0 above the other as shown in the figure. Finally, we turn the figure thus obtained by $\pi/4$ counterclockwise. Observe that this procedure preserves the continuity of the map from C_H to Ξ.

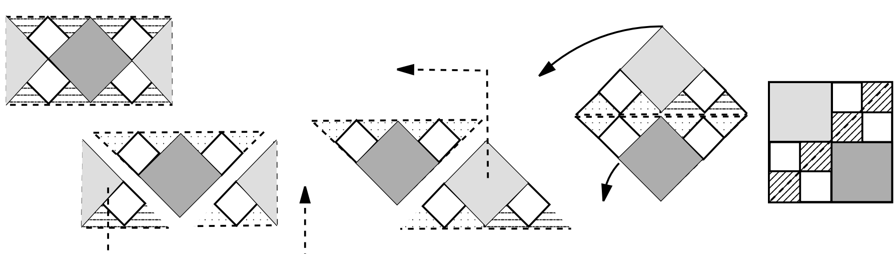

Figure 2: Correspondence between C_H and Ξ.

Remark. The complementary $\mathbf{f}_c$ of the form $\mathbf{f}$ in C_H is represented by a point with the same ordinate as $\mathbf{f}$ and shifted by π in the horizontal direction, while the conjugate, adjoint, and antipodal forms are symmetric with respect to $\mathbf{f}$ as shown in Fig. 1-III.

The following relations hold among the pairs (ξ^+, ξ^-) of the triples obtained from the triple $\mathbf{f} = (m, n, k)$ by all the considered involutions.

$\mathbf{f}$	(m,n,k)	ξ^+	ξ^-	$\mathbf{f}_c = -\mathbf{f}^*$	$(n,m,-k)$	$-1/\xi^+$	$-1/\xi^-$
$\overline{\mathbf{f}}$	$(m,n,-k)$	$-\xi^-$	$-\xi^+$	$\overline{\mathbf{f}}_c = -\overline{\mathbf{f}}^*$	(n,m,k)	$1/\xi^-$	$1/\xi^+$
$\mathbf{f}^*$	$(-n,-m,k)$	$-1/\xi^-$	$-1/\xi^+$	$\mathbf{f}^*_c = -\mathbf{f}$	$(-m,-n,-k)$	ξ^-	ξ^+
$\overline{\mathbf{f}}^*$	$(-n,-m,-k)$	$1/\xi^+$	$1/\xi^-$	$\overline{\mathbf{f}}^*_c = -\overline{\mathbf{f}}$	$(-m,-n,k)$	$-\xi^+$	$-\xi^-$

Table 1

Therefore, in Ξ, the complementary of the forms in H^0 are obtained by moving H^0 by a translation over H^0_R, and *vice versa*, and the forms outside H^0 and outside H^0_R are obtained by moving the upper-right quarter of Ξ over the lower-left, and *vice versa*. The conjugation becomes the reflection with respect to the diagonal $\xi^+ = -\xi^-$, whereas the antipodal symmetry, which is a reflection with respect to the center of H_0 and H^0_R, becomes the reflection with respect to the point $(1, -1)$ or $(-1, 1)$.

We recall that

$$\mathbf{A} = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \mathbf{B} = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), \quad \text{and} \quad \mathbf{R} = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$$

denote the generators of $SL(2, \mathbb{Z})$ acting on the xy plane and that A, B, and R denote the corresponding generators of $\mathcal{T}$.

Definition. The operators α, β, and σ acting on $\mathbb{R}P^1$ are defined in terms of the operators A, B, and R of $\mathcal{T}$ in this way

$$\alpha(\xi^\pm(\mathbf{f})) = \xi^\pm(A(\mathbf{f})), \quad \beta(\xi^\pm(\mathbf{f})) = \xi^\pm(B(\mathbf{f})), \quad \sigma(\xi^\pm(\mathbf{f})) = \xi^\pm(R(\mathbf{f})). \quad (7)$$

Lemma 4.1. *The actions of the operators α, β, and σ on the roots $\xi^\pm$ coincide with those of the inverse of the homographic operators* A, B, *and* R *defined by the generators* **A**, **B**, *and* **R** *of $SL(2, \mathbb{Z})$.*

Proof. The actions of the inverse homographic operators A^{-1} and B^{-1} (see eq. (13) of Part I) are

$$\mathrm{A}^{-1} : \xi \to \frac{\xi - 1}{1}, \qquad \mathrm{B}^{-1} : \xi \to \frac{1}{-1 + 1/\xi}, \qquad \mathrm{R}^{-1} = \mathrm{R} : \xi \to -\frac{1}{\xi}. \quad (8)$$

On the other hand, if $\xi^\pm$ are the two roots of $f(x, y)|_{y=1} = 0$, then $\alpha(\xi^\pm)$ are by definition the corresponding roots of $f(x + y, y)|_{y=1} = f(x + 1, 1) = 0$ and are hence equal to $\xi^\pm - 1$.

By definition, $\beta(\xi^+)$ is the first root of the equation $f(x, x + y)|_{y=1} = 0$. We note that $1/\xi^+ = (-k - \sqrt{\Delta})/2n$ is the second root w^- of the equation $f(1, y) = 0$. Hence $\beta(w^-)$ by the above definition is the second root of $f(x, y + x)|_{x=1} = f(1, y + 1) = 0$. We hence have $\beta(1/\xi^+) = 1/\xi^+ - 1$. The first root of the equation $f(x, x + y)|_{y=1} = 0$ is therefore equal to $1/\beta(1/\xi^+) = 1/(-1 + 1/\xi^+)$. The proof for $\beta(\xi^-)$ is analogous (exchanging ξ^+ with ξ^- and *first* with *second*).

Finally, we note that $-1/\xi^\pm = (k \pm \sqrt{\Delta})/2n$ are exactly the first and second roots of $f(-y, x)|_{y=1}$, i.e., $-1/\xi^\pm = \xi^\pm(R(\mathbf{f}))$, and are hence equal to $\sigma(\xi^\pm(\mathbf{f}))$ by definition. $\square$

Remark. The above lemma defines an isomorphism between the group $\mathcal{T}$ acting on the space of forms and the group generated by α, β, and σ acting on Ξ.

The actions of the operators α^{-1} and β^{-1} are

$$\alpha^{-1}(\xi) = \xi + 1, \qquad \beta^{-1}(\xi) = \frac{1}{1 + 1/\xi}.$$

The following lemma holds for all real numbers.

We write α^n and β^n for the respective nth iterations of α and β.

Lemma 4.2. *If $\xi > 1$ with continued fraction $\xi = [a, b, c, \dots]$, then*

$$\alpha^a(\xi) = [0, b, c, \dots].$$

If $0 < \xi < 1$ and $\xi = [0, d, e, g, \dots]$, then

$$\beta^d(\xi) = [e, g, \dots].$$

Proof. By Lemma 4.1,

$$\alpha(\xi) = \xi - 1 \quad \text{and} \quad \alpha^a(\xi) = \xi - a.$$

Hence, if

$$\xi = a + \frac{1}{b + \frac{1}{c + \frac{1}{\dots}}},$$

then

$$\alpha^a(\xi) = \frac{1}{b + \frac{1}{c + \frac{1}{\dots}}} = [0, b, c, \dots].$$

Moreover,

$$\beta(\xi) = \frac{1}{-1 + \frac{1}{\xi}} \quad \text{and} \quad \beta^d(\xi) = \frac{1}{-d + \frac{1}{\xi}}.$$

Hence, if

$$\xi = \frac{1}{d + \frac{1}{e + \frac{1}{g + \frac{1}{\dots}}}},$$

then

$$\beta^d(\xi) = \frac{1}{-d + \frac{1}{\frac{1}{d + \frac{1}{e + \frac{1}{g + \frac{1}{\dots}}}}}} = \frac{1}{-d + d + \frac{1}{e + \frac{1}{g + \frac{1}{\dots}}}} = e + \frac{1}{g + \frac{1}{\dots}} = [e, g, \dots].$$

$\square$

Remark. If the continued fraction of $x > 0$ is $[a_1, a_2, a_3 \dots]$, then it is convenient for our aim to let the continued fraction of $-x < 0$ be denoted simply by $-[a_1, a_2, a_3 \dots]$. We have:

$$-\big[a_1, a_2, a_3 \dots\big] = \big[-a_1, -a_2, -a_3 \dots\big].$$

Indeed, if

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\dots + \frac{1}{a_n}}}}}.$$

then

$$-x = -a_1 - \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\dots + \cfrac{1}{a_n + \frac{1}{\dots}}}}}} = (-a_1) + \cfrac{1}{(-a_2) + \cfrac{1}{(-a_3) + \cfrac{1}{(-a_4) + \cfrac{1}{\dots + \cfrac{1}{(-a_n) + \frac{1}{\dots}}}}}}.$$

Lemma 4.3. *If $\xi < -1$ and its continued fraction is $\xi = -[a, b, c, \dots]$, then*

$$\alpha^{-a}(\xi) = -[0, b, c, \dots].$$

If $-1 < \xi < 0$ with continued fraction $\xi = -[0, d, e, g, \dots]$, then

$$\beta^{-d}(\xi) = -[e, g, \dots].$$

Proof. By Lemma 4.1 and the remark following it,

$$\alpha^{-a}(\xi) = \xi + a.$$

Hence, if $\xi = -[a, b, c, \dots]$, i.e.,

$$\xi = -a - \cfrac{1}{b + \cfrac{1}{c + \frac{1}{\dots}}},$$

then

$$\xi + a = -\cfrac{1}{b + \cfrac{1}{c + \frac{1}{\dots}}}.$$

Moreover,

$$\beta^{-b}(\xi) = \frac{1}{d + 1/\xi}.$$

Hence, if $\xi = -[0, d, e, g, \dots]$, i.e.,

$$\xi = -\frac{1}{d + \frac{1}{e + \frac{1}{g + \frac{1}{\dots}}}},$$

then

$$\frac{1}{d + 1/\xi} = -\frac{1}{e + \frac{1}{g + \frac{1}{\dots}}}.$$

$\square$

We group some observations in the following lemma.

Notation. A black or a white arrow from the point $\mathbf{f}$ to the point $\mathbf{g}$ in C_H indicates respectively that $A\mathbf{f} = \mathbf{g}$ or $B\mathbf{f} = \mathbf{g}$.

Remark. The arrows provide any sequence of points belonging to a cycle or to a chain in H^0 with an orientation.

Lemma 4.4. *If the points* $\mathbf{f}$ *and* $\mathbf{g}$ *in* C_H *are joined by an arrow from* $\mathbf{f}$ *to* $\mathbf{g}$*, then*

1. *the points* $\mathbf{f}^*$ *and* $\mathbf{g}^*$ *in* C_H *symmetric with respect to the horizontal line of points* $\mathbf{f}$ *and* $\mathbf{g}$ *are joined by an arrow from* $\mathbf{g}^*$ *to* $\mathbf{f}^*$ *of the opposite color* (see Fig. 3-I,II)*;*
2. *the points* $\overline{\mathbf{f}}$ *and* $\overline{\mathbf{g}}$ *in* C_H *symmetric with respect to the vertical line* $k = 0$ *of points* $\mathbf{f}$ *and* $\mathbf{g}$ *are related by an arrow from* $\overline{\mathbf{g}}$ *to* $\overline{\mathbf{f}}$ *of the same color* (see Fig. 3-II,III)*;*
3. *the points* $\overline{\mathbf{f}}^*$ *and* $\overline{\mathbf{g}}^*$ *in* C_H *symmetric with respect to the center of* H^0 *of points* $\mathbf{f}$ *and* $\mathbf{g}$ *are related by an arrow from* $\overline{\mathbf{f}}^*$ *to* $\overline{\mathbf{g}}^*$ *of the opposite color* (see Fig. 3-II-IV).

Proof. The proofs of the corresponding identities

$$\begin{aligned}
&1.\quad (A\mathbf{f})^* = B^{-1}\mathbf{f}^*, \quad && (B\mathbf{f})^* = A^{-1}\mathbf{f}^*;\\
&2.\quad \overline{A\mathbf{f}} = A^{-1}\,\overline{\mathbf{f}}, && \overline{B\mathbf{f}} = B^{-1}\,\overline{\mathbf{f}};\\
&3.\quad (\overline{A\mathbf{f}})^* = B\overline{\mathbf{f}}^*, && (\overline{B\mathbf{f}})^* = A\overline{\mathbf{f}}^*.
\end{aligned} \tag{9}$$

are given in Lemma 1.3 of Part I.

Remark. The cycle of the forms in H^0 is oriented by the direction of the arrows. Every point of the cycle has one and only one successor, which is the only one of its images by A and B that lie inside H^0, and one and only one predecessor, which is the only one of its images by A^{-1} and B^{-1} that lie inside H^0 (see Lemma 4.6 of Part I).

4.2 Classes non representing zero

Because of the one-to-one map between C_H and Ξ, we let the same symbols denote the regions in C_H and in Ξ, as in Fig. 1.

Definition. The form $\mathbf{f} = (m, n, k)$ is a *turning point* iff $mn < 0$ and $|m + n| < |k|$.

The adjective *turning* of a point $\mathbf{f}$ in H^0 (or in H^0_R) means that if $\mathbf{g}$ and $\mathbf{h}$ are respectively the predecessor and the successor of $\mathbf{f}$ in the cycle containing $\mathbf{f}$, then the operators T_1 and T_2 satisfying $\mathbf{f} = T_1\mathbf{g}$ and $\mathbf{h} = T_2\mathbf{f}$ are different, that is, at $\mathbf{f}$, the incoming arrow and the outcoming arrow have different colors (see Lemma 4.6 of Part I).

Remark. If $\mathbf{f}$ is a turning point in H^0, then the roots $\xi^{\pm}(\mathbf{f})$, besides the relation $\xi^{+}(\mathbf{f}) > 0$, $\xi^{-}(\mathbf{f}) < 0$, holding in H^0, satisfy: either $\xi^{+}(\mathbf{f}) > 1$ and $-1 < \xi^{-}(f) < 0$ or $0 < \xi^{+}(\mathbf{f}) < 1$ and $\xi^{-}(f) < -1$.

Definition. The continued fraction of ξ is said *immediately periodic* if it has no nonzero elements before the period, i.e., either $\xi = [[a_1, a_2, a_3, \dots, a_p]]$, or $\xi = [0, [a_1, a_2, a_3, \dots, a_p]]$.

Proof of Theorem 3.1

Lemma 4.5. *The continued fractions of $\xi^{\pm}(\mathbf{h})$ are immediately periodic if $\mathbf{h}$ is a turning point.*

Proof. By definition, the turning point $\mathbf{h}$ belongs to a cycle $\gamma_{\mathbf{h}}(T_1, \dots T_t)$ in H^0, where either $T_i = A$ or $T_i = B$. Let $T_1 = A$. We can write $T\mathbf{h} = \mathbf{h}$, where $T = B^{a_p} A^{a_{p-1}} \cdots B^{a_2} A^{a_1}$ is a product of p alternating powers of A and B, such that the exponents a_i satisfy $\sum_{i=1}^{p} a_i = t$.

For the cycle of $\mathbf{h}$, we write

$$\gamma_{\mathbf{h}}\left(A^{a_1}, B^{a_2}, \dots, A^{a_{p-1}}, B^{a_p}\right),$$

and $\mathbf{h}_{a_1} := A^{a_1}\mathbf{h}$, $\mathbf{h}_{a_1+a_2} = B^{a_2}A^{a_1}\mathbf{h}$, and so on, until $\mathbf{h}_t = T\mathbf{h} = \mathbf{h}$. Let (ξ^+, ξ^-) be the pair of roots associated with $\mathbf{h}$. Using Lemma 4.1, we construct the operator τ, obtained from T by translating A into α and B into β, which satisfies

$$\tau(\xi^+) = \xi^+, \qquad \tau(\xi^-) = \xi^-. \tag{10}$$

Since $\mathbf{h} \in H^0$, $\xi^+(\mathbf{h})$ is positive and $\xi^-(\mathbf{h})$ is negative. We write $\xi^+ = [b_1, b_2, \dots]$. The image of $\mathbf{h}$ by A^{a_1+1} is outside H^0, by Lemma 4.6 of Part I. We thus obtain $\alpha^{a_1}\xi^+ = [b_1 - a_1, b_2, \dots] = [0, b_2, \dots]$ and hence $b_1 = a_1$. Analogously, the image of $A^{a_1}\mathbf{h}$ by B^{a_2+1} is outside H^0, and $\beta^{a_2} \circ \alpha^{a_1}(\xi^+) = [b_3, b_4, \dots]$, i.e., $b_2 = a_2$. In the same way we obtain $b_i = a_i$ for $i = 1, \dots, p$. But by eq. (10), at the end of the cycle we get

$$\tau\big(\xi^+\big) = \big[b_{p+1}, b_{p+2}, \dots\big] = \big[a_1, a_2, \dots, a_p, b_{p+1}, b_{p+2}, \dots\big].$$

Similarly, applying the jth iteration τ^j of τ to ξ^+, for every natural number j,

$$\tau^j\big(\xi^+\big) = \big[b_{jp+1}, b_{jp+2}, \dots\big] = \big[a_1, a_2, \dots, a_p, a_1, a_2, \dots\big],$$

we find that $[a_1, a_2, \dots, a_{p-1}, a_p]$ are the first p elements of the continued fraction obtained from ξ^+ canceling the first jp elements and hence concluding that

$$\xi^+ = \big[[a_1, a_2, a_3, \dots, a_p]\big],$$

i.e., the continued fraction of ξ^+ is immediately periodic. Note that p is even because T begins with a power of B and ends with a power of A.

For ξ^-, it is convenient to write the second equation in (10) as

$$\xi^- = \tau^{-1}\big(\xi^-\big),$$

where $\tau^{-1} = \alpha^{-a_1} \circ \beta^{-a_2} \cdots \alpha^{-a_{p-1}} \circ \beta^{-a_p}$. Again, by Lemma 4.6 of Part I, the image of $\mathbf{h}$ by A^{-1} is outside H^0, i.e, $\alpha^{-1}(\xi^-) = \xi^- + 1 > 0$, and hence $0 > \xi^- > -1$, i.e., $\xi^- = -[0, c_1, c_2, \dots]$. Similarly, again using Lemma 4.3, we find that $\beta^{-a_p}(\xi^-) = -[c_2, c_3, \dots]$ and hence $c_1 = a_p$. Now, applying $A^{-a_{p-1}}$ to $B^{-a_p}\mathbf{h}$, we obtain $\alpha^{-a_{p-1}} \circ \beta^{-a_p}(\xi^-) = -[c_3, c_4, \dots]$, i.e., $c_2 = a_{p-1}$. In the same way we obtain $c_i = a_{p+1-i}$ for $i = 1, \dots, p$. At the end of the cycle, by eq. (10), we get

$$\tau^{-1}\big(\xi^-\big) = -\big[c_{p+1}, c_{p+2}, \dots\big] = -\big[a_p, a_{p-1}, \dots, a_2, a_1, c_{p+1}, c_{p+2}, \dots\big]$$

and hence also find that $a_p, a_{p-1}, \dots, a_2, a_1$ are the first p elements of the continued fraction of any iteration of τ^{-1} on ξ^-, i.e.:

$$\tau^{-j}\big(\xi^-\big) = -\big[c_{jp+1}, c_{jp+2}, \dots\big] = -\big[a_p, a_{p-1}, \dots, a_1, a_p, a_{p-1}, \dots\big].$$

We hence conclude that

$$\xi^- = -\big[0, [a_p, a_{p-1}, \dots, a_2, a_1]\big].$$

$\square$

We have proved that the roots $\xi^\pm(\mathbf{h})$ are periodic and their periods are mutually inverse when $\mathbf{h} \in H^0$ is a turning point, i.e., the operator T of $\mathcal{T}^+$ satisfying $T\mathbf{h} = \mathbf{h}$ starts with A and ends with B (or *vice versa*).

In the cycle of $\mathbf{h}$ between $\mathbf{h}$ and $\mathbf{h}_{a_1}$ (whenever $a_1 > 1$), a nonturning point in H^0 is evidently obtained as

$$\mathbf{h}_j = A^j \mathbf{h}$$

for any $j < a_1$ and satisfies

$$\xi^+\big(\mathbf{h}_j\big) = \big[a_1 - j, a_2, \dots\big] = \big[a_1 - j, [a_2, a_3, \dots, a_p, a_1]\big],$$

i.e., it has the same period as $\xi^+(\mathbf{h})$, being the period defined up to cyclic permutations. Moreover,

$$\xi^-\big(\mathbf{h}_j\big) = -\big[j, [a_p, a_{p-1}, \dots, a_1]\big].$$

Applying this reasoning to every point of the cycle between two turning points, we obtain that the periods of the continued fractions of $\xi^\pm(\mathbf{h})$ are mutually inverse for every $\mathbf{h} \in H^0$.

To complete the proof of Theorem 3.1, we must consider the points outside H^0. Every point belongs to an orbit, and every orbit has a representative inside H^0, by the results of Part I. Moreover, every point of any orbit can be written as $\mathbf{p} = T\mathbf{h}$ with $\mathbf{h} \in H^0$ and $T \in \mathcal{T}$. Every element T of the group can be written as a finite product of the generators A, B, and R. By Lemma 4.1, we translate T into τ composed of the corresponding generators α, β, and σ. The action of each of these generators on a continued fraction obviously affects only its initial elements, and τ hence affects only a finite initial part of $\xi^\pm(\mathbf{h})$. Therefore, the periods of $\xi^\pm(\tau(\mathbf{h}))$ remain unchanged, since they are defined up to cyclic permutations.

For instance, the periods of the continued fractions of $\xi^\pm(\mathbf{h}_c)$ are the same as those of $\xi^\pm(\mathbf{h})$ because of the relations shown in Table 1 and because $\mathbf{h}_c = R\mathbf{h}$. We have completed the proof of Theorem 3.1. $\square$

Remark. For every class $C(m, n, k)$ of indefinite forms with a discriminant different from a square number, the above proof shows that the number of points of the cycle in H^0 (Theorem 4.11 of Part I) is deducible from the periods of the continued fractions of $\xi^\pm(m, n, k)$. As we remarked, the number p

of sequences of arrows of the same colors that alternate in the cycle is necessarily even. But the length P of the period of the continued fraction can be odd. In this case, the cycle indeed corresponds to the double of the period, and $p = 2P$, as we will see in detail in Theorems 3.3.b and 3.3.d.

Proof of Theorem 3.2. By Lemma 4.5, the roots $\xi^+(\mathbf{f})$ and $\xi^-(\mathbf{f})$ are immediately periodic if $\mathbf{f}$ is a turning point of H^0, and every class of forms with a discriminant different from a square number has a representative in H^0 that is a turning point.

Note that ξ^+ is positive and ξ^- is negative in H^0, and *vice versa* in H^0_R. We also recall that inverting the order of the pair of roots (and hence of periods) corresponds to inverting the signs of the coefficients of the equation (i.e., of $\mathbf{f}$).

Given a sequence $s = (a_1, \dots, a_P)$, we firstly suppose that $[[a_1, \dots, a_P]]$ is the first root of a quadratic equation with integer coefficients. To determine such an equation, we write

$$\xi = a_1 + \cfrac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_P + 1/\xi}}}.$$

We denote its coefficients (m, n, k) that define the form $\mathbf{f} := (m, n, k)$.

If we suppose that the first root is less than 1, i.e., $\xi = [0, [a_1, a_2, \dots, a_P]]$, we obtain the equation,

$$\xi = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_P + \xi}}}},$$

whose coefficients correspond to $\overline{\mathbf{f}}^*$ (see Table I).

Supposing that the first root is negative and greater than -1, i.e., it is equal to $-[0, [a_1, a_2, \dots, a_P]]$, we obtain an equation, whose coefficients correspond to $-\mathbf{f}^*$.

Finally, supposing that the first root is negative and less than -1, i.e., it is equal to $-[[a_1, a_2, \dots, a_P]]$, we obtain another equation, with coefficients corresponding to $-\overline{\mathbf{f}}$.

Since $-\mathbf{f}^* = \mathbf{f}_c$ and $-\overline{\mathbf{f}} = \overline{\mathbf{f}}^*_c$, the four triples of coefficients that we have obtained belong to only two classes, related by the antipodal symmetry. $\square$

Here we prove the theorem on the symmetries of the periods. We illustrate Theorem 3.3 in Fig. 3, where the coordinate of the horizontal axis is $-k$ and the coordinate of the vertical axis is $(m + n)$. The cycles in H^0 indeed correspond to the periods related to the forms considered in the examples. In these figures, the small circles indicate the forms. Black circles correspond to turning points.

The black arrow indicates the operator A and the white arrow indicates the operator B.

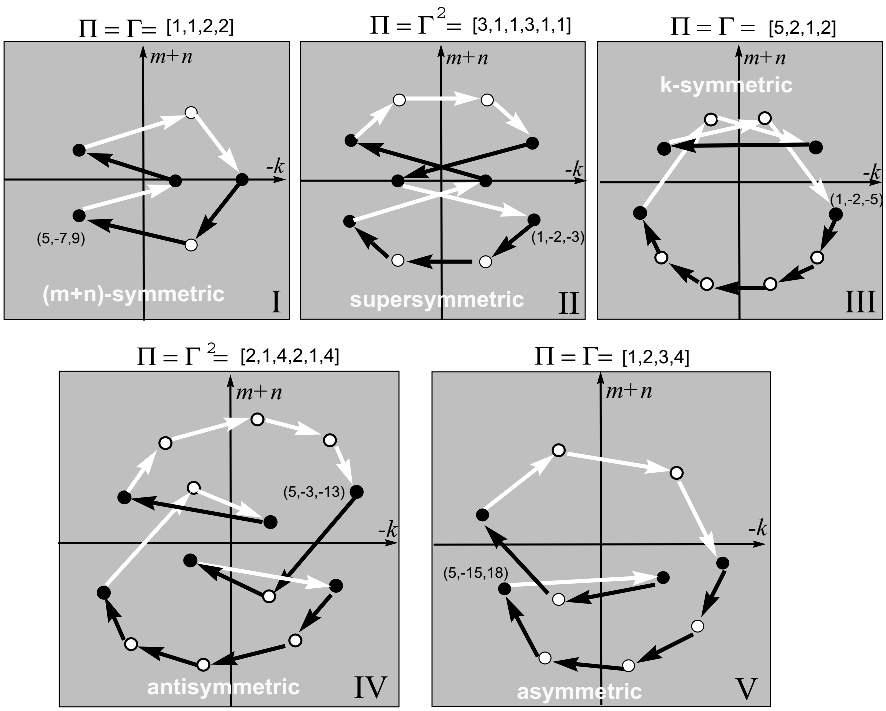

Figure 3: The elements of the period Π are equal to the numbers of arrows between two consecutive turning points (black circles).

Lemma 4.6. *If a cycle contains two points related by some symmetry, then the cycle has that symmetry.*

Proof. By Lemma 4.6 of Part I, a point of a cycle uniquely determines both its following and its preceding point, and hence all points of the cycle. Because of the relations stated in Lemma 4.4 between the arrows entering and exiting from two symmetric points, the symmetry of a pair of points determines the symmetry of the pairs of their neighboring points in the cycle and hence the symmetry of the entire cycle. □

Note that the symmetry of a cycle in H^0 as invariance under some reflection concern *only its points*: the directions of the arrows and their colors are necessarily related by the rules given by Lemma 4.4.

Lemma 4.7. *There are no arrows connecting points with the same value of k in a cycle containing more than two points.*

Proof. Points with the same value of k lie in a vertical line. Suppose that a vertical white arrow connects two symmetric points $\mathbf{f}$ and $\mathbf{f}^*$. By Lemma 4.6, a black arrow must connect $\mathbf{f}^*$ to $(\mathbf{f}^*)^* = \mathbf{f}$. Hence, either these two points form a cycle, or they cannot be joined by an arrow. $\square$

Lemma 4.8. *An $(m+n)$-symmetric cycle contains exactly two self-adjoint points. The self-adjoint points are turning points.*

Proof. Let $\mathbf{f}$ and $\mathbf{f}^*$ be two different points of the cycle (they lie in the same vertical line). Let $\mathbf{g}$ follow $\mathbf{f}$ (i.e., either $A\mathbf{f} = \mathbf{g}$ or $B\mathbf{f} = \mathbf{g}$). The symmetric $\mathbf{g}^*$ belongs to the cycle and is related to $\mathbf{f}^*$ by $B^{-1}\mathbf{f}^*$ or $A^{-1}\mathbf{f}^*$. Following the arrows after $\mathbf{g}$ and their symmetric arrows after $\mathbf{g}^*$, we must close that part of the cycle from $\mathbf{f}$ to $\mathbf{f}^*$. But because there are no arrows between adjoint points by Lemma 4.7, for some pair of adjoint points $\mathbf{h}$ and $\mathbf{h}^*$, we must have $A\mathbf{h} = \mathbf{j}$ and $B^{-1}\mathbf{h}^* = \mathbf{j}$, where $\mathbf{j}$ is self-adjoint (belonging to the line $(m+n) = 0$). The point $\mathbf{j}$ is therefore a turning point. In the part of the cycle from $\mathbf{f}^*$ to $\mathbf{f}$ there is a self-adjoint point by the same argument. We must prove that there are not other self-adjoint points. By the above argument, starting with any pair of adjoint points and reaching the successive pairs following the arrows according to their directions, when we reach the second self-adjoint point, we close the cycle. Observe that a self-adjoint point lies necessarily in H^0. Since a cycle cannot visit twice any of its points, and all points in H^0 of the orbit belong to the cycle, no other self-adjoint points are possible. $\square$

Lemma 4.9. *If an orbit contains a self-adjoint point, then it is symmetric with respect to the plane $(m+n) = 0$.*

Proof. Let $\mathbf{h} = \mathbf{h}^*$ be a self-adjoint point. It follows from the arguments in the proof of the preceding lemma that if the successor of $\mathbf{h}$ is $A\mathbf{h}$, then the predecessor of $\mathbf{h}$ is $B^{-1}\mathbf{h}$. Then $A\mathbf{h}$ and $B^{-1}\mathbf{h}$ are a pair of adjoint points. By Lemma 4.6, the orbit is at least $(m+n)$-symmetric. $\square$

Proof of Theorem 3.3.a (see Fig. 3-I)**.** Suppose that the class is $(m+n)$-symmetric. This means that the cycle is invariant under reflection with respect to the plane $m + n = 0$. We choose one of the two self-adjoint points of the cycle (which exist by Lemma 4.8), for example, $\mathbf{h}$. We consider all points of the cycle between $\mathbf{h}$ and the second self-adjoint point $\mathbf{h}'$ following the arrows.

The product of generators A and B corresponding to the sequence of the arrows from $\mathbf{h}$ to $\mathbf{h}'$ is an operator $M \in \mathcal{T}^+$ satisfying

$$M\mathbf{h} = \mathbf{h}'.$$

The points between $\mathbf{h}'$ and $\mathbf{h}$ forming the other part of the cycle are the adjoints of the points from $\mathbf{h}$ to $\mathbf{h}'$. By Lemma 4.4,

$$\hat{M}\mathbf{h} = \mathbf{h}',$$

where $\hat{M}$ is obtained from M by changing A to B^{-1} and B to A^{-1}. Equivalently, we can write

$$\hat{M}^{-1}\mathbf{h}' = \mathbf{h},$$

where $\hat{M}^{-1}$ is obtained from M by reversing the order of the factors and exchanging A and B. But then $\hat{M}^{-1} = M^\vee$, i.e., the transpose of M, because the transpose of A and B are respectively B and A. Hence,

$$MM^\vee\mathbf{h} = \mathbf{h}.$$

Therefore, if M is a word, product of q words of type A^i and B^j,

$$M = A^{a_1}B^{a_2}\cdots A^{a_{q-1}}B^{a_q},$$

then the operator $T = MM^\vee$ defining the cycle is the product

$$A^{a_1}B^{a_2}\cdots A^{a_{q-1}}B^{a_q}A^{a_q}B^{a_{q-1}}\cdots B^{a_2}A^{a_1}.$$

The roots $\xi^\pm(\mathbf{h})$ satisfying $\tau(\xi^\pm) = \xi^\pm$ are hence

$$\xi^+ = \big[[a_1, a_2, \dots, a_q, a_q, \dots, a_2, a_1]\big],$$

$$\xi^- = \big[0, [a_q, a_{q-1}, \dots, a_1, a_1, a_2, \dots, a_q]\big].$$

Their periods have the length $P = 2q$ and are palindromic.

On the other hand, given a period Γ of length P, we associate a cycle to it, as explained in the proof of Theorem 3.1, using Lemma 4.2. Now, if Γ is even and palindromic, then the corresponding product of powers of operators A and B defines an operator T, satisfying $T\mathbf{h} = \mathbf{h}$ for some turning point $\mathbf{h} \in H^0$. T is composed by an even number of alternating powers A^i and B^j, whose exponents are the elements of the palindromic period Γ. Therefore we can write $T = MM^\vee$ and

$$\mathbf{h} = MM^\vee\mathbf{h} = (M^\vee)^{-1}M^{-1}\mathbf{h}.$$

By the second equality above, we have

$$\mathbf{h} = \hat{M}M^{-1}\mathbf{h} \quad \textit{and} \quad \mathbf{h}^* = MM^\vee\mathbf{h}^*,$$

because $M^{\vee -1} = \hat{M}$ and $\hat{M}^{-1} = M^\vee$. Il follows that $\mathbf{h} = \mathbf{h}^*$ is self-adjoint. By Lemma 4.9, the cycle, and hence the class, is $(m+n)$-symmetric. $\square$

Proof of Theorem 3.3.b (see Fig. 3-II). If a class is supersymmetric, then its cycle in H^0 is invariant under reflection in the horizontal plane ($m + n = 0$) and also in the vertical plane $k = 0$. A supersymmetric orbit, being, in particular, $(m+n)$-symmetric, has two self-adjoint points by Lemma 4.6, satisfying $m + n = 0$. Since, by the same lemma, the cycle cannot contain more than two selfadjoint points, and these points are conjugate, i.e. symmetric with respect to the vertical plane $k = 0$, the cycle is supersymmetric. We call these points $\mathbf{h}$ and $\bar{\mathbf{h}}$. Being self-adjoint, they are turning points. Let $A\mathbf{h}$ and $B^{-1}\mathbf{h}$ be the successor and predecessor of $\mathbf{h}$. By Lemma 4.4, the successor and predecessor of $\bar{\mathbf{h}}$ are $B\bar{\mathbf{h}}$ and $A^{-1}\bar{\mathbf{h}}$, and

$$\overline{A\mathbf{h}} = A^{-1}\bar{\mathbf{h}}, \qquad \overline{B^{-1}\mathbf{h}} = B\bar{\mathbf{h}}.$$

If we follow the successive points of the cycle after $\mathbf{h}$ and $A\mathbf{h}$, we must reach the point $\bar{\mathbf{h}}$, and $A^{-1}\bar{\mathbf{h}}$ necessarily precedes it. To each of these points from $\mathbf{h}$ and $\bar{\mathbf{h}}$, say $\mathbf{f} = T\mathbf{h}$, there corresponds the conjugate point $\bar{\mathbf{f}} = \bar{T}\bar{\mathbf{h}}$ in the same arc of the cycle from $\mathbf{h}$ to $\bar{\mathbf{h}}$, where $\bar{T}$ is obtained from T by changing each A to A^{-1} and each B to B^{-1}. Therefore, we can write

$$\bar{\mathbf{h}} = M\mathbf{h}, \qquad \mathbf{h} = \bar{M}\bar{\mathbf{h}},$$

obtaining $\bar{M} = M^{-1}$, i.e., M is a palindromic word:

$$M = A^{a_1}B^{a_2}\cdots B^{a_2}A^{a_1}.$$

We note that the sequence must end with A or B if it starts respectively with A or B; therefore, the number of powers of A and B in M is odd.

Using Theorem 3.3.a, the operator obtained as the product of the generators corresponding to the remaining part of the cycle from $\bar{\mathbf{h}}$ to $\mathbf{h}$, is $M^\vee$, which is composed of the same sequence of q groups of generators as M in reversed order with B and A exchanged. Since M itself is palindromic, $M^\vee$ is obtained from M by simply exchanging A and B.

Using Lemma 4.2 we thus find that the periods of the continued fractions of $\xi^\pm(\mathbf{h})$ are odd and, being palindromic, coincide.

Conversely, if we have a continued fraction with an odd palindromic period

$$\big[a_1, a_2, \ldots, a_{q-1}, a_q, a_{q-1}, \ldots a_2, a_1\big],$$

we associate to it an operator $M = A^{a_1}B^{a_2}\cdots B^{a_2}A^{a_1}$. The form $\mathbf{h}$ satisfying $MM^\vee\mathbf{h} = \mathbf{h}$ is self-adjoint, as we have seen proving Theorem 3.3.a. The fact that the word M is palindromic, implies, applying Lemma 4.4, that the cycle of $\mathbf{h}$ is also invariant under conjugation, and hence it is supersymmetric. The entire orbit is therefore supersymmetric. $\square$

Proof of Theorem 3.3.c (see Fig. 3-III)**.** Let the class be k-symmetric. This means that the cycle is invariant under reflection with respect to the vertical plane $k = 0$. Consider a pair $\mathbf{h}$ and $\overline{\mathbf{h}}$ of conjugate points in the cycle. Let $A\mathbf{h}$ and $B^{-1}\mathbf{h}$ be the successor and predecessor of $\mathbf{h}$. We follows the same arguments applied in Theorem 3.3.b to the conjugate points $\mathbf{h}$ and $\overline{\mathbf{h}}$ (that here do not lie in the plane $m + n = 0$) and to the arc of the cycle between them, obtaining

$$\overline{\mathbf{h}} = M\mathbf{h}, \qquad \mathbf{h} = \bar{M}\overline{\mathbf{h}}.$$

Therefore $M = \bar{M}^{-1}$, i.e., M is a palindromic sequence

$$M = A^{a_1} B^{a_2} A^{a_3} \cdots A^{a_3} B^{a_2} A^{a_1}.$$

We note that the sequence must end with A or B if it starts respectively with A or B. The number of powers of A and of B composing the word M is $2q - 1$ and is therefore odd. Applying the same argument to the arc from $\overline{\mathbf{h}}$ to $\mathbf{h}$, we obtain $\mathbf{h} = N\overline{\mathbf{h}}$, where N (if M starts with A) is

$$N = B^{b_1} A^{b_2} B^{b_3} \cdots B^{b_3} A^{b_2} B^{b_1}.$$

As before, the number of powers of A and of B is odd, say $2r - 1$.

We thus obtain

$$\mathbf{h} = NM\mathbf{h},$$

where $T = NM$ defining the cycle is composed of two sequences that are palindromic and odd:

$$\big[(a_1, \ldots, a_q, \ldots, a_1)(b_1, \ldots, b_r, \ldots, b_1)\big].$$

By Theorem 3.1, the resulting sequence is the period of the root of the quadratic equation associated with $\mathbf{h}$, and this period is by consequence even and bipalindromic.

Conversely, given a bipalindromic period, we subdivide it into two palindromic odd sequences $(a_1, \ldots, a_q, \ldots, a_1)$ and $(b_1, \ldots, b_r, \ldots, b_1)$ of respective lengths $2q - 1$ and $2r - 1$ and build the operators $M = A^{a_1} B^{a_2} \cdots A^{a_1}$ and $N = B^{b_1} A^{b_2} \cdots B^{b_1}$. By Theorem 3.1, the period $[a_1, \ldots, a_q, \ldots, a_1, b_1, \ldots, b_r, \ldots, b_1]$ is the period of the root $\xi^+(\mathbf{f})$, where $\mathbf{f}$ satisfies

$$\mathbf{f} = NM\mathbf{f}.$$

Lemma 4.4 implies that the points $\mathbf{f}$ and $M\mathbf{f}$, which are in the cycle, are conjugate and that all the points $A\mathbf{f}$, $A^2\mathbf{f}, \ldots, A^{a^1}\mathbf{f}$, $BA^{a_1}\mathbf{f}, \ldots$ belonging to that part of the cycle between $\mathbf{f}$ and $M\mathbf{f}$ are the conjugate points of the corresponding points $A^{-1}M\mathbf{f}$, $A^{-2}M\mathbf{f}, \dots, A^{-a_1}M\mathbf{f}$, $B^{-1}A^{-a_1}M\mathbf{f}, \dots$ belonging to the same part of the cycle. Observe that if a_q is even, then there is a point (the central point in the sequence from $\mathbf{f}$ to $M\mathbf{f}$) on the plane $k = 0$ that is therefore self-conjugate. Similarly, the pairs of points obtained in the other part of the cycle, from $M\mathbf{f}$ to $NM\mathbf{f} = \mathbf{f}$, are conjugate, and there is possibly a central point self-conjugate, if b_r is even. We thus find that the cycle is k-symmetric because it contains pairs of conjugate points and at most two isolate selfconjugate points. The entire orbit is therefore k-symmetric. □

Proof of Theorem 3.3.d (see Fig. 3-IV)**.** Let the class we consider be antisymmetric. This means that its cycle is invariant under reflection in the center of H^0. We choose any pair of antipodal points in the cycle and call them $\mathbf{h}$ and $\overline{\mathbf{h}}^*$. Let $A\mathbf{h}$ and $B^{-1}\mathbf{h}$ be the successor and predecessor of $\mathbf{h}$. By Lemma 4.4, the successor and predecessor of $\overline{\mathbf{h}}^*$ are $B\overline{\mathbf{h}}^*$ and $A^{-1}\overline{\mathbf{h}}^*$, and

$$\overline{(A\mathbf{h})}^* = B\overline{\mathbf{h}}^*, \qquad \overline{B^{-1}\mathbf{h}^*} = A^{-1}\overline{\mathbf{h}}^*. \tag{11}$$

If we consider the successive points of the cycle following $\mathbf{h}$ and $A\mathbf{h}$, we must reach the point $\overline{\mathbf{h}}^*$. To each point $\mathbf{f} = T\mathbf{h}$ after $\mathbf{h}$, there corresponds a symmetric point $\overline{\mathbf{f}} = \check{T}\overline{\mathbf{h}}$ in the antipodal part of the cycle from $\mathbf{h}$ to $\overline{\mathbf{h}}$, where $\check{T}$ is obtained from T by changing each A to B and each B to A. Hence, we finally write

$$\overline{\mathbf{h}}^* = M\mathbf{h}, \qquad \mathbf{h} = \check{M}\overline{\mathbf{h}}^*.$$

Eq. (11) implies that the last operator in M must be the same as the initial operator (in this case A). We thus obtain

$$\mathbf{h} = \check{M}M\mathbf{h},$$

where the word M is composed by an odd number q of sub-words, i.e., the alternating powers of A and B. The operator defining the cycle is therefore

$$\check{M}M = B^{a_1}A^{a_2}\cdots B^{a_q}A^{a_1}B^{a_2}\cdots A^{a_q}.$$

Hence, the roots $\xi^{\pm}(\mathbf{h})$ have odd periods

$$\big[a_1, a_2, \dots, a_q\big], \quad \big[a_q, a_{q-1}, \dots, a_1\big].$$

The lack of any symmetry among the points belonging to the half of the cycle between $\mathbf{h}$ and $\overline{\mathbf{h}}^*$ implies that the period $[a_1, a_2, \dots, a_q]$ is non palindromic.

Conversely, suppose that for some form $\mathbf{g}$, $\xi^+(\mathbf{g})$ has odd non palindromic period $[a_1, a_2, \dots, a_q]$. By Theorem 3.1, there is a form $\mathbf{h} \in H^0$ in the same class of $\mathbf{g}$ satisfying $T\mathbf{h} = \mathbf{h}$, where $T = \check{M}M$, and

$$M = A^{a_q}B^{a_{q-1}}\cdots B^{a_2}A^{a_1}, \quad \check{M} = B^{a_q}A^{a_{q-1}}\cdots A^{a_2}B^{a_1}.$$

We define $\mathbf{f} := M\mathbf{h}$. From $\mathbf{h} = \check{M}M\mathbf{h}$ we obtain

$$\mathbf{f} = M\check{M}M\mathbf{h} = M\check{M}\mathbf{f},$$

and, using Eq. 11,

$$h^* = M\check{M}\overline{\mathbf{h}}^*.$$

Therefore $\overline{\mathbf{h}}^* = \mathbf{f}$, that is, since $\mathbf{f} = M\mathbf{h}$, the form $\mathbf{h}^*$ is in the same class of $\mathbf{h}$ and the orbit of $\mathbf{h}$ possesses the antipodal symmetry. The orbit cannot be supersymmetric because in this case the period $[a_1, a_2, \ldots, a_q]$ should be palindromic, according to Theorem 3.3.b. Hence the orbit is antisymmetric. $\square$

Proof of Theorem 3.3.e (see Fig. 3-V)**.** The asymmetric case is the simplest. The number p of alternating powers of A and B, factors of the operator $T \in \mathcal{T}^+$ satisfying $T\mathbf{h} = \mathbf{h}$, where $\mathbf{h}$ is a turning point of the cycle, is necessarily even. Since there is no symmetry relating the points of the cycle of $\mathbf{h}$, the periods of the roots $\xi^{\pm}(\mathbf{h})$ are non palindromic and contain an even number p of elements. On the other hand, if the continued fractions of $\xi^{\pm}(\mathbf{g})$ have even non palindromic period, by Theorem 3.1 there is a form $\mathbf{h} \in H^0$ in the same class of $\mathbf{g}$ satisfying $T\mathbf{h} = \mathbf{h}$, where T is made of alternating powers of A and B, whose exponents are the elements of the periods of the continued fractions of $\xi^{\pm}(\mathbf{g})$. The lack of symmetry in these periods implies the asymmetricity of the cycle and hence of the entire corresponding class of forms. $\square$

Proof of Corollary 3.4. By Theorem 3.2, the sequence s uniquely defines a primitive class iff this class is invariant under the antipodal symmetry. Such a class is either supersymmetric or antisymmetric. By Theorem 3.3, these cases are the only cases where the periods are odd. $\square$

Proof of Corollary 3.6. Suppose that the cycle in H^0 has the n-cyclic symmetry, i.e., that there exists an operator M of $\mathcal{T}^+$ and a point $\mathbf{h} \in H^0$ such that $\mathbf{h} = M^n\mathbf{h}$. By the results of Part I and Theorem 3.1, all points $\mathbf{h}_i$ of the cycle satisfy $\widetilde{M^n}_i\mathbf{h}_i = \mathbf{h}_i$ for some $\widetilde{M^n}_i$ obtained from M^n by a cyclic permutation of its factors. Among them, there is a turning point $\mathbf{f}$ such that $\xi^+(\mathbf{f})$ is immediately periodic,

$$\xi^+(\mathbf{f}) = \big[[(a_1, \ldots, a_p)_1, (a_1 \ldots, a_p)_2, \ldots, (a_1, \ldots, a_p)_n]\big],$$

where $\widetilde{M^n} = (B^{a_p}A^{a_{p-1}} \cdots B^{a_2}A^{a_1})^n$ with even p. But $\xi^+(\mathbf{f})$ satisfies

$$\mu\big(\xi^+(\mathbf{f})\big) = \xi^+(\mathbf{f}),$$

where μ is obtained from $B^{a_p} A^{a_{p-1}} \cdots B^{a_2} A^{a_1}$ by translating A into α and B into β. By Lemma 4.1, $\mathbf{f}$ also satisfies

$$\mathbf{f} = B^{a_p} A^{a_{p-1}} \cdots B^{a_2} A^{a_1} \mathbf{f},$$

and therefore belongs to a cycle of p elements. Since a point of the cycle cannot be visited twice, no other points can belong to the same cycle, and the cycle has not the n-cyclic symmetry. □

Proof of Theorem 3.7. By Corollary 4.12 of Part I, the number of points of a class $C(m, n, k)$ inside every domain of G_A and of $G_{\bar{A}}$ is equal to the number t_B, which is the total number of times that B appears as factor in the operator $T \in \mathcal{T}^+$ that defines the cycle in H^0. Proving Theorem 3.1, we have seen that the elements of $\Pi = [a_1, a_2, \ldots, a_p]$ are the exponents of the alternating powers A and B forming T. It follows that t_B is equal to either t_{odd} or t_{even}, depending which generator among A and B have the powers a_j with odd j.

If the period Γ is even palindromic, then P is even and $a_i = a_{P+1-i}$. Moreover, $(P + 1 - i)$ is odd if i is even, and *vice versa*. Hence, the values t_{odd} and t_{even}, given by eq. (3), coincide and are equal to $t/2$ because their sum is equal to t.

If the period is even nonpalindromic, then $p = P$, and the values t_{odd} and t_{even} may not coincide.

If the period Γ is odd, then P is odd. In Π, by definition, $a_{i+P} = a_i$ for all $i = 1, \ldots, P$, so that i and $(i + P)$ have opposite parity, for every i. Also in this case, $t_{\text{odd}} = t_{\text{even}}$. □

4.3 Classes representing 1

Proof of Corollary 3.5. We prove this corollary by showing that the class of forms representing 1 is either k-symmetric or supersymmetric[4]. The statements follow from Theorems 3.3.c and 3.3.b. Indeed, in such a class, there is a representive $\mathbf{f}$ with the coefficients $(1, n, k)$, i.e., a form

$$f = x^2 + kyx + ny^2.$$

[4]Alternative proof: The class of the quadratic form $C(1, n, k)$ is the sole class representing 1 among the classes with the discriminant $k^2 - 4n$. This class is the identity of the group of classes. Because the inversion in the class group corresponds to the conjugation, the identity class is self-conjugate and is hence invariant under reflection in the axis $k = 0$. The sole classes having this symmetry are the supersymmetric classes (with an odd palindromic period) and the k-symmetric classes (with an even bipalindromic period).

Let $T = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$. If $v = (x, y)$, then $g(v) = f(Tv) = x^2 - kxy + ny^2$. We observe that $\mathbf{g} = (1, n, -k) = \overline{\mathbf{f}}$ and hence $\overline{\mathbf{f}} \in C(1, n, k)$. Therefore, the class $C(1, n, k)$ is either k-symmetric or supersymmetric. □

4.4 Classes representing 0

The classes of forms representing 0 have a discriminant equal to a square number (see §4.5 of Part I). The roots of the corresponding quadratic equations are rational and therefore have a finite continued fraction.

Proof of Theorem 3.8. The rational number k/m is equal to $\xi^+(\mathbf{h})$, $\mathbf{h} = (m, 0, -k)$. The quadratic form $\mathbf{h}$ for which $\xi^+(\mathbf{h}) > 0$ and $\xi^-(\mathbf{h}) = 0$ belongs to the set $F_{\bar{A}}$ (see Sec. 4.5 of Part I). By Theorem 4.13 of Part I, $\mathbf{h}$ is one of the k forms with discriminant $\Delta = k^2$ and is the sole form of its class in F_A.

The classes $C(m, 0, k)$ and $C(m, 0, -k)$ are conjugate, and have therefore the same number of points in each domain.

The form $\mathbf{f} = A\mathbf{h}$ is in the interior of H^0 and satisfies $A^{-1}\mathbf{f} \in F_{\bar{A}}$. By Theorem 4.17 of Part I, it is the starting point of the chain containing all the t points in H^0 of its class. The final point of that chain is the form $\mathbf{g}$ satisfying $\mathbf{g} = T\mathbf{f}$ for some operator $T \in \mathcal{T}^+$, the product of $t-1$ generators.

By hypothesis, $[a_1, a_2, \ldots, a_N]$ is the even continued fraction of $\xi^+(\mathbf{h})$. Using Lemmas 4.1 and 4.2, we obtain $\xi^+(\mathbf{f}) = [a_1 - 1, a_2, \ldots, a_N]$ and $\xi^+(\mathbf{g}) = [1]$.

We define τ by the equation $\xi^+(\mathbf{g}) = \tau(\xi^+(\mathbf{f}))$. Since N is even,

$$\tau = \beta^{a_N - 1} \circ \alpha^{a_{N-1}} \cdots \beta^{a_2} \circ \alpha^{a_1 - 1}.$$

Hence, the operator T sending $\mathbf{f}$ to $\mathbf{g}$ is equal to

$$T = B^{a_N - 1} A^{a_{N-1}} \cdots B^{a_2} A^{a_1 - 1}.$$

According to Corollary 4.18 of Part I, the number of points of $C(m, n, k)$ in the interior of each domain in G_A and in $G_{\bar{A}}$ is equal to the total number ($t_B(T)$) of times that B appears as factor in T and is therefore equal to $\hat{t}_{\text{even}}$ given by eq. (4), and the number of points in the interior of each domain in G_B and in $G_{\bar{B}}$ is equal to the total number ($t_A(T)$) of times that A appears as factor in T and is therefore equal to $\hat{t}_{\text{odd}}$.

When the class is supersymmetric or $(m+n)$-symmetric, it is symmetric with respect to the horizontal plane $(m + n) = 0$. The number of points in the interior of the domains in G_A and $G_{\bar{A}}$ coincides with that in the interior of the domains in G_B and $G_{\bar{B}}$. Since $\hat{t}_{\text{odd}} + \hat{t}_{\text{even}} = \hat{t} - 1$, we have

$$\hat{t}_{\text{odd}} = \hat{t}_{\text{even}} = (\hat{t} - 1)/2. \tag{12}$$

The number of points in the interior of H^0 is hence odd when $t_A(T) = t_B(T)$. The theorem is proved. $\square$

Remark. The case of $\mathbf{f} = (0, 0, k)$ is not concerned by the above theorem, where $m > 0$. In this case the orbit is supersymmetric by Lemma 4.15 of Part I, and $t = t_A = t_B = 0$.

Proof of Theorem 3.9. For $m = 1, \dots, k-1$, the rational number k/m is the nonzero root of the equation $m\xi^2 - k\xi =$, i.e., $\xi^+(\mathbf{h}) = k/m$, and $\mathbf{h} = (m, 0, -k)$ is a representative of the class $C(m, 0, -k)$. Observe that the form $(m, 0, k)$ is the conjugate of $\mathbf{h}$, and its class has the same type of symmetry as the class of $\mathbf{h}$.

Item i: Lemma 4.15 of Part I states that $C(0, 0, k)$ is supersymmetric. If $m = k/2$, let $\mathbf{h} = (m, 0, -2m)$. The form $\mathbf{f} := A\mathbf{h} = (m, -m, 0)$ is then the central point of H^0, satisfying $m + n = 0$ and $k = 0$. By Lemma 4.4, $B\mathbf{f} = \mathbf{h}^*$, $A\mathbf{f} = \overline{\mathbf{h}}$, and $B^{-1}\mathbf{f} = \overline{\mathbf{h}}^*$, therefore they belong to the boundary of H^0, as $\mathbf{h}$. Hence, $\mathbf{f}$ is the only point of the orbit in the interior of H^0, and the orbit is supersymmetric. The fact that if $m \neq 0$ then $m = k/2$ if the orbit $C(m, 0, k)$ is supersymmetric is proved by the following reasoning. The initial point $\mathbf{i}$ of the chain in H^0 is reached by two arrows from $A^{-1}\mathbf{i}$ and $B^{-1}\mathbf{i}$, and the final point $\mathbf{f}$ is joined by two arrows to $A\mathbf{f}$ and $B\mathbf{f}$. Since the orbit is supersymmetric, the arrows from $A^{-1}\mathbf{i}$ to $\mathbf{i}$ and the arrow from $B^{-1}\mathbf{i}$ to $\mathbf{i}$ must be symmetric with respect to the center, as well as the arrows from $\mathbf{f}$ to $A\mathbf{f}$ and from $\mathbf{f}$ to $B\mathbf{f}$. Therefore, the only possibility is that both the initial and the final point coincide with the central point of H^0.

Item ii: Let $\mathbf{h} = (m, 0, -k)$, $m > 0$, with $m \neq k/2$, and let $m/k = [a_1, \dots, a_N]$ with even N. Let $C(m, 0, -k)$ be $(m+n)$-symmetric. By the same arguments proving Theorem 3.8, we obtain

$$\beta^{a_N} \circ \alpha^{a_{N-1}} \cdots \beta^{a_2} \circ \alpha^{a_1} \left(\frac{m}{k}\right) = 0,$$

while by the same arguments proving Theorem 3.3.a,

$$B^{a_N} A^{a_{N-1}} \cdots B^{a_2} A^{a_1} \mathbf{h} = \mathbf{h}^*. \tag{13}$$

The final point $\mathbf{g}$ of the chain staring at $\mathbf{f} = A\mathbf{h}$ must be sent by B^{-1} to $\mathbf{h}^*$ because the orbit is $(m+n)$-symmetric. Similarly, the adjoint of the successor of $\mathbf{f}$, say $A\mathbf{f}$, is the predecessor of $\mathbf{g}$, $B^{-1}\mathbf{g}$, and so on. In this way we obtain that the sequence of powers of A and B in (13) is palindromic. As in the proof of

Theorem 3.3.a, we conclude that every $(m+n)$-symmetric chain contains exactly one self-adjoint point in H^0, satisfying $(m+n)=0$. On the other hand, if the even continued fraction of (m/k) is palindromic, then using Lemma 4.2, we reach a point on the boundary of H^0, and by the symmetry of the corresponding operator in $\mathcal{T}^+$, we conclude that this point is $\mathbf{h}^*$ and the orbit is hence at least $(m+n)$-symmetric. In fact, such orbit could be supersymmetric, but this is excluded by item *i*.

Item iii: Let $\mathbf{h}=(m,0,-k)$, $m>0$, with $m\neq k/2$, and let $m/k=[a_1,\dots,a_N]$ with odd N. Let $C(m,0,-k)$ be k-symmetric. By the same arguments proving Theorem 3.8, we obtain

$$\alpha^{a_N}\circ\beta^{a_{N-1}}\cdots\beta^{a_2}\circ\alpha^{a_1}\left(\frac{m}{k}\right)=0,$$

while by the same arguments proving Theorem 3.3.b,

$$A^{a_N}B^{a_{N-1}}\cdots B^{a_2}A^{a_1}\mathbf{h}=\overline{\mathbf{h}}. \tag{14}$$

Indeed, the final point $\mathbf{g}$ of the chain starting at $\mathbf{f}=A\mathbf{h}$ must be sent by A to $\overline{\mathbf{h}}$ because the orbit is k-symmetric. Each point of the chain, obtained as $T\mathbf{h}$ for some $T\in\mathcal{T}^+$, has its conjugate in the chain, and the arrow, A or B, between two consecutive points $\mathbf{g}$ and $\mathbf{j}$, is sent by the conjugation to the arrow, A^{-1} or B^{-1} respectively, between the consecutive points $\bar{\mathbf{j}}$ and $\bar{\mathbf{g}}$ according to Lemma 4.4. In this way we obtain that the sequence of powers of A and B in (14) palindromic. On the other hand, if the odd continued fraction m/k is palindromic, then using Lemma 4.2, we reach a point on the boundary of H^0, and by the symmetry of the corresponding operator in $\mathcal{T}^+$, we conclude that this point is $\overline{\mathbf{h}}$ and the class has hence at least the k-symmetry. In fact, it is k-symmetric, since item *i* excludes the possibility of being supersymmetric.

Item iv: The remaining possibility is that the chain and also the orbit $C(m,0,k)$ are asymmetric. This occurs iff neither the odd nor the even continued fraction of k/m are palindromic.

To conclude the proof, we observe that for an antisymmetric orbit we have $t_A=t_B$ because the points in G_A and in G_B of an antisymmetric class are related by the antipodal symmetry, as well as the points in $G_{\bar{A}}$ and $G_{\bar{B}}$. By Theorem 3.8, the number of points in H^0 must therefore be odd. An antisymmetric chain in H^0 containing an odd number of points must contain the center of H^0, but the orbit would then be supersymmetric by item *i*. Therefore the class $C(m,0,k)$ is never antisymmetric. $\square$

5 Reduction theory

We present here a corollary of Theorem 3.1 that provides a reduction procedure allowing to transform an indefinite form $f = mx^2 + ny^2 + kxy$ with $mn \geq 0$ into a form $m'x^2 + n'y^2 + k'xy$ of the same class with $m'n' \leq 0$.

Moreover, in this section, we show how such reduction procedure is related to the "classical reduction theory" [5], [6], [7].

Corollary 5.1. *Let* $\mathbf{f} = (m, n, k)$ *with* $mn > 0$ *and* Δ *not a square number. Let* $[\alpha_0, \alpha_1, \ldots, \alpha_N, [a_1, \ldots, a_p]]$ *be the continued fraction of the root* $\xi^+(\mathbf{f})$. *Then the form* $\mathbf{f}'$ *in the same class as* $\mathbf{f}$*:*

$$\mathbf{f}' := (m', n', k') = T(\mathbf{f})$$

satisfies $m'n' < 0$, *where*

$$T = C^{\alpha_N} \cdots A^{\alpha_2} B^{\alpha_1} A^{\alpha_0} \tag{15}$$

with $C = A$ *if* N *is even and* $C = B$ *if* N *is odd.*

If the discriminant Δ *of* $\mathbf{f} = (m, n, k)$ *with* $mn > 0$ *is a square number, and* $[\alpha_0, \alpha_1, \ldots, \alpha_N]$ *is the continued fraction of the rational root* $\xi^+(\mathbf{f})$, *then the form* $\mathbf{f}' = T\mathbf{f}$, *with* T *defined by* (15), *satisfies* $m'n' = 0$.

Proof. The expression for T follows from Lemma 4.2. Observe that α_0 may be zero. Moreover, if $\xi^+(\mathbf{f}) < 0$, then the elements of the continued fraction are all negative. We obtain

$$\xi^+(T\mathbf{f}) = \big[[a_1, \ldots, a_P]\big]$$

if N is even, otherwise

$$\xi^+(T\mathbf{f}) = \big[0, [a_1, \ldots, a_P]\big].$$

Since $\xi^+(T\mathbf{f})$ is immediately periodic, we conclude that $T\mathbf{f} \in H^0$ or $T\mathbf{f} \in H^0_R$ and hence that the form $T\mathbf{f}$ satisfies $m'n' < 0$. In the case when $\xi^+(\mathbf{f})$ is rational, then $\xi^+(T\mathbf{f}) = 0$, and the form $T\mathbf{f}$ satisfies $m'n' = 0$, i.e., belong to the boundary of H^0 or H^0_R. Observe that the condition $\xi^+(\mathbf{f}) > 0$ means that $\mathbf{f}$ is in either $G_{\bar{A}}$ or $G_{\bar{B}}$ (where $m > 0$ and $n > 0$), see Fig. 1. The operator T indeed belongs to $\mathcal{T}^+$, and its sequence of powers of A and B can be read as a path from $\mathbf{f}$ toward H^0 (or its boundary, in the rational case) entirely in $G_{\bar{A}}$ or in $G_{\bar{B}}$. This path is unique by Theorem 4.2 in Part I. Similarly, if $\xi^+(\mathbf{f}) < 0$, $\mathbf{f}$ is in either either G_A or G_B (where $m < 0$ and $n < 0$). The operator T belongs to $\mathcal{T}^-$, and its sequence of powers of A^{-1} and B^{-1} can be read as a path from $\mathbf{f}$ toward H^0 (or its boundary, in the rational case) entirely in G_A or in G_B. $\square$

5.1 Classical reduction theory in the Poicaré tiling of the de Sitter world

Definition. The *minus continued fraction* of the real number $\xi > 0$ is the (finite or infinite) sequence of integers $(b_0, b_1, b_2, \dots)$, $b_0 \geq 1$, $b_i \geq 2$ $\forall i > 0$, such that:

$$\xi = b_0 - \cfrac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\dots}}}. \tag{16}$$

We call for short *minued fraction* a minus continued fraction.

Example. The minued fraction of $1/4$ (whose continued fraction is $[0, 4]$) is $(1, 2, 2, 2)$:

$$\frac{1}{4} = 1 - \cfrac{1}{2 - \frac{1}{2 - \frac{1}{2}}}.$$

Any irrational number has an infinite minued fraction, and a quadratic surd has a periodic minued fraction, whose period is defined exactly as for the periodic continued fractions (see Definition 3). A periodic minued fraction with period of length L is denoted by

$$\big(b_0, b_1, \dots, b_{N-1}, (b_N, b_{N+1}, \dots, b_{N+L-1})\big). \tag{17}$$

The minued fraction x if said *immediately periodic* if $x = ((b_0, b_1, b_2, \dots, b_{L-1}))$.

The following theorems represent a synthesis of the classical reduction theory. The proofs that we give here show a geometric description of this theory in the Poincaré tiling of the de Sitter world.

Definition. An indefinite form $\mathbf{h} = (m, n, k)$ such that $C(m, n, k)$ does not represent zero is said to be *reduced* iff it satisfies $m > 0$, $n > 0$, $k < 0$, and $m + n < |k|$.

Theorem 5.2. *Let $\mathbf{h} = (m, n, k)$ be a reduced form. Then the minued fraction of $\xi^+(\mathbf{h})$ is immediately periodic, and the number of elements of its period is equal to the number of reduced forms of the class $C(m, n, k)$. The first roots ξ^+ associated with the other reduced forms are given by the cyclic permutations of the elements of $\xi^+(\mathbf{f})$.*

Theorem 5.3. *Let $\mathbf{f} = (m, n, k)$ satisfy $m > 0$, $n > 0$ and $k < 0$. Then the minued fraction of $\xi^+(\mathbf{f})$ is periodic,*

$$\xi^+(\mathbf{f}) = \big(b_0, b_1, \dots, b_M, (c_1, c_2, \dots, c_L)\big),$$

and the form $\mathbf{h} = T\mathbf{f}$*, where*

$$T = RA^{b_M} \cdots RA^{b_2} RA^{b_1} RA^{b_0},$$

is reduced.

The proof of Theorem 5.2 follows from the results of this work and from the following lemma.

Lemma 5.4. *Let* $\mathbf{f} = (m, n, k)$ *and* $\xi^+(\mathbf{f}) = (a, b, c, d, \dots) > 0$*. Then* $\xi^+(RA^a\mathbf{f}) = (b, c, d, \dots)$*. I.e., canceling the first element in the minued fraction of* $\xi^+(\mathbf{f})$ *corresponds to acting on* $\mathbf{f}$ *by the operator* A *iterated a number of times equal to that element and then acting by* R*.*

Proof. By Lemma 4.1,

$$\xi^+(A^a\mathbf{f}) = \alpha^a(\xi^+(\mathbf{f})) = -\frac{1}{b - \frac{1}{c - \frac{1}{d - \frac{1}{\dots}}}}$$

and

$$\xi^+(RA^a\mathbf{f}) = \sigma \circ \alpha^a(\xi^+(\mathbf{f})) = b - \frac{1}{c - \frac{1}{d - \frac{1}{\dots}}}.$$

$\square$

A reduced form $\mathbf{h}$ of a class $C(m, n, k)$ not representing zero belongs to $H_{\bar{A}}$. It is represented in Ξ by a point with $\xi^+ > 1$ and $0 < \xi^- < 1$.

The point $\mathbf{f} := A\mathbf{h}$ is in H^0 (see Fig. 4). We now consider all the successive points $A^2\mathbf{h}$, $A^3\mathbf{h}$, etc., that belong to H^0. For some power, say A^{c_1} with $c_1 \geq 3$, the point $A^{c_1}\mathbf{h}$ exits from H^0 and is necessarily in H_A. At this point, we apply the operator R, going back to $H_{\bar{A}}$. The point $RA^{c_1}\mathbf{h}$ coincides with $\mathbf{h}$ if and only if the period of the continued fraction of $\xi^+(\mathbf{f})$ consists of two elements. Indeed, if $RA^{c_1}\mathbf{h} = \mathbf{h}$, then the point $\mathbf{f} = A\mathbf{h}$ in H^0 satisfies $ARA^{c_1-1}\mathbf{f} = \mathbf{f}$. Using the relation $R = A^{-1}BA^{-1}$, we obtain

$$BA^{c_1-2}\mathbf{f} = \mathbf{f},$$

hence concluding that the cycle in H^0 contains only two turning points and that, by Theorem 3.7, the orbit has only one element in $H_{\bar{A}}$.

If $RA^{c_1}\mathbf{h}$ differs from $\mathbf{h}$, then we repeat the above procedure until getting

$$RA^{c_L} RA^{c_{L-1}} \cdots RA^{c_1}\mathbf{h} = \mathbf{h}.$$

For the same reason, this occurs when all points in $H_{\bar{A}}$, as well as in H^0, are visited (see Figure 4). By Lemma 5.4, using the same arguments as in the proof of Theorem 3.1, we find that the minued fraction of $\xi^+(\mathbf{h})$ is periodic, namely,

$$\xi^+(\mathbf{h}) = \big((c_1, c_2, \ldots, c_L)\big).$$

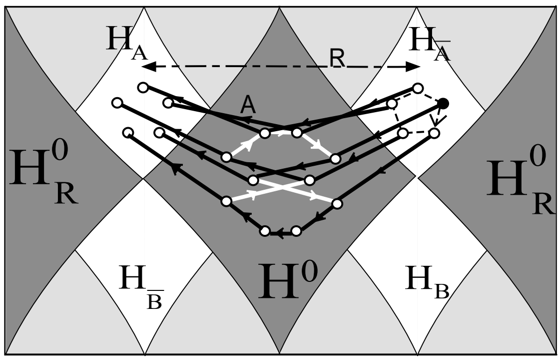

Figure 4: Relation between a cycle in H^0 with geometric period [1, 1, 3, 1, 1, 3] and a cycle (dotted line) in H^A with period (3, 5, 3, 2, 2).

We have thus proved that the minued fraction of the first root of an equation corresponding to a form in $H_{\bar{A}}$ is immediately periodic and the length L of its period is equal to the number of points of the class in $H_{\bar{A}}$, i.e., the number of reduced forms. $\square$

Proof of Theorem 5.3. Observe that a form $\mathbf{f}$ satisfying the conditions of the theorem lies in $G_{\bar{A}}$. By Theorem 4.2 of Part I, there is a form $\mathbf{h}$ in $H_{\bar{A}}$ such that $\mathbf{f} = T^{-1}\mathbf{h}$, where $T \in \mathcal{T}^+$. Hence we write $\mathbf{f} = T\mathbf{h}$, where T is a word in the operators A and B. We rewrite T as a product of the operators A and R by the following procedure: we replace each operator B in T with $AA^{-1}BA^{-1}A$ and then each $A^{-1}BA^{-1}$ with R. In the obtained expression of T every factor B^i has been replaced with $\underbrace{(ARA)(ARA)\cdots(ARA)}_{i \text{ times}}$. Observe that the expression of T in terms of R and A is unique. By Theorem 5.2, the minued fraction of $\xi^+(\mathbf{h})$ is immediately periodic. Using Lemma 5.4, we conclude that

$$T = RA^{b_M}RA^{b_{M-1}}\cdots RA^{b_1}RA^{b_0}$$

if and only if $\xi^+(\mathbf{f}) = (b_0, b_1, \ldots, b_M, (c_1, c_2, \ldots, c_L))$. $\square$

Corollary 5.5. *If* $\Pi = [a_1, a_2, \ldots, a_p]$ *is the geometric period of the continued fraction of* $\xi^+(\mathbf{h})$*,* $\mathbf{h} \in H_{\bar{A}}$*, then the period of the minued fraction of* $\xi^+(\mathbf{h})$ *is obtained as follows:*

1. *the element* a_{2i+1} *of the period is replaced with* $a_{2i+1} + 2$ *and*
2. *the element* a_{2i} *of the period is replaced with with a sequence* $2, 2, \ldots, 2$ *formed by the number* 2 *repeated* $a_{2i}-1$ *times.*

Remark. Let $\xi^+(\mathbf{f})$ be the root associated to a form $\mathbf{f}$ non representing zero. Then the *length* of the period of the minued fractions of $\xi^+(\mathbf{f})$ is equal to the number of reduced forms of the class of $\mathbf{f}$ in H_A, whereas the number of forms in H^0 is equal to the *sum* of all elements of the geometric period of the continued fraction of $\xi^+(\mathbf{f})$.

Example. The supersymmetric class represented in Fig. 4 is that of the example of Theorem 3.3.b. Let $\mathbf{f} = (2, -1, -3)$. We have $\xi^+(\mathbf{f}) = 3/4 + \sqrt{17}/4 = [[1, 1, 3]]$. Hence, $\Pi = [1, 1, 3, 1, 1, 3]$. There are ten points inside H^0. The point $\mathbf{h} = A^{-1}\mathbf{f} = (2, 4, -7)$ is in $H_{\bar{A}}$ and $\xi^+(\mathbf{h}) = 7/4 + \sqrt{17}/4$ has the minued fraction

$$\xi^+(\mathbf{h}) = ((3, 5, 3, 2, 2)).$$

There are indeed five points in $H_{\bar{A}}$.

We are now able to prove Theorem 3.10.

Proof of Theorem 3.10. By Corollary 5.5, with a given period $(c_1, c_2, \ldots, c_L)$ of a minued fraction, we associate the period $[a_1, a_2, \ldots, a_p]$ of the corresponding continued fraction, replacing each element $c_i > 2$ with the element $a_j = c_i - 2$ and replacing each sequence of r ($r \geq 0$) successive elements $c_{i+1}, c_{i+2}, \ldots, c_{i+r}$ all equal to 2 with the element $a_{j+1} = r+1$ (hence, $a_{j+1} = 1$ if $r = 0$, i.e., $c_{i+1} > 2$). We obtain a period of p elements with even p, which is the geometric period Π of the continued fraction. By Theorem 3.7, the sum of the odd-indexed elements of the continued fraction is equal to the sum of even-indexed elements is also equal to the number of points of the orbit in H_A and in $H_{\bar{A}}$ if the class has the mentioned symmetries. By Theorem 5.2, the number of points in $H_{\bar{A}}$ of the orbit equals the number L of elements of the period of the considered minued fraction. Since both the sums of the odd-indexed and of the even-indexed elements of the period Π are obtained by subtracting 2 to the elements c_i of the minued fraction, we obtain the equation

$$\sum_{i=1}^{L}(c_i - 2) = L,$$

and hence the statement of the theorem. $\square$

Remark. For a nonsymmetric class, the minued fractions corresponding to ξ^+ and ξ^-, which have inverse geometric periods, are different.

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Appendix:
Tables of classes of indefinite forms with a small discriminant

The following tables contain a representative of every class with $\Delta \leq 100$; moreover, if Δ is not a square integer, we give the period Γ, its length P, the number $t^{\uparrow}$ of points inside each domain in G_A and $G_{\bar{A}}$, the number $t^{\downarrow}$ of points inside each domain in G_B and $G_{\bar{B}}$, and the type of symmetry. A star indicates that the class is nonprimitive, i.e., is obtained from a primitive class by multiplying by an integer greater than 1.

If Δ is equal to a square number, then instead of Γ and P, we give the continued fraction of k/m, its length N, the number t of points in the interior of H^0 and of H^0_R, the number $t^{\uparrow}$ of points in the interior of each domain in G_A and $G_{\bar{A}}$, the number $t^{\downarrow}$ of points in the interior of each domain in G_B and $G_{\bar{B}}$, and the type of symmetry.

Remark. The classes of forms not representing zero are either supersymmetric or k-symmetric if $\Delta \leq 100$.

Indeed, the first class (i.e., with a minimal discriminant) not representing zero and having the $(m+n)$-symmetry has the period [1, 1, 3, 3, 1, 1] and the discriminant 136. The first antisymmetric class has the period [1, 2, 3] and the discriminant 148, and the first asymmetric class has the period [1, 1, 1, 2, 3, 5] and the discriminant 316.

The tables show that there are classes of forms representing zero with $\Delta \leq 100$ that have all the possible types of symmetry.

Moreover, in the following figure, we plot the fractions of the total number of classes with a given discriminant versus the discriminant $< 10^4$ corresponding to the different types of symmetries (indicated by different symbols).

Updating Reference. A Reference more precise than [2] in Part I is

V.I.Arnold, G. Capitanio, R. Uribe-Vargas, *Geometry*, Springer, pages 255–290, to appear.

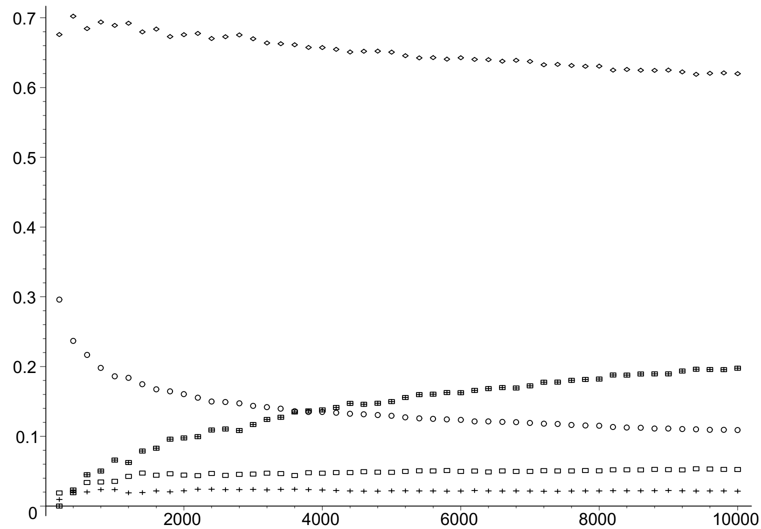

Figure 5: Rhombus, k-symmetric; circle, supersymmetric; cross, $(m+n)$-symmetric; window, asymmetric; square, antisymmetric.

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Δ	m	n	k	Γ	P	$t^{\uparrow} - t_{\downarrow}$	symm.	n.p.
5	1	-1	1	[1]	1	1-1	super	
8	1	-1	2	[2]	1	2-2	super	
12	2	-1	2	[2, 1]	2	2-1	k	
	1	-2	2	[1, 2]	2	1-2	k	
13	1	-1	3	[3]	1	3-3	super	
17	2	-2	1	[1, 3, 1]	3	5-5	super	
20	1	-1	4	[4]	1	4-4	super	
	2	-2	2	[1]	1	1-1	super	$\star$
21	3	-1	3	[3, 1]	2	3-1	k	
	1	-3	3	[1, 3]	2	1-3	k	
24	2	-1	4	[4, 2]	2	4-2	k	
	1	-2	4	[2, 4]	2	2-4	k	
28	3	-2	2	[1, 1, 4, 1]	4	5-2	k	
	2	-3	2	[1, 4, 1, 1]	4	2-5	k	
29	1	-1	5	[5]	1	5-5	super	
32	2	-2	4	[2]	1	2-2	super	$\star$
	1	-4	4	[1,4]	2	1-4	k	
	4	-1	4	[4,1]	1	4-1	k	
33	2	-1	5	[5, 2, 1, 2]	4	6-4	k	
	1	-2	5	[2, 1, 2, 5]	4	4-6	k	
37	3	-3	1	[1, 5, 1]	3	7-7	super	
40	3	-3	2	[1, 2, 1]	3	4-4	super	
	1	-1	6	[6]	1	6-6	super	
41	2	-2	5	[2, 1, 5, 1, 2]	5	11-11	super	
44	2	-1	6	[6, 3]	2	6-3	k	
	1	-2	6	[3, 6]	2	3-6	k	
45	3	-3	3	[1]	1	1-1	super	$\star$
	1	-5	5	[1, 5]	2	1-5	k	
	5	-1	5	[5, 1]	2	5-1	k	
48	1	-3	6	[2, 6]	2	2-6	k	
	3	-1	6	[6, 2]	2	6-2	k	
	4	-2	4	[2, 1]	2	1-2	k	$\star$
	2	-4	4	[1, 2]	2	1-2	k	$\star$
52	3	-3	4	[1, 1, 6, 1, 1]	5	10-10	super	
	2	-2	6	[3]	1	3-3	super	$\star$
53	1	-1	7	[7]	1	7-7	super	
56	5	-2	4	[2, 1, 6, 1]	4	8-2	k	
	2	-5	4	[1, 6, 1, 2]	4	2-8	k	
57	4	-3	3	[1, 1, 3, 7, 3, 1]	6	7-9	k	
	3	-4	3	[1, 3, 7, 3, 1, 1]	6	9-7	k	

Tables of classes not representing zero with $\Delta < 100$.

Δ	m	n	k	Γ	P	$t^{\uparrow} - t_{\downarrow}$	symm.	n.p.
60	2	-3	6	[2, 3]	2	2-3	k	
	3	-2	6	[3, 2]	2	3-2	k	
	1	-6	6	[1, 6]	2	1-6	k	
	6	-1	6	[6, 1]	2	6-1	k	
61	3	-3	5	[2, 7, 2]	3	11-11	super	
65	4	-4	1	[1, 7, 1]	3	9-9	super	
	2	-2	7	[3, 1, 3]	3	7-7	super	
68	1	-1	8	[8]	1	8-8	super	
	4	-4	2	[1, 3, 1]	3	5-5	super	$\star$
69	5	-3	3	[1, 1, 7, 1]	4	8-2	k	
	3	-5	3	[1, 7, 1, 1]	4	2-8	k	
72	3	-3	6	[2]	1	2-2	super	$\star$
	1	-2	8	[4, 8]	2	8-4	k	
	2	-1	8	[8, 4]	2	8-4	k	
73	4	-4	3	[1, 2, 3, 1, 7, 1, 3, 2, 1]	9	21-21	super	
76	3	-1	8	[8, 2, 1, 3, 1, 2]	6	10-7	k	
	1	-3	8	[2, 1, 3, 1, 2, 8]	6	7-10	k	
77	1	-7	7	[1, 7]	2	1-7	k	
	7	-1	7	[7, 1]	2	7-1	k	
80	4	-4	4	[1]	1	1-1	super	$\star$
	2	-2	8	[4]	1	4-4	super	$\star$
	1	-4	8	[2, 8]	2	2-8	k	
	4	-1	8	[8, 2]	2	8-2	k	
84	6	-2	6	[3, 1]	2	3-1	k	$\star$
	2	-6	6	[1, 3]	2	1-3	k	$\star$
	4	-3	6	[2, 1, 1, 8, 1, 1]	6	4-10	k	
	3	-4	6	[1, 1, 8, 1, 1, 2]	6	10-4	k	
85	3	-3	7	[2, 1, 2]	3	5-5	super	
	1	-1	9	[9]	1	9-9	super	
88	2	-3	8	[2, 1, 8, 1, 2, 4]	6	12-6	k	
	4	-2	8	[4, 2, 1, 8, 1, 2]	6	6-12	k	
89	4	-4	5	[1, 1, 4, 9, 4, 1, 1]	7	21-21	super	
92	1	-7	8	[1, 3, 1, 8]	4	2-11	k	
	7	-1	8	[8, 1, 3, 1]	4	11-2	k	
93	1	-3	9	[3, 9]	2	3-9	k	
	3	-1	9	[9, 3]	2	9-3	k	
96	5	-3	6	[2, 1, 1, 1]	4	3-2	k	
	3	-5	6	[1, 1, 1, 2]	4	2-3	k	
	2	-4	8	[2, 4]	2	2-4	k	$\star$
	4	-2	8	[4, 2]	2	4-2	k	$\star$
	1	-8	8	[1, 8]	2	1-8	k	
	8	-1	8	[8, 1]	2	8-1	k	
97	2	-2	9	[4, 1, 2, 2, 9, 2, 2, 1, 4]	9	27-27	super	

Tables of classes not representing zero with $\Delta < 100$ (continuation).

Δ	m	n	k	k/m	N	t	$t^{\uparrow} - t_{\downarrow}$	symm.	n.p.
1	0	0	1	0	0	0	0-0	super	
4	0	0	2	0	0	0	0-0	super	$\star$
	1	0	2	[2]	1	1	0-0	super	
9	0	0	3	0	0	0	0-0	super	$\star$
	1	0	3	[3]	1	2	1-0	k	
	2	0	3	[1, 1, 1]	3	2	0-1	k	
16	0	0	4	0	0	0	0-0	super	$\star$
	1	0	4	[4]	1	3	2-0	k	
	2	0	4	[2]	1	1	0-0	super	$\star$
	3	0	4	[1, 2, 1]	3	3	0-2	k	
25	0	0	5	0	0	0	0-0	super	$\star$
	1	0	5	[5]	1	4	3-0	k	
	2	0	5	[2, 2]	2	3	1-1	$m+n$	
	3	0	5	[1, 1, 1, 1]	4	3	1-1	$m+n$	
	4	0	5	[1, 3, 1]	3	4	0-3	k	
36	0	0	6	0	0	0	0-0	super	$\star$
	1	0	6	[6]	1	5	4-0	k	
	2	0	6	[3]	1	2	1-0	k	$\star$
	3	0	6	[2]	1	1	0-0	super	$\star$
	4	0	6	[1, 1, 1]	3	2	0-1	k	$\star$
	5	0	6	[1, 4, 1]	3	5	0-4	k	
49	0	0	7	0	0	0	0-0	super	$\star$
	1	0	7	[7]	1	6	5-0	k	
	2	0	7	[3, 1, 1]	3	4	2-1	asym	
	3	0	7	[2, 2, 1]	3	4	1-2	asym	
	4	0	7	[1, 1, 2, 1]	4	4	2-1	asym	
	5	0	7	[1, 2, 1, 1]	4	4	1-2	asym	
	6	0	7	[1, 5, 1]	3	6	0-5	k	
64	0	0	8	0	0	0	0-0	super	$\star$
	1	0	8	[8]	1	7	6-0	k	
	2	0	8	[4]	1	3	2-0	k	$\star$
	3	0	8	[2, 1, 2]	3	4	2-1	k	
	4	0	8	[2]	1	1	0-0	super	$\star$
	5	0	8	[1, 1, 1, 1, 1]	5	4	1-2	k	
	6	0	8	[1, 2, 1]	3	3	0-2	k	$\star$
	7	0	8	[1, 6, 1]	3	7	0-6	k	

Tables of classes not representing zero with $\Delta \leq 100$.

Δ	m	n	k	k/m	N	t	$t^{\uparrow} - t_{\downarrow}$	symm.	n.p.
81	0	0	9	0	0	0	0-0	super	⋆
	1	0	9	[9]	1	8	7-0	k	
	2	0	9	[4, 1, 1]	3	5	3-1	asym	
	3	0	9	[3]	1	2	1-0	k	⋆
	4	0	9	[2, 3, 1]	3	5	1-3	asym	
	5	0	9	[1, 1, 3, 1]	4	5	3-1	asym	
	6	0	9	[1, 1, 1]	3	2	0-1	k	⋆
	7	0	9	[1, 3, 1, 1]	4	5	1-3	asym	
	8	0	9	[1, 7, 1]	3	8	0-7	k	
100	0	0	10	0	0	0	0-0	super	⋆
	1	0	10	[10]	1	9	8-0	k	
	2	0	10	[5]	1	4	3-0	k	⋆
	3	0	10	[3, 3]	2	5	2-2	$m+n$	
	4	0	10	[2, 2]	2	3	1-1	$m+n$	⋆
	5	0	10	[2]	1	1	0-0	super	⋆
	6	0	10	[1, 1, 1, 1]	4	3	1-1	$m+n$	⋆
	7	0	10	[1, 2, 2, 1]	4	5	2-2	$m+n$	
	8	0	10	[1, 3, 1]	3	4	0-3	k	⋆
	9	0	10	[1, 8, 1]	3	9	0-8	k	

Tables of classes not representing zero with $\Delta \leq 100$ (continuation).