

Codimension one generic homoclinic classes with interior

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Abstract. We study C^1 -generic diffeomorphisms with a homoclinic class with non empty interior and in particular those admitting a codimension one dominated splitting. We prove that if in the finest dominated splitting the extreme subbundles are one dimensional then the diffeomorphism is partially hyperbolic and from this we deduce that the diffeomorphism is transitive.

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1 Introduction

It is a main problem in generic dynamics to understand the structure of homoclinic classes, this has become most important after some results in [4] which raised the interest in the study of chain recurrence classes and in particular homoclinic classes which are, generically, the chain recurrence classes containing periodic points. This results, as most of the results (including the ones here presented) are in the C^1 category, very little is known about C^r generic diffeomorphisms with $r > 1$ (see [15]).

A lot is known in the case of an isolated homoclinic class $H(p, f)$ of a hyperbolic periodic point p (i.e., it is maximal invariant in a neighborhood) of a *generic diffeomorphism* f , see for instance [5], chapter 10 (we remark here that the genericity of f implies that the continuation $H(p_g, g)$ is also isolated for g in a neighborhood of f). The key point of knowing that the class is isolated is that, after perturbation, orbits that remains in a neighborhood must belong to the continuation of the homoclinic class $H(p_g, g)$. However, if one does not know *a priori* that the homoclinic class is isolated (in other words, the class

might be wild, i.e., non isolated) only very sparse results have been obtained (see for example [2] where they prove that generic homoclinic classes are index complete). The main difficulty is to overcome the fact that after perturbations one cannot ensure that the perturbed points remain in the class.

For example, it is not known whether the non-wandering set of a C^1 -generic diffeomorphism may have nonempty interior and not coincide with the whole manifold. We treat this problem in this paper solving it in some particular cases and also giving some results which may help to obtain a general solution. We remark that it is not difficult to construct examples of diffeomorphisms such that its non-wandering set has non empty interior and doesn't coincide with the whole manifold.

1.1 Definitions and statement of results

Let M be a compact connected boundaryless manifold of dimension d and let $\text{Diff}^1(M)$ be the set of diffeomorphisms of M endowed with the C^1 topology. We shall say that a property (or a diffeomorphism) is *generic* if and only if there exists a residual (G_δ -dense) set \mathcal{R} of $\text{Diff}^1(M)$ for which for every $f \in \mathcal{R}$ satisfies that property.

The main result of this paper concerns the following conjecture of [3], we shall denote $\Omega(f)$ to the nonwandering set of f :

Conjecture 1. *There is a residual set $\mathcal{R} \subset \text{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and $\text{int}(\Omega(f)) \neq \emptyset$ then f is transitive.*

For a hyperbolic periodic point $p \in M$ of some diffeomorphism f we denote its *homoclinic class* by $H(p, f)$, defined as the closure of the transversal intersections between the stable and unstable manifolds of the orbit of p . For generic diffeomorphisms, if the nonwandering set has nonempty interior, then there is a homoclinic class with nonempty interior (see [3] or [4]). So, the conjecture is reduced to the study of homoclinic classes with nonempty interior.

Some progress has been made towards the proof of this conjecture (see [3] and [1]), in particular, it has been proved in [3] that isolated homoclinic classes as well as homoclinic classes admitting a strong partially hyperbolic splitting (we shall define this concept later) verify the conjecture. Also, they proved that a homoclinic class with non empty interior must admit a dominated splitting (see Theorem 8 in [3]). In [1] the conjecture was proved for surface diffeomorphisms.

In [3] the question about whether within the finest dominated splitting the extremes subbundles should be volume hyperbolic was posed. We give a pos-

itive answer when the class admits codimension one dominated splitting. This gives also new situations where the above conjecture holds and weren't known.

Let us recall the definition of dominated splitting: a compact set H invariant under a diffeomorphism f admits dominated splitting if the tangent bundle over H splits into two Df invariant subbundles $T_H M = E \oplus F$ such that there exist $C > 0$ and $0 < \lambda < 1$ such that for all $x \in H$:

$$\|Df^n_{|E(x)}\| \|Df^{-n}_{|F(f^n(x))}\| \leq C\lambda^n$$

we say in this case that F dominates E .

Let us remark that Gourmelon ([9]) proved that there always exists an adapted metric for which $C = 1$.

We shall say that a bundle F is uniformly expanding (contracting) if there exists $n_0 < 0$ ($n_0 > 0$) such that $\|Df^{n_0}_{|F(x)}\| < 1/2 \forall x \in H$.

The main theorem of this paper is the following

Theorem 1. *Let f be a generic diffeomorphism with a homoclinic class H with non empty interior and admitting a codimension one dominated splitting $T_H M = E \oplus F$ where $\dim(F) = 1$. Then, the bundle F is uniformly expanding for f .*

As a consequence of our main theorem we get the following easy corollaries.

Recall that a compact invariant set H is strongly partially hyperbolic if it admits a three ways dominated splitting $T_H M = E^s \oplus E^c \oplus E^u$ (that is, $E^s \oplus E^c$ is dominated by E^u and E^s is dominated by $E^c \oplus E^u$), where E^s is non trivial and uniformly contracting and E^u is non trivial and uniformly expanding.

Corollary 1. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E^1 \oplus E^2 \oplus E^3$ is a dominated splitting for f and $\dim(E^1) = \dim(E^3) = 1$. Then, H is strongly partially hyperbolic and $H = M$.*

Proof. The class should be strongly partially hyperbolic because of the previous theorem (applied to f and to f^{-1}). Corollary 1 of [3] (page 185) implies that $H = M$. \square

We say that a homoclinic class H is far from tangencies if there is a neighborhood of f such that there are no homoclinic tangencies associated to periodic points in the continuation of H . The tangencies are of index i if they are associated to a periodic point of index i , that is, its stable manifold has dimension i . We get the following result following the generalization of the results of [18] in [10] (see also [2]):

Corollary 2. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that H is far from tangencies of index 1 and $n - 1$ and has index 1 and $n - 1$ periodic points. Then, $H = M$.*

Proof. Since the class is far from tangencies, and the classes for generic diffeomorphisms either coincide or are disjoint (see [4]) we have that using [10] the class must admit a dominated splitting with one dimensional extremal subbundles (see also [2] Corollary 3), thus, by using Corollary 1 we get the result. \square

In fact, the previous corollary can be compared to a corollary of a new result of Yang ([20]) on Lyapunov stable homoclinic classes far from tangencies. Yang's result (Theorem 3 in [20]) implies that a generic homoclinic class with nonempty interior and far away from (any) tangencies must be strongly partially hyperbolic or contained in the closure of the set of sinks and sources, and thus (using the results of [3]) the whole manifold.

Incidentally, we also give a new proof in the two dimensional case:

Corollary 3. *Let f be a generic surface diffeomorphism having a homoclinic class with nonempty interior. Then f is conjugated to a linear Anosov diffeomorphism in \mathbb{T}^2 .*

Proof. Since the class must admit dominated splitting (Theorem 8 of [3]), this should be into 2 one dimensional subbundles. So, the class must be hyperbolic and thus, since the conjecture holds for hyperbolic homoclinic classes f is Anosov (the rest follows from classical theory of Anosov diffeomorphisms). \square

1.2 Idea of the proof

The idea of the proof is the following.

First we prove that if the homoclinic class has interior, the periodic points in the class (which are all saddles) should have eigenvalues (in the F direction) exponentially (with the period) far from 1. Otherwise we manage to obtain a sink or a source inside the interior of the class and thus contradicting the fact that the interior of the homoclinic class for generic diffeomorphisms is, roughly speaking, robust (Theorem 4 of [3]).

Then, using the previous fact and some results of [12] and [16] we manage to prove that the center manifolds integrating a one dimensional extreme subbundle should have nice dynamical properties. For this we also use the connecting lemma for pseudo orbits of [4].

Finally, in the event that the extreme subbundle is not hyperbolic, we manage to obtain (using dynamical properties and a Lemma of Liao) periodic points near the class with bad contraction or expansion in those extreme subbundles. Using Lyapunov stability of the homoclinic class (which is generic, see [3] and [6]) we ensure that the periodic points we found belong to the class and thus reach a contradiction.

2 Preliminary results

In this section we shall state some results we are going to use in the proof of the main theorem. It can be skipped and used as reference when the results are used.

Some generic properties of diffeomorphisms are contained in the following Theorem (see [3] and references therein):

Theorem 2. *There exists a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$*

- a1) *f is Kupka Smale (that is, all its periodic points are hyperbolic and their invariant manifolds intersect transversally).*
- a2) *The periodic points of f are dense in the chain recurrent set of f ⁽¹⁾. Moreover, homoclinic classes coincide with those chain recurrent classes which contain periodic points.*
- a3) *Every homoclinic class with non empty interior of f is Lyapunov stable for f and f^{-1} ⁽²⁾. This implies that the stable and unstable set of any point in the class is contained in the class.*
- a4) *For every periodic point p of f , $H(p, f) = \overline{W^s(p)} \cap \overline{W^u(p)}$.*
- a5) *Given a homoclinic class H of a periodic point p , if U is an open set such that $\bar{U} \subset \text{int}(H)$ then there exists \mathcal{U} neighborhood of f such that for every $g \in \mathcal{U} \cap \mathcal{R}$, $U \subset H(p_g, g)$ is satisfied (where p_g is the continuation of p for g).*
- a6) *Homoclinic classes vary continuously with the Hausdorff distance with respect to f . This means, that given $p \in H$ a periodic point and $\varepsilon >$*

¹The chain recurrent set is the set of points x satisfying that for every $\varepsilon > 0$ there exist an ε -pseudo orbit from x to x , that is, there exist points $x = x_0, x_1, \dots, x_k = x$, $k > 0$ such that $d(f(x_i), x_{i+1}) < \varepsilon$.

²Lyapunov stability of Λ means that $\forall U$ neighborhood of Λ there is $V \subset U$ neighborhood of Λ such that $f^n(V) \subset U \forall n \geq 0$.

0, there exists \mathcal{U} a neighborhood of f such that for every $g \in \mathcal{U}$, the homoclinic class of the continuation of p lies within less than ε from H in the Hausdorff distance.

To obtain dynamical properties of the center manifolds we shall use the following results from [12] and [16]. First recall that if $T_H M = E \oplus F$ is a dominated splitting then, Theorem 5.5 of [11] gives us a local f -invariant manifolds W_ε^F tangent to F .

Local f -invariance means that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$f^{-1}(W_\delta^F(x)) \subset W_\varepsilon^F(f(x)).$$

Taking f^{-1} we have an analog for E .

Theorem 3 (Main Theorem of [12]). *Let Λ a compact invariant set of a generic diffeomorphism f admitting a codimension one dominated splitting $T_\Lambda M = E \oplus F$ with $\dim(F) = 1$. Assume that $\overline{\text{Per}(f|_\Lambda)} = \Lambda$. Then, $\forall x \in \Lambda$ and $\forall \varepsilon > 0$ there exists $\delta > 0$ such that*

$$f^{-n}(W_\delta^F(x)) \subset W_\varepsilon^F(f^{-n}(x)) \quad \forall n \geq 0$$

In particular, $W_\delta^F(x) \subset \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon\}$.

Remark 2.8 of [7] gives a similar result that is enough in our context.

If there is a dominated splitting for H of the form $T_H M = E \oplus F$, then, there exists V , a neighborhood of H such that if a point z satisfies that $f^n(z) \in V \quad \forall n \in \mathbb{Z}$ then we can define the splitting for z and it will be dominated (see [5]). Such a neighborhood will be called adapted.

If I is an interval, we denote by

$$\omega(I) = \bigcup_{x \in I} \omega(x), \quad \text{and by} \quad W_\varepsilon^{ss}(I) = \bigcup_{x \in I} W_\varepsilon^{ss}(x)$$

its strong stable manifold. Also $\ell(I)$ denote its length. We shall state the following result which is an immediate Corollary of Theorem 3.1 of [16] for generic dynamics.

Theorem 4 ([16]). *Let $f \in \text{Diff}^1(M)$ a generic diffeomorphism and Λ compact invariant set admitting a codimension one dominated splitting $T_\Lambda M = E \oplus F$ (where $\dim(F) = 1$). Then, there exists δ_0 such that if I is an interval integrating the subbundle F satisfying $\ell(f^n(I)) < \delta < \delta_0 \quad \forall n \geq 0$ and that its orbit remains in an adapted neighborhood V of Λ , then, only one of the following holds:*

1. $\omega(I)$ is contained in the set of periodic points of f restricted to V of Λ one of which is an attractor.
2. I is wandering (that is, $W_\varepsilon^{ss}(f^n(I)) \cap W_\varepsilon^{ss}(f^m(I)) = \emptyset$ for all $n \neq m$). This implies that $\ell(f^n(I)) \rightarrow 0$ as $|n| \rightarrow \infty$.

Other result we shall use is the following well known Lemma of Franks:

Theorem 5 (Frank's Lemma [8]). *Let $f \in \text{Diff}^1(M)$. Given $\mathcal{U}(f)$, a C^1 neighborhood of f , then $\exists \mathcal{U}_0(f)$ and $\varepsilon > 0$ with the following property: if $g \in \mathcal{U}_0(f)$, $\theta = \{x_1, \dots, x_m\}$ and*

$$L : \bigoplus_{x_i \in \theta} T_{x_i} M \rightarrow \bigoplus_{x_i \in \theta} T_{g(x_i)} M \quad \text{such that} \quad \|L - Dg|_{\bigoplus T_{x_i} M}\| < \varepsilon$$

are given, then there exists $\tilde{g} \in \mathcal{U}(f)$ such that $D\tilde{g}_{x_i} = L|_{T_{x_i} M}$. Moreover if R is a compact set disjoint from θ we can consider $\tilde{g} = g$ in $R \cup \theta$.

Finally we state the following Lemma of Liao. A proof can be found (with the same notation) in [19]. We shall state the result in the particular case of index one dominated splitting with an adapted metric (which always exist because of [9]), but it holds in a wider context. Recall also that for linear maps A_i in one dimensional spaces it holds that $\prod_i \|A_i\| = \|\prod_i A_i\|$.

Lemma 1 (Liao [13]). *Let Λ be a compact invariant set of f with dominated splitting $T_H M = E \oplus F$ such that $\|Df|_{E(x)}\| \|Df|_{F(x)}^{-1}\| < \gamma \forall x \in \Lambda$ and $\dim(F) = 1$. Assume that*

1. *There is a point $b \in \Lambda$ such that $\|Df^{-n}|_{F(b)}\| \geq 1 \forall n \geq 0$.*
2. *There exists $\gamma < \gamma_1 < \gamma_2 < 1$ such that given $x \in \Lambda$ satisfying*

$$\|Df|_{F(x)}^{-n}\| \geq \gamma_2^n \forall n \geq 0$$

we have that there is $y \in \omega(x)$ satisfying

$$\|Df|_{F(y)}^{-n}\| \leq \gamma_1^n \forall n \geq 0.$$

Then, for any $\gamma_2 < \gamma_3 < \gamma_4 < 1$ and any neighborhood U of Λ there exists a periodic point p of f whose orbit lies in U , is of the same index as the dominated splitting and satisfies $\|Df|_{F(p)}^{-n}\| < \gamma_4^n \forall n \geq 0$ and $\|Df|_{F(p)}^{-n}\| \geq \gamma_3^n \forall n \geq 0$.

3 Proof of the main theorem

For $p \in \text{Per}(f)$, $\pi(p)$ denotes the period of p .

Lemma 2. *Let H be a homoclinic class with interior of a generic diffeomorphism f such that $T_H M = E \oplus F$ is a dominated splitting with $\dim F = 1$. Then, there exists $\lambda < 1$ such that for all $p \in \text{Per}(f|_H)$ the following holds:*

$$\|Df_{/F(p)}^{-\pi(p)}\| \leq \lambda^{\pi(p)}$$

Proof. Arguing by contradiction assume that the conclusion does not hold, that is, for every $\lambda < 1$ there exists $p \in \text{Per}(f|_H)$ such that $\|Df_{/F(p)}^{-\pi(p)}\| \geq \lambda^{\pi(p)}$ which is equivalent to $\|Df_{/F(p)}^{\pi(p)}\| \leq \lambda^{-\pi(p)}$ since F is one dimensional. When the class is isolated this is enough since one can perturb the orbit in order to create a sink contradicting the isolation. Here, since the class may be wild, the creation of the sink represents no contradiction, so we must use the persistence of the interior given by generic property *a5*) of Theorem 2 and create a sink there.

Let U be an open set such that $\overline{U} \subset \text{int}(H)$. Since f is generic, property *a5*) of Theorem 2 ensure us the existence of a neighborhood \mathcal{U} of f such that for every g in a residual subset of \mathcal{U} we have $U \subset H_g$ (H_g is the continuation of H for g , from *a5*) of Theorem 2 this continuation makes sense since it will be the only class containing U).

Frank's Lemma implies the existence of $\varepsilon > 0$ such that if we fix an arbitrary finite set of points, we can perturb the diffeomorphism as near as we want of those points obtaining a new diffeomorphism with arbitrary derivatives (ε -close to the originals) in those points and such that the diffeomorphism lies inside \mathcal{U} .

Let us fix $1 > \lambda > 1 - \varepsilon/2$ and let $p \in \text{Per}(f|_H)$ as before. Since f is generic, the periodic points of the same index as p are dense in H so, we can choose $q \in U \cap \text{Per}(f)$ homoclinically related to p .

Let $x \in W^s(p) \cap W^u(q)$ and $y \in W^s(q) \cap W^u(p)$, we get that the set $\Lambda = \mathcal{O}(p) \cup \mathcal{O}(q) \cup \mathcal{O}(x) \cup \mathcal{O}(y)$ hyperbolic.

Consider the following periodic pseudo orbit contained in Λ ,

$$\{\dots, p, f(p), \dots, f^{N\pi(p)-1}(p), f^{-n_0}(y), \dots, f^{n_0}(y), f^{-n_0}(x), \dots, f^{n_0}(x), p, \dots\}$$

which we shall denote as \wp^N . Clearly, given $\beta > 0$ there exists n_0 such that \wp^N is a β -pseudo orbit. At the same time, if we choose N large enough we

obtain a pseudo orbit which stays near p much longer than of q and then inherit the behavior of the derivative of p rather than that of q .

The shadowing lemma for hyperbolic sets (see [17]) implies that for every $\alpha > 0$ there exists β such that every closed β -pseudo orbit is α -shadowed by a periodic point. So, let us choose α in such a way that the following conditions are satisfied:

$$(a) \quad B_{2\alpha}(q) \subset U.$$

(b) If $d(z, w) < \alpha$ and x, y are in an adapted neighborhood of H then,

$$\frac{\|Df_{/F(z)}\|}{\|Df_{/F(w)}\|} < 1 + c$$

$$(c \text{ verifies } (1 + c)\left(1 - \frac{\varepsilon}{2}\right)^{-1} < 1 + \varepsilon).$$

Let $\beta < \alpha$ be given from the Shadowing Lemma for that α and let n_0 be such that \wp^N is a β -pseudo orbit. Therefore there exists a periodic orbit r of period $\pi(r) = N\pi(p) + 4n_0$ which α -shadows \wp^N . Therefore, setting $k = \sup_{x \in M} \|Df_x\|$, we have

$$\begin{aligned} \|Df_{/F(r)}^{N\pi(p)+4n_0}\| &\leq k^{4n_0}(1+c)^{N\pi(p)}\|Df_{/F(p)}^{\pi(p)}\|^N \\ &\leq k^{4n_0}\left((1+c)\left(1-\frac{\varepsilon}{2}\right)^{-1}\right)^{N\pi(p)} \\ &< (1+\varepsilon)^{\pi(r)} \end{aligned}$$

where the last inequality holds provided N is large enough. Notice that the orbit of r passes through U . On the other hand, by domination, we have that

$$\|Df_{/E(r)}^{\pi(r)}\| < \|Df_{/F(r)}^{\pi(r)}\|.$$

Since E and F are invariant we conclude that any eigenvalue of $Df_r^{\pi(r)}$ is less than $(1 + \varepsilon)^{\pi(r)}$.

Now, if we compose in the orbit of r its derivatives with homoteties of value $(1 + \varepsilon)^{-1}$ we obtain, by using Frank's Lemma, a diffeomorphism g so that all the eigenvalues associated to the periodic orbit r are less than 1, that is, r is a periodic attractor (sink). This contradicts the generic assumption, since the sink is persistent, so every residual $\mathcal{R} \in \mathcal{U}$ will have diffeomorphisms with a sink near r , thus contained in U , and thus contradicting that the interior is persistent. \square

Lemma 3. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E \oplus F$ is a dominated splitting with $\dim(F) = 1$. Then, there exists ε_0 such that for all $\varepsilon \leq \varepsilon_0$ there exists δ such that $\forall x \in H$,*

$$W_\delta^F(x) \subset W_\varepsilon^{uu}(x) := \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon ; \\ d(f^{-n}(x), f^{-n}(y)) \rightarrow 0\}.$$

Proof. First we shall prove the Lemma for periodic points and then, using this fact prove the general statement. Let $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(H)$ is contained in the adapted neighborhood of H and such that if $d(x, y) < \varepsilon_0$ then

$$\frac{\|Df_{/F(x)}^{-1}\|}{\|Df_{/F(y)}^{-1}\|} < \lambda^{-1}$$

where λ is given by Lemma 2. Let $\varepsilon \leq \varepsilon_0$ and let $\delta > 0$ from Theorem 3 corresponding to this ε .

Let $p \in \text{Per}(f|_H)$ for which there is $y \in W_\delta^F(p)$ such that $d(f^{-n}(y), f^{-n}(p)) \not\rightarrow 0$. Since $W_\delta^F(p)$ is one dimensional, $W_\delta^F(p) \setminus \{p\}$ is a disjoint union of two intervals. Denote I_δ the connected component of $W_\delta^F(p) \setminus \{p\}$ that contains y . By Theorem 3 we have either $f^{2\pi(p)}(I_\delta) \subset I_\delta$ or $f^{2\pi(p)}(I_\delta) \supset I_\delta$. In any event, since $y \in I_\delta$ we conclude that there exists a point $z_0 \in W_\varepsilon^F(p)$ fixed under $f^{2\pi(p)}$ and such that $\|Df_{/F(z_0)}^{2\pi(p)}\| \leq 1$.

This contradicts the previous Lemma, since by the way ε was chosen we get (since we know that $d(f^i(p), f^i(z_0)) < \varepsilon$ for all i) that

$$\begin{aligned} \|Df_{/F(p)}^{2\pi(p)}\| &= \prod_{i=0}^{2\pi p-1} \|Df_{/F(f^i(p))}\| < \lambda^{-2\pi(p)} \prod_{i=0}^{2\pi p-1} \|Df_{/F(f^i(z_0))}\| \\ &= \lambda^{-2\pi(p)} \|Df_{/F(z_0)}^{2\pi(p)}\| < \lambda^{-2\pi(p)} \end{aligned}$$

Now, let's prove the general statement. Let us suppose that for every $\varepsilon > 0$ there exist $x \in H$ and a small arc $I \subset W_\delta^F(x)$ containing x such that $\ell(f^{-n}(I)) \not\rightarrow 0$. We know, because of Theorem 3 that $\ell(f^{-n}(I)) \leq \varepsilon$, then, taking $n_j \rightarrow +\infty$ such that $\gamma \leq \ell(f^{-n_j}(I)) \leq \varepsilon$ and taking limits, we obtain an arc J integrating F such that $\ell(f^n(J)) \leq \varepsilon \forall n \in \mathbb{Z}$ and containing a point $z \in J \cap H$ (a limit point of $f^{-n_j}(x)$).

Now, we shall use Theorem 4 to reach a contradiction. It is not difficult to discard the first possibility in the Theorem because it will contradict what we have proved for periodic points.

On the other hand, if J is wandering, we know that it can not be accumulated by periodic points. Since f is generic, we reach a contradiction if we prove that the points in J are chain recurrent (see property $a2$) of Theorem 2). Theorem 4, implies that, $\ell(f^n(J)) \rightarrow 0$ ($|n| \rightarrow +\infty$), then, since $z \in H \cap J$, if we fix ϵ , and $y \in J$, then, for some future iterate k_1 and a past one $-k_2$, we know that $f^{k_1}(y)$ is ϵ -near of $f^{k_1}(z)$ and $f^{-k_2}(y)$ is ϵ -near $f^{-k_2}(z)$. Since homoclinic classes are chain recurrent classes, there is an ϵ pseudo orbit from $f^{k_1}(z)$ to $f^{-k_2}(z)$ and then, y is chain recurrent, a contradiction. \square

Corollary 4. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E \oplus F$ is a dominated splitting with $\dim(F) = 1$. Then, F is uniquely integrable.*

Proof. It follows from the fact that the center stable manifold is dynamically defined (see [11]). \square

Uniqueness of the center manifolds imply that one can know that if you have a point $y \in W_\delta^F(x) \cap H$ then there exists $\gamma < \delta$ such that $W_\gamma^F(y) \subset W_\delta^F(x)$.

Corollary 5. *Let $H = H(p, f)$ be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E \oplus F$ is a dominated splitting with $\dim(F) = 1$. Then, for all $L > 0$ and $l > 0$ there exists n_0 such that if I is a compact arc integrating F whose length is smaller than L , then $\ell(f^{-n}(I)) < l \forall n > n_0$.*

Proof. It is easy to see that every compact arc integrating F should have its iterates of length going to zero in the past because of Theorem 3 (it is enough to consider a finite covering of I where the Theorem applies).

Lets suppose then that there exists L and l such that for every $j > 0$ there is an arc I_j integrating F of length smaller than L and $n_j > j$ such that $\ell(f^{-n_j}(I_j)) \geq l$. We can suppose without loss of generality that $\ell(I_j) \in (L/2, L)$.

Also, we can assume (maybe considering subsequences) that I_j converges uniformly to an arc J integrating F and verifying $L/2 \leq \ell(J) \leq L$.

Since the length of J is finite and it integrates F we know that $\ell(f^{-n}(J)) \rightarrow 0$ with $n \rightarrow +\infty$.

Let $\epsilon = l/2$ and δ given by Theorem 3 which ensures that $W_\delta^2(x) \subset W_\epsilon^u(x) \forall x$.

Let n_0 such that $\forall n \geq n_0$ we have $\ell(f^{-n}(J)) < \delta/4$. Let also be γ small enough such that if $x \in B_\gamma(J)$ then $d(f^{-k}(x), f^{-k}(J)) < \delta/4 \forall 0 \leq k \leq n_0$.

Now, if we consider j large enough (in particular $j > n_0$) such that $I_j \subset B_\gamma(J)$ we obtain $\ell(f^{-n_0}(I_j)) < \delta$ and so $\ell(f^{-n}(I_j)) < \varepsilon < l \forall n \geq n_0$, so, $n_j < n_0$ which is a contradiction. \square

We are ready to give the proof of our main theorem:

Theorem 6. *Let H be a homoclinic class with non empty interior for a generic diffeomorphism f such that $T_H M = E \oplus F$ is a dominated splitting with $\dim(F) = 1$. Then, F is uniformly expanding*

Proof. Because of the existence of an adapted norm for the dominated splitting (see [9]) we can assume that $\|Df|_{E(x)}\| \|Df^{-1}|_{F(f(x))}\| < \gamma$ (for the sake of simplicity).

Suppose the theorem is not true. Thus, for every $0 < \nu < 1$ there exists some $x \in H$ such that $\|Df_{/F(x)}^{-n}\| \geq \nu$, $\forall n \geq 0$ (otherwise for every x there would be some $n_0(x)$ which would be the first one for which $\|Df_{/F(x)}^{-n}\| < \nu$ and by compactness, the $n_0(x)$ are uniformly bounded, then F would be hyperbolic). If we choose points x_m satisfying $\|Df_{/F(x)}^{-n}\| \geq 1 - 1/m \forall n \geq 0$, a limit point x will satisfy $\|Df_{/F(x)}^{-n}\| \geq 1 \forall n \geq 0$.

First of all, we consider the case where we cannot use the Shifting Lemma of Liao (Lemma 1). It is not difficult to see that this implies (using Pliss' Lemma, see also [19]) that $\forall \gamma < \gamma_1 < \gamma_2 < 1$, there exists $x \in H$ such that

$$\|Df_{/F(x)}^{-n}\| \geq \gamma_2^n \quad \forall n \geq 0$$

but, $\forall y \in \omega(x)$ we have that

$$\|Df_{/F(y)}^{-n}\| \geq \gamma_1^n \quad \forall n \geq 0$$

So, if we work in $\omega(x)$ which is a closed invariant set, we have that the subbundle E will be hyperbolic since the dominated splitting implies that $\forall z \in \omega(x)$

$$\|Df_{/E(z)}\| < \frac{\gamma}{\|Df_{/F(f(z))}^{-1}\|} < \frac{\gamma}{\gamma_1} < 1.$$

This implies that, since we have dynamical properties for the manifolds integrating the subbundle F , that we can shadow recurrent orbits. Indeed, if we have a recurrent point $y \in \omega(x)$, for every small ε (in particular, such that the stable and unstable manifolds of y are well defined) we can consider n large enough so that $d(f^n(y), y) \leq \varepsilon/3$, $f^n(W_\varepsilon^E(y)) \subset W_{\varepsilon/3}^E(f^n(y))$ and $f^{-n}(W_\varepsilon^F(f^n(y))) \subset W_{\varepsilon/3}^F(y)$ which gives us (using classical arguments) a

periodic point p of f which verifies that has period n and remains ε -close to the first n iterates of y . It is not difficult to see that we can consider this periodic point to be of index 1 and such that its stable manifold intersects the unstable manifold of y . So, $p \in \overline{W^u(H)}$ and using Lyapunov stability of H we know $\overline{W^u(H)} \subset H$ (see [6] Lemma 2.1), so $p \in H$.

Since γ_1 was arbitrary, we can choose it to satisfy $\gamma_1 > \lambda$ where λ is as in Lemma 2. Also, we can choose ε small so that $\|Df_{/F(p)}^{-n}\| > \lambda^n$ contradicting Lemma 2.

Now, we shall study what happens if Liao's shifting Lemma can be applied. That is, there exists $\gamma < \gamma_1 < \gamma_2 < 1$ such that for all $x \in H$ satisfying

$$\|Df_{/F(x)}^{-n}\| \geq \gamma_2^n \quad \forall n \geq 0$$

there exists $y \in \omega(x)$ such that

$$\|Df_{/F(x)}^{-n}\| \leq \gamma_1^n \quad \forall n \geq 0$$

So, using the Shifting Lemma we have that for every $\gamma_2 < \gamma_3 < \gamma_4 < 1$ we have a periodic orbit p_U of f contained in any neighborhood U of Λ and satisfying that

$$\|Df_{/F(p)}^{-n}\| \leq \gamma_4^n \quad \text{and} \quad \|Df_{/F(f^i(p))}^{-n}\| \geq \gamma_3^n$$

for some $i \in 0, \dots, \pi(p)$ (remember that F is one dimensional, so the product of norms is the norm of the product). But since this periodic points are not very contracting in the direction F , if we choose $\gamma_3 > \lambda$ (as before) and U sufficiently small to ensure that the stable manifold of some periodic point will intersect the unstable one of a point in H we reach the same contradiction as before. \square

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References

- [1] F. Abdenur, C. Bonatti, S. Crovisier and L. Diaz. *Generic diffeomorphisms on compact surfaces*. Fund. Math., **187** (2005), 127–159.
- [2] F. Abdenur, C. Bonatti, S. Crovisier, L. Diaz and L. Wen. *Periodic points and homoclinic classes*. Ergodic Theory Dynam. Systems, **27** (2007), 1–22.

- [3] F. Abdenur, C. Bonatti and L. Díaz. *Nonwandering sets with non empty interior*. Nonlinearity, **17** (2004), 175–191.
- [4] C. Bonatti and S. Crovisier. *Recurrence et Genericite*. Inventiones Math., **158** (2004), 33–104.
- [5] C. Bonatti, L.J. Díaz and M. Viana. *Dynamics Beyond Uniform Hyperbolicity*. Springer-Verlag (2005).
- [6] C.M. Carballo, C.A. Morales and M.J. Pacifico. *Homoclinic classes for generic C^1 vector fields*. Ergodic Theory Dynam. Systems, **23** (2003), 403–415.
- [7] S. Crovisier. *Partial hyperbolicity far from homoclinic bifurcations*. Preprint Arxiv (2008).
- [8] J. Franks. *Necessary conditions for stability of diffeomorphisms*. Transactions of the A.M.S., **158** (1971), 301–308.
- [9] N. Gourmelon. *Adapted metrics for dominated splitting*. Ergod. Th. and Dyn. Sys., **27** (2007), 1839–1849.
- [10] N. Gourmelon. *Generation of homoclinic tangencies by C^1 perturbations*. Disc. and Cont. Dynamical Systems A, **26** (2010), 1–42.
- [11] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds*. Springer Lecture Notes in Math., **583** (1977).
- [12] P. Lessa and M. Sambarino. *Invariant Manifolds for codimension one dominated splitting*. Preprint.
- [13] S. Liao. *On the stability conjecture*. Chinese Annals of Math., **1** (1980), 9–30.
- [14] R. Mañé. *A proof of the C^1 stability conjecture*. Publications del IHES (1987).
- [15] E. Pujals. *Some simple questions related to the C^r stability conjecture*. Nonlinearity, **21** (2008).
- [16] E.R. Pujals and M. Sambarino. *Integrability on codimension one dominated splitting*. Bull. Braz. Math. Soc., N.S., **38** (2007), 1–19.
- [17] M. Shub. *Global Stability of Dynamical Systems*. Springer-Verlag (1987).
- [18] L. Wen. *Homoclinic tangencies and dominated splitting*. Nonlinearity (2002).
- [19] L. Wen. *The selecting lemma of Liao*. Disc. and Cont. Dyn. Sys., **20** (2008), 159–175.
- [20] J. Yang. *Lyapunov stable homoclinic classes*. Preprint Arxiv (2008).

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