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On the topological classification of rarefaction curves in systems of three conservation laws

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Abstract. In this paper we study the topological properties of integral curves of a system of implicit differential equations associated to rarefaction curves of a system of three conservation laws. This system of equations becomes singular at the points of eigenvalue of multiplicity greater or equal to two. We focus our attention to the generic case of multiplicity two and three. We give local weak topological models for these equations.

Keywords: system of implicit differential equations, rarefaction curves, systems of three conservation laws.

Mathematical subject classification: 34A09, 34C20, 37C15, 35LXX, 35L65, 57R45, 58J45.

1 Introduction

Implicit differential equations is a subject that appears in many contexts and there is a vast literature about it, for instance, [4], [3], [5], [8], [2], [11]. It also has many facets and the techniques used depend on the type of situation. The purpose of this work is to study the topological properties of a system of implicit differential equations associated to rarefaction curves of a system of three conservation laws.

Let

$$U_t + H(U)_x = 0 \tag{1.1}$$

be a system of n conservation laws in one space dimension x, where $U(x,t) \in \mathbb{R}^n$ and $H : \mathbb{R}^n \to \mathbb{R}^n$, is a flux function depending smoothly on U. For smooth

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solutions, system (1.1) is equivalent to system

$$\frac{\partial U}{\partial t} + DH(U)\frac{\partial U}{\partial x} = 0, \tag{1.2}$$

where $DH(U)$ is the $n \times n$ jacobian matrix of the flux function. Solutions of the form $U(x,t) = \tilde{U}(\lambda)$, $\lambda = x/t$, are wave solutions of (1.1) where each $\tilde{U}(\lambda)$ propagates in space x with speed λ. Substituting this solution in (1.2) one obtains

$$DH(U)\dot{U} = \lambda \dot{U}, \tag{1.3}$$

where $\dot{U} = dU/d\lambda$, i.e., solutions of types $\tilde{U}(\lambda)$ are the integral curves of the line field defined by the eigenspaces of the jacobian matrix of the flux function. These integral curve are called *rarefaction curves*. *Rarefaction waves* are arcs of rarefaction curves such that the propagation speed (corresponding eigenvalue) λ increases from the left to the right side of the wave.

Our strategy in this paper is as follows:

By eliminating λ we can associate to system (1.3) an implicit differential equation: each row of the system leads to an equation $\sum \frac{\partial H_i}{\partial U_j} dU_j = \lambda dU_i$, for $i = 1, 2 \ldots, n$. Let us suppose (without loss of generality) that $dU_1 \neq 0$. By eliminating λ gives $n-1$ quadratic form:

$$\sum \frac{\partial H_i}{\partial U_j} dU_j dU_1 - \sum \frac{\partial H_1}{\partial U_j} dU_j dU_i = 0, \quad i = 2, 3, \ldots, n.$$

Dividing these equations by $dU_1{}^2$ and denoting $\frac{dU_i}{dU_1} = r_i$ we obtain a system of equations, quadratic in the variables r_i:

$$\frac{\partial H_i}{\partial U_1} + \sum_{j=2}^{n} \frac{\partial H_i}{\partial U_j} r_j - \frac{\partial H_1}{dU_1} r_i - \sum_{j=2}^{n} \frac{\partial H_1}{\partial U_j} r_j r_i = 0, \quad i = 2, 3, \ldots, n.$$

For $n = 2$, rarefaction curves were studied in [10] for quadratic flux function. In that paper it is shown that the natural locus to study rarefaction curves is a 2-dimensional surface, called *characteristic surface*. The main result is that the configuration of rarefaction curves is structuraly stable under C^3 Whitney perturbation of the flux function.

For $n = 3$, rarefaction curves were studied in [7], [9]. In [7], for a flux function H such that the two first coordinates are quadratic polynomials as in [10] and the third one is a homogeneous linear function, it is obtained the global structure of rarefaction curves. The fact that the curves are contained on leaves of a 2-dimensional foliation was helpful to describe the structure of rarefaction.

In [9] rarefaction curves were studied for generic flux function. In that paper the structure of rarefaction curves is considered in a neigborhood of the set where there are coincidence of eigenvalues (characteristic speeds) of the derivative of flux function. This set forms a 2-dimensional surface called *boundary of the elliptic region*; in [7] this surface is a elliptic cylinder. The structure is described near regular and exceptional points of this surface where two eigenvalues coincide.

In this paper we consider generic flux functions to describe the structure of rarefaction near regular points of the *singular set* (boundary of the elliptic region), as well as near the subset of this surface where the eigenvalues of DH has algebraic multiplicity 3 and geometric multiplicity 1.

If the eigenvalues of $DH(0)$ are all simple, then, by the Implicit Function Theorem there are three linearly independent smooth vector fields defined in a neighborhood of the origin. Each vector field gives rise to one family of rarefaction curves, so we obtain three distinct branches of rarefaction curves passing through each point.

This situation changes completely when two or more eigenvalues coincide. The goal of this work is to describe the rarefaction curves when $DH(0)$ presents one eigenvalue of algebraic multiplicity greater or equal to two.

More specifically, in section 2, we describe the local structure of rarefaction curves where the jacobian matrix $DH(0)$ presents an eigenvalue with algebraic multiplicity two and geometric multiplicity one.

In section 3, we consider the case where $DH(0)$ presents one eigenvalue of algebraic multiplicity three and geometric multiplicity one.

In both cases we impose some generic conditions on the flux function H and prove structurally stability in a weak sense.

As usual, the word *generic* means that the condition is satisfied for a residual subset in the C^2 topology. Actually the hypohesis depend only on the two jet, $j^2H(0)$.

As we will see in the following sections, in the cases studied here, the origin belongs to the boundary of elliptic region. This surface separates the space into two open regions: the one with simple eigenvalues (and three branches of rarefaction curves) and another one with just one family of rarefaction curves.

The term *weak* referred above means that we prove the existence of local homeomorphisms in the ambient space that preserves each branch of rarefaction curves. The construction obtained also allows to parametrize the rarefaction curves even after it reaches the boundary of the elliptic region.

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2 Double eigenvalue

Let us consider the line fields defined by the eigenvectors of the jacobian matrix of a map $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Equivalently, let us consider the system of line fields defined implicitly by

$$DH(p)dp = \lambda dp. \tag{2.1}$$

When we make a smooth change of coordinates $p = \phi(u)$ and substitute into equation (2.1) we get an equivalent system

$$\begin{gathered} DH(\phi(u))D\phi(u)du = \lambda D\phi(u)du \\ \text{or} \\ D\phi(u)^{-1}DH(\phi(u))D\phi(u)du = \lambda du. \end{gathered} \tag{2.2}$$

By equivalent systems we mean that the solutions $u(t)$ of one system are mapped onto solutions of the other $p(t) = \phi(u(t))$. The equivalence provides a change of coordinates that sends solutions of one system to solutions of the other.

In particular, a linear change of coordinates $p = Pu$, P an $n \times n$ invertible matrix, leads to the system

$$P^{-1}DH(\phi(u))Pdu = \lambda du$$

Therefore if $H(p) = Bp + Q(p)$, with $Q(0) = 0$, $DQ(0) = 0$, then by a linear change of coordinates we get

$$\left\{P^{-1}BP + P^{-1}DQ(Pu)P\right\}du = \lambda du. \tag{2.3}$$

In this way, there is no loss of generality if we assume that B is in Jordan canonical form.

Hypothesis 1. *$DH(0)$ has an eigenvalue of algebraic multiplicity two and geometric multiplicity one (one eigenvector).*

Using a linear change of coordinates, if necessary, we assume that the matrix $DH(0)$ is in the Jordan canonical form:

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \tag{2.4}$$

with $\lambda_1 \neq \lambda_2$. Hence by a linear change of coordinates, system (2.1) is equivalent to $A(p)dp = \lambda dp$, with $A(p) = J + R(p)$, where $R(p)$ is a 3×3 matrix with entries depending on p such that $R(0) = 0$.

If $F(\lambda, x, y, z)$ is the characteristic polynomial of A then $F(\lambda, 0, 0, 0) = (\lambda - \lambda_2)(\lambda - \lambda_1)^2$. It follows that $F(\lambda_2, 0, 0, 0) = 0$ and $\frac{\partial F}{\partial \lambda}(\lambda_2, 0, 0, 0) \neq 0$. By the Implicit Function Theorem, there is a neighborhood of $(0, 0, 0)$ in $\mathbb{R}^3$ and a smooth function $a(x, y, z)$ such that $F(a(x, y, z), x, y, z) = 0$, with $a(0, 0, 0) = \lambda_2$. Therefore, we may write

$$F(\lambda, x, y, z) = (\lambda - a(x, y, z))\big(\lambda^2 + \alpha(x, y, z)\lambda + \beta(x, y, z)\big).$$

Proposition 1. *The above system $A(p)dp = \lambda dp$ is equivalent to the system $\hat{A}(u)du = \lambda du$ with $\hat{A}(u) = J + \hat{Q}(u)$ and*

$$\hat{Q}(u) = \begin{bmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{bmatrix},$$

where $\hat{Q}(0) = 0$, and we are omitting the variable u of each entry of the matrix.

Proof. Let $Y(x, y, z)$ be a non-singular vector field defined in a neighborhood of $(0, 0, 0)$ which generates $ker[A(p) - a(p)I]$, I the identity matrix, such that

$$Y(0, 0, 0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3.$$

By the Flow Box Theorem, there exists a local diffeomorphism $\Phi(u_1, u_2, u_3) = (x, y, z)$ such that $\Phi(0, 0, 0) = (0, 0, 0)$, $D\Phi(0, 0, 0) = I$ and

$$D\Phi(u_1, u_2, u_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = Y\big(\Phi(u_1, u_2, u_3)\big).$$

In other words, Y is conjugate to the vertical vector field:

$$Y(\Phi(u)) = D\Phi(u)e_3.$$

By construction $A(\Phi(u))Y(\Phi(u)) = a(\Phi(u))Y(\Phi(u))$, so substituting $Y(\Phi(u))$ by $D\Phi(u)e_3$, we get

$$A(\Phi(u))D\Phi(u)e_3 = a(\Phi(u))D\Phi(u)e_3$$

or

$$D\Phi(u)^{-1}A(\Phi(u))D\Phi(u)e_3 = a(\Phi(u))e_3\,.$$

Let $\hat{A}(u) = D\Phi(u)^{-1}A(\Phi(u))D\Phi(u)$, then $\hat{A}(0) = A(0) = J$ and the last column of $\hat{A}(u)$ is of the desired form:

$$\begin{bmatrix} 0 \\ 0 \\ a(\Phi(u)) \end{bmatrix}. \qquad \square$$

In this way we have reduced the degree of the implicit differential equation associated to the system separating the singular from the non singular part

$$\begin{bmatrix} \lambda_1 + a_1 & 1 + a_2 & 0 \\ b_1 & \lambda_1 + b_2 & 0 \\ c_1 & c_2 & \lambda_2 + c_3 \end{bmatrix} \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \lambda \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix}. \tag{2.5}$$

To simplify the notation we use $p = (x, y, z)$ instead of (u_1, u_2, u_3) and write system (2.5) in the form;

$$(\lambda_1 + a_1)dx + (1 + a_2)dy = \lambda dx$$
$$b_1 dx + (\lambda_1 + b_2)dy = \lambda dy$$
$$c_1 dx + c_2 dy + (\lambda_2 + c_3)dz = \lambda dz.$$

Elimination of λ gives the following implicit differential equation:

$$\begin{cases} (1 + a_2)w^2 + (a_1 - b_2)w - b_1 = 0 \\ (\lambda_1 - \lambda_2 + a_1 - c_3)r + (1 + a_2)rw - (c_2 w + c_1) = 0 \\ dy - w dx = 0 \\ dz - r dx = 0 \end{cases} \tag{2.6}$$

We assume that $\frac{\partial b_1}{\partial x}(0, 0, 0) \neq 0$. Since $\lambda_1 - \lambda_2 \neq 0$, the first two equations of system (2.6) define a three dimensional surface S in a neighborhood of $(0, 0, 0, 0, 0)$ in $\mathbb{R}^3 \times \mathbb{R}(P^2)$ ($\mathbb{R}(P^2)$ is the real projective plane with affine coordinates $(1, w, r)$) parametrized by

$$(y, z, w) \mapsto \left(\chi(y, z, w), y, z, w, \frac{c_1 + c_2 w}{\big(\lambda_1 - \lambda_2 + a_1 - c_3\big) + \big(1 + a_2\big)w}\right).$$

Moreover, it follows from $\frac{\partial\chi}{\partial w}(0,0,0) = 0$ and $\frac{\partial^2\chi}{\partial w^2}(0,0,0) \neq 0$ that the restriction of the canonical projection $(x, y, z, w, r) \mapsto (x, y, z)$ to the manifold S has a generic singularity of fold type, with a smooth surface defined by $\frac{\partial\chi}{\partial w}(y, z, w) = 0$ as singular set. We denote this singular set by Σ. Since $\frac{\partial^2\chi}{\partial w^2}(0,0,0) \neq 0$, the surface Σ is parametrized as a graph $(y, z, a(y, z))$, in a neighborhood of the origin.

The singular set Σ can also be defined as the fixed point set of the involution σ that relates the roots of the quadratic equation $(1 + a_2)w^2 + (a_1 - b_2)w - b_1 = 0$, which is given by $\sigma(y, z, w) = \left(y, z, -w - \frac{(a_1-b_2)}{(1+a_2)}\right)$ where the arguments are $(\chi(y, z, w), y, z)$.

The intersection of the kernel of the one forms $dy - w dx$ and $dz - r dx$ with the tangent space of S defines a line field which is tangent to the following vector field:

$$X(y, z, w) = w\frac{\partial\chi}{\partial w}\frac{\partial}{\partial y} + r\frac{\partial\chi}{\partial w}\frac{\partial}{\partial z} + \left(1 - w\frac{\partial\chi}{\partial y} - r\frac{\partial\chi}{\partial z}\right)\frac{\partial}{\partial w}.$$

X is a non singular vector field transversal to the surface Σ.

As the last component of this vector field never vanishes in a neighborhood of $(0, 0, 0)$, it generates a flow that may be parametrized by w. In other words, by reparametrizing the integral curves using w as parameter, the flow generated by the vector field X may be written in the form $\varphi_w(y, z, w)$.

This allows to define $\Psi(y, z, w) = \varphi_w(y, z, a(y, z))$, a local diffeomorphism in a neighborhood of $(0, 0, 0)$, that sends the plane $w = 0$ to the surface Σ and brings the vector field X to the vertical vector field $\frac{\partial}{\partial w}$.

Therefore, the first three components of Ψ parametrizes the rarefaction curves in such way that given a point $P = (x, y, z)$ we have three possibilities:

- If $x < \chi(y, z)$ there is no rarefaction curve passing through P.
- If $x > \chi(y, z)$ there are two rarefaction curves parametrized by w passing through P.
- If $x = \chi(y, z)$, P is in the image of the fold set which is the set of singular points of the rarefaction curves.

We also have the induced involution $\hat{\sigma} = \Psi^{-1} \circ \sigma \circ \Psi$, which has the following properties:

a) the plane $w = 0$ is its set of fixed points.

b) its derivative $d\hat{\sigma}$ at $w = 0$ sends the vertical direction to the vertical direction.

It follows from the above description that two equations associated to rarefaction problem satisfying the generic conditions are smoothly (or topologically) equivalent if and only if there is a smooth diffeomorphism (resp. homeomorphism) that preserves the vertical foliation and conjugates the respective involutions.

This imposes conditions which in general are not satisfied. For instance, if we denote by $\mathcal{F}$ the vertical foliation, then any equivalence must also preserve its image $\hat{\sigma}(\mathcal{F})$. Actually, $\mathcal{F}$ represents one branch of the rarefaction curves while the other branch is represented by $\hat{\sigma}(\mathcal{F})$. The subset $x \geq \chi(y, z)$ is diffeomorphic to the quotient map that identifies a point $Q = (y, z, w)$ with its image $\hat{\sigma}(Q)$. Hence it is uniquely determined by the intersection of the two leaves $\mathcal{F}_Q \cap \hat{\sigma}(\mathcal{F}_{\hat{\sigma}(Q)})$.

This, of course, requires several restrictions upon the diffeomorphism in the base space of variables (y, z) and its analysis would will take us apart from the goal of this work.

However the above construction provides immediately an *weak equivalence*, in the following sense: there is a smooth diffeomorphism that it sends each branch of the rarefaction curves, to the corresponding one. More precisely we have proven the following:

Proposition 2. *Given a map* $H\colon \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $DH(0)$ *has an eigenvalue of algebraic multiplicity two and geometric multiplicity one (one eigenvector). Assume that* $\frac{\partial b_1}{\partial x}(0, 0, 0) \neq 0$*, then the foliation by rarefaction curves is weakly stable,i.e., there is an open neighborhood* $\mathcal{U}$ *of* H *in* $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ *such that, if* $G \in \mathcal{U}$ *there is a local diffeomorphism* ψ *defined in a neighborhood of the origin, that sends one branch of the rarefaction foliation associated to the system* $DH(p)dp = \lambda dp$ *to one branch of rarefaction foliation of* $DG(p)dp = \lambda dp$*. Moreover, if we denote by* $\hat{\sigma}$ *and* $\hat{\sigma}_1$ *the respective involutions, then the diffeomorphism* $\hat{\sigma}_1 \circ \psi \circ \hat{\sigma}$ *defines an equivalence that preserves the other branch of the rarefaction foliation.*

3 Triple eigenvalue

In this section we consider the following case:

Hypothesis 2. $DH(0)$ *has an eigenvalue of algebraic multiplicity three and geometric multiplicity one (one eigenvector).*

We assume, using a linear change of coordinates, if necessary, that the matrix $DH(0)$ is in the Jordan Cannonical Form:

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}.$$

Hence by a linear change of coordinates, system (2.1) is equivalent to the system $A(p)dp = \lambda dp$, with $A(p) = J + R(p)$ and $R(0) = 0$, which is associated to a system of implicit differential equations, as follows:

If

$$A(p) = \begin{bmatrix} \lambda_1 + A_1(p) & 1 + A_2(p) & A_3(p) \\ B_1(p) & \lambda_1 + B_2(p) & 1 + B_3(p) \\ C_1(p) & C_2(p) & \lambda_1 + C_3(p) \end{bmatrix},$$

then

$$\begin{cases} F = B_1 + (B_2 - A_1)w + (1 + B_3)r - A_3wr - (1 + A_2)w^2 = 0 \\ G = C_1 + (C_3 - A_1)r + C_2w - (1 + A_2)wr - A_3r^2 = 0 \\ dy - wdx = 0 \\ dz - rdx = 0. \end{cases} \tag{3.1}$$

Let us denote by $\mathcal{F}$ the map from $\mathbb{R}^3 \times \mathbb{R}(P^2)$ to $\mathbb{R}^2$ defined by (F, G), i.e., $\mathcal{F} = (F, G)$. We assume that the matrix $D\mathcal{F}(0)$ has rank 2, so that $\mathcal{F}^{-1}(0)$ is locally a three dimensional smooth submanifold. For that, it is enough to assume $dC_1(0) \neq 0$.

Hypothesis 3. *In this paper we assume that $dC_1(0) \neq 0$. This means that the associated implicit differential equation defines a 3-dimensional submanifold of $\mathbb{R}^3 \times \mathbb{R}(P^2)$, denoted by $\mathcal{F}^{-1}(0)$.*

However, taking also into account that the plane fields defined by the last two equations of system (3.1), we require that the matrix

$$\begin{bmatrix} B_{1x}(0) & B_{1y}(0) & B_{1z}(0) & 0 & 1 \\ C_{1x}(0) & C_{1y}(0) & C_{1z}(0) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

has rank 4. For that, it suffices to suppose that $C_{1x}(0) \neq 0$.

Hypothesis 4. *In this paper we assume that* $C_{1x}(0) \neq 0$. *In other words, the tangent plane of* $\mathcal{F}^{-1}(0)$ *is transversal to the plane fields defined by* $dy - w dx = 0$ *and* $dz - r dx = 0$.

Of course this is a *generic* condition on the coefficients of the original map H.

As the first equation in system (3.1) is non-singular with respect to the variable r, solving this equation for r gives

$$r = \frac{\big(1 + A_2\big)w^2 - B_1 - \big(B_2 - A_1\big)w}{1 + B_3 - A_3 w}. \tag{3.2}$$

By substituting this expression for r in the second equation, we obtain a third degree equation in the variable w that, after algebraic simplifications, can be written as:

$$f(x, y, z, w) = w^3 + \alpha(x, y, z)w^2 + \beta(x, y, z)w + \gamma(x, y, z) = 0. \tag{3.3}$$

The hypothesis $C_{1x} \neq 0$ implies that $\frac{\partial \gamma}{\partial x}(0, 0, 0) \neq 0$. So we conclude that $E = f^{-1}(0)$ is locally a graph $x = \mathcal{V}(y, z, w)$, with

$$\mathcal{V}(0, 0, 0) = \mathcal{V}_w(0, 0, 0) = \mathcal{V}_{ww}(0, 0, 0) = 0 \quad \text{and} \quad \mathcal{V}_{www}(0, 0, 0) \neq 0.$$

Hence if we require some additional generic hypothesis, the restriction of the projection $\pi(x, y, z, w) = (x, y, z)$ to E, namely, the map $\Pi(y, z, w) = (\mathcal{V}(y, z, w), y, z)$ will be a generic stable map with a cusp singularity at the origin.

Hypothesis 5. *Let us assume that the mapping*

$$(x, y, z) \longmapsto \big(\alpha(x, y, z), \beta(x, y, z), \gamma(x, y, z)\big)$$

is a local diffeomorphism in a neighborhood of $(0, 0, 0)$.

With this hypothesis the system of equations $f = 0$, $f_w = 0$ defines locally a surface Σ, such that the singular set contains a smooth curve C, the *cusp curve*, defined by $f = 0$, $f_w = 0$ and $f_{ww} = 0$.

Remark 1. *Hypothesis 5 implies that the restriction of the canonical projection* $\Pi : \mathbb{R}^3 \times \mathbb{R}(P^2) \longrightarrow \mathbb{R}^3$ *is a generic cusp map. This hypothesis is equivalent to the requirement that the map* $(x, y, z) \to A(x, y, z)$ *unfolds generically the matrix* J *in the sense of Arnold in* [1].

The image $S_f = \Pi(\Sigma)$ corresponds to double roots and C is the set of triple roots of the polynomial f in equation (3.3).

The curve C divides the singular set into two components $\Sigma = \Sigma^+ \cup \Sigma^-$ with $C = \Sigma^+ \cap \Sigma^-$, its image $\Pi(C)$ is classically known as *cuspidal edge* (see figures 1 and 2).

The pre-image $\Pi^{-1}(S_f)$ is the set $\Pi^{-1}(S_f) = \Sigma \cup \Delta$, where Δ is characterized by the following property: $(x_1, y_1, z_1, w_1) \in \Delta$ if and only if $f(x_1, y_1, z_1, w_1) = 0$ and there exists $w_2 \neq w_1$ such that $f(x_1, y_1, z_1, w_2) = 0$ and $f_w(x_1, y_1, z_1, w_2) = 0$. In other words w_1 and w_2 are roots of the same polynomial, but w_2 is a double root. It is easy to see that Δ is the union of two surfaces $\Delta^+ \cup \Delta^-$. Figure 1 illustrates the pre-image of $\Pi^{-1}(S_f)$.

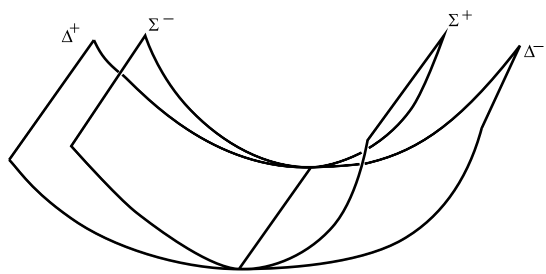

Figure 1: The pre-image $\Pi^{-1}(S_f)$.

The map $\Pi(y, z, w) = (\mathcal{V}(y, z, w), y, z)$ is generic as a map of $\mathbb{R}^3$ and it is equivalent by a change of coordinates in the source and in the target spaces, to $(u, v, w) \longmapsto (w^3 + uw, v, u))$.

For future references, we let

$$\Lambda_1 = \{(x, y, z);\ f(x, y, z, w) = 0 \text{ has exactly one real solution}\},$$
$$\Lambda_2 = \{(x, y, z);\ f(x, y, z, w) = 0 \text{ has one simple and double solution}\},$$
$$\Lambda_3 = \{(x, y, z);\ f(x, y, z, w) = 0 \text{ has three distinct solutions}\}.$$

The image $\Pi(\Sigma) = S_f$ is a singular two dimensional surface. The singular set of S_f is the set $\Pi(C)$, see figure 2. It separates a neighborhood of the origin in $\mathbb{R}^3$ into two connected components $\Lambda_1 \cup \Lambda_3$. Clearly $\Lambda_2 = \Pi(\Sigma - C)$.

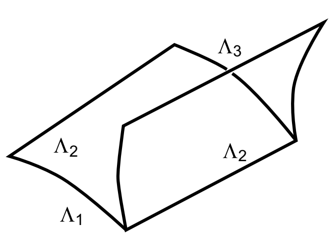

Figure 2: The image $\Pi(\Sigma) = S_f$.

Next result gives the construction of a vector field $\hat{X}$, associated to the implicit differential equation (3.1).

Lemma 2. *The intersection of the kernel of the 1-forms $dy - wdx$ and $dz - rdx$ with the tangent plane of E induces a line field tangent to the vector field:*

$$\hat{X}(x, y, y, z, w) = \big(f_w, wf_w, rf_w, -(f_x + wf_y + rf_z)\big).$$

Proof. From hypothesis 5, the plane fields defined by $dy - wdx = 0$ and $dz - rdx = 0$ are transverse to the tangent planes of E. Hence their intersections with each tangent plane define a line field.

If we write $\hat{X} = (X_1, X_2, X_3, X_4)$, then, of course, $X_2 = wX_1$, $X_3 = rX_1$. In order to be tangent to E, we have $df \cdot \hat{X} = 0$ or $f_w X_4 = -X_1(f_x + wf_y + rf_z)$. Hence, it is enough to define $X_1 = f_w$ to obtain the desired expression for $\hat{X}$. □

Since by hypothesis $f_x(0, 0, 0) \neq 0$, the vector field $\hat{X}$ is nonvanishing in a neighborhood of the origin $(0, 0, 0, 0)$. Moreover, by construction, the surface E is invariant under the flow of $\hat{X}$, therefore we may consider the induced vector field X defined by $X(y, z, w) = \hat{X}(\Upsilon(y, z, w), y, z, w)$.

The vector field X is transversal to the singular set Σ and to the surface Δ described above, except at points on C. Moreover, if $p \in C$ then $X(p)$ and the tangent vector T_pC are always linearly independent.

Combining these properties of the vector field X with the Flow Box Theorem and the properties of the map Π, we obtain the following qualitative description of the projection of the trajectories of X:

Let $S_f^+ = \Pi(\Sigma^+)$ and $S_f^- = \Pi(\Sigma^-)$ be the components of the complement of cuspidal edge in S.

If $q \in S_f^+$, then its trajectory enters the open set Λ_3 until it reaches the surface S_f^-. At this point, the orbit is reflected, opposite to the incoming direction, until it reaches the surface S_f^+ again. The trajectory is reflected again, returning to the component S_f^- where it leaves Λ_3 and enters the set Λ_1. The region Λ_1 is foliated by the trajectories of the projected vector field. Figure 3 shows the trajectory by q in a cross section of S_f.

In this way the projection of trajectories of the vector field X defines two return maps, one on each component S_f^- and S_f^+. Notice that the analysis of the dynamics of these mappings is essential for the description of the topological properties of the solutions of the implicit differential equation associated to system (3.1).

These mappings may also be described directly in the surface E as follows:

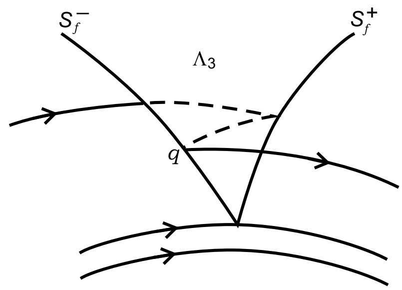

Figure 3: Cross section of S_f.

Take an initial condition a in the component Σ^- and follow its trajectory until it reaches a point $b \in \Delta^-$. By definition, there is a unique point a_- such that $\Pi(b) = \Pi(a_-)$. The return map is defined by $\phi_-(a) = a_-$. Analogously we define a return map $\phi_+ \colon \Sigma^+ \longrightarrow \Sigma^+$, now going backwards in time. Clearly both maps can be extended continously to the curve C as fixed points (recall that X is tangent to the surface Σ only at points C and a trajectory starting at such points never intersects Σ). Figure 4 illustrates the return map ϕ_- in E.

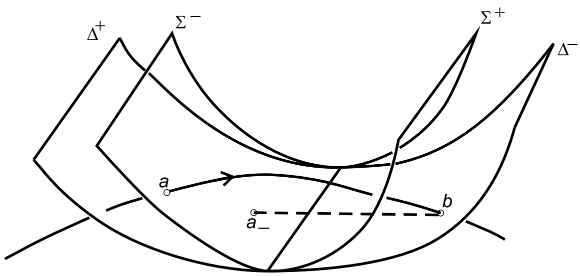

Figure 4: Return map ϕ_-.

Before we state the main result of this section we make the following additional hypothesis:

Hypothesis 6. *The cuspidal edge is normally attracting (reppeling) for both return maps ϕ_- and ϕ_+.*

Remark 3. *In the proof, it will be clear that hypothesis 6 is a condition depending only on $j^2H(0)$. We will also show that ϕ_- and ϕ_+ are topologically conjugate. Actually it is enough to require that the submanifold E is normally attracting or reppeling.*

Theorem 4. *Given two generic systems $DH_1(p)dp = \lambda dp$ and $DH_2 dp = \lambda dp$ satisfying the hypotheses (2)-(6), there exists a homeomorphism h defined* *in a neighborhood U of $0 \in \mathbb{R}^3$ which is a topological equivalence between the respective projected vector fields $\Pi_*(X^1) = (X_1^1, X_2^1, X_3^1)$ and $\Pi_*(X^2) = (X_1^2, X_2^2, X_3^2)$ in the complement of $U \cap \Sigma$. In other words, h sends trajectory of $\Pi_*(X^1)$ to trajectory of $\Pi_*(X^2)$, preserving the sense of trajectories outside the region of three eigenvalues.*

The theorem is proven by showing that there are two topological models for the trajectories. The distinction is made based on the dynamical properties of the return map ϕ_+.

We will show that, in the generic case, the line of fixed point is either normally attracting or repelling. Once we obtain this, we get the conjugacy between the return maps by standard methods.

Proof of Theorem 4: The proof has several steps and we will use a generic H to denote the flux function.

Step 1: defining a sequence of blowing-ups in order to obtain a conjugacy between the return maps. Recall that $f(x, y, z, w) = w^3 + \alpha(x, y, z)w^2 + \beta(x, y, z)w + \gamma(x, y, z) = 0$ defines locally $x = \mathcal{V}(y, z, w)$. Let us consider the elementary symmetric functions

$$\begin{aligned}
\sigma_1\big(w_1, w_2, w_3\big) &= w_1 + w_2 + w_3,\\
\sigma_2\big(w_1, w_2, w_3\big) &= w_1w_2 + w_1w_3 + w_2w_3\\
\sigma_3\big(w_1, w_2, w_3\big) &= w_1w_2w_3.
\end{aligned}$$

Define $\sigma(w_1, w_2, w_3) = (\hat{\sigma_1}, \hat{\sigma_2}, w_1)$, with $y = \hat{\sigma_1}(w_1, w_2, w_3)$, $z = \hat{\sigma_2}(w_1, w_2, w_3)$ defined implicitly by the following equations:

$$\begin{aligned}
&\alpha\big(\mathcal{V}(y, z, w), y, z\big) = -\sigma_1\big(w_1, w_2, w_3\big)\\
&\beta\big(\mathcal{V}(y, z, w), y, z\big) = \sigma_2\big(w_1, w_2, w_3\big)\\
&w = w_1.
\end{aligned}$$

Notice that

$$\begin{aligned}
f_w\big(\mathcal{V}(\hat{\sigma_1}, \hat{\sigma_2}, w_1)(\hat{\sigma_1}, \hat{\sigma_2})\big) &= 3w_1^2 - 2\sigma_1 w_1 + \sigma_2\\
&= (w_1 - w_2)(w_1 - w_3).
\end{aligned} \tag{3.4}$$

Differentiating the above expressions and using that

$$\mathcal{V}_y = -\frac{f_y}{f_x}, \quad \mathcal{V}_z = -\frac{f_z}{f_x} \quad \text{and} \quad \mathcal{V}_w = -\frac{f_w}{f_x},$$

we get

$$\begin{bmatrix} \dfrac{\partial y}{\partial w_1} & \dfrac{\partial y}{\partial w_2} & \dfrac{\partial y}{\partial w_3} \\ \dfrac{\partial z}{\partial w_1} & \dfrac{\partial z}{\partial w_2} & \dfrac{\partial z}{\partial w_3} \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \frac{1}{f_x(w_2 - w_3)} B$$

$$\times \begin{bmatrix} -\alpha_x f_y + \alpha_y f_x & -\alpha_x f_z + \alpha_z f_x & \alpha_x f_w \\ -\beta_x f_y + \beta_y f_x & -\beta_x f_z + \beta_z f_x & \beta_x f_w \\ 0 & 0 & f_x \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & 0 & 1 \\ -(w_1 + w_2) & -1 & w_3 - w_1 \\ w_1 + w_3 & 1 & w_1 - w_2 \end{bmatrix}.$$

For further details of calculations see [12].

It is easy to verify the following properties:

i) $\Delta := \sigma(w_2 - w_3)$

ii) $\sigma(w_1 = w_2) \cup \sigma(w_1 = w_3) = \Sigma$

iii) σ is a folding map with folding set equals to $w_2 = w_3$.

Item (i) follows from the definition of Δ and equation (3.4); item (ii) follows from equation (3.4) and item (iii) is a straighforward computation.

It follows from (*iii*) that if q does not belong to Δ, then its pre-image $\sigma^{-1}(q)$ has cardinality zero or two.

Let $Z = d\sigma^{-1}\hat{X}(\sigma)$ be the induced vector field.

A straightfoward computation shows that $Z = \frac{1}{(w_2 - w_3) f_x}(Z_1, Z_2, Z_3)$ with:

$$Z_1 = (w_3 - w_2) f_x (f_x + w_1 f_y + r f_z)(\sigma);$$

$$Z_2 = -\left[(w_1 + w_2)M + N\right] f_w - (w_3 - w_1) f_x (f_x + w_1 f_y + r f_z)(\sigma);$$

$$Z_3 = \left[(w_1 + w_3)M + N\right] f_w - (w_1 - w_2) f_x (f_x + w_1 f_y + r f_z)(\sigma),$$

where $M(y,z,w) = (\alpha_y f_x - \alpha_x f_y)w + (\alpha_z f_x - \alpha_x f_z)r + \alpha_x(f_x + w_1 f_y + r f_z)$ or $M(x,y,z) = f_x(\alpha_y w + \alpha_z r + \alpha_x)$ and $N(y,z,w) = (\beta_y f_x - \beta_x f_y)w + (\beta_z f_x - \beta_x f_z)r + \beta_x(f_x + w_1 f_y + r f_z)$ or $N(x,y,z) = f_x(\beta_y w + \beta_z r + \beta_x)$.

We define the extended vector field $Y(w_1,w_2,w_3) = (w_2 - w_3)Z(w_1,w_2,w_3)$, which, of course, has the same phase portrait as Z except at $w_2 = w_3$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = f_x(f_x + w_1 f_y + r f_z) \begin{bmatrix} w_3 - w_2 \\ w_1 - w_3 \\ w_2 - w_1 \end{bmatrix} + f_w \begin{bmatrix} 0 \\ -(w_1 + w_2)M - N \\ (w_1 + w_3)M + N \end{bmatrix}. \tag{3.5}$$

Let us describe the properties of the vector field Y:

i) the line $w_1 = w_2 = w_3$ is the set of singularities;

ii) if $\tau_1(w_1,w_2,w_3) = (w_1,w_3,w_2)$ denotes the permutation (transposition) that fixes the variable w_1, then $\tau_1^*(Y) = -Y$;

iii) all the regular trajectories of Y are closed and in the opposite sense of the vector field X;

iv) the return map ϕ_+ is described as follows: take an initial condition ξ in the plane $w_1 = w_2$ and follow its trajectory, in the positive sense, until it encounters the plane $w_2 = w_3$ at some point ξ_1. Then $\phi_+(\xi) = \tau_2(\xi_1)$ where τ_2 is the permutation that fixes the variable w_2.

Proposition 5. *ϕ_- and ϕ_+ are topologically conjugated.*

Proof. This follows from the symmetry properties of the vector field Y:

Let τ_1, τ_2, τ_3 be the reflections:

$$\tau_1\big(w_1,w_2,w_3\big) = \big(w_1,w_3,w_2\big),$$
$$\tau_2\big(w_1,w_2,w_3\big) = \big(w_3,w_2,w_1\big),$$
$$\tau_3\big(w_1,w_2,w_3\big) = \big(w_2,w_1,w_3\big).$$

Notice that $\tau_i^{-1} = \tau_i,\ i = 1,2,3.$

If we define the following vector fields: $\tau_j^*(Y) = Y_j$, for $j = 2,3$, then, it is easy to verify that $\tau_1^*(Y_2) = -Y_3$, $\tau_1^*(Y) = -Y$ and that the time spent by a trajectory of Y_2 to go from the plane $w_2 = w_3$ until the plane $w_1 = w_3$ is equal the time spent for a trajectory of Y_3 to go from $w_1 = w_2$ to $w_2 = w_3$.

Using these observations we prove that:

$$\phi_- = \tau_1 \circ \phi_+ \circ \tau_1.$$

Indeed, if Y_j^t denotes the flow generated by the vector field Y_j and Y^t denotes the flow of Y, then $\phi_-(p) = Y_3^s \circ Y^{-t}(p)$ and $\phi_+(p) = Y_2^{-s} \circ Y^t(p)$, for t and s depending on w_1, w_2, w_3. Therefore,

$$\tau_1 \circ \phi_- \circ \tau_1 = \tau_1 \circ Y_3^s \circ Y^{-t}\tau_1 = Y_2^{-s} \circ \tau_1 \circ Y^{-t}\tau_1 = Y_2^{-s} \circ Y^t = \phi_+.$$

Concluding the proof. □

Step 2: Performing two linear changes of coordinates in order to put the linear part of Y in cannonical Jordan form:

Let $u_1 = w_1 - \frac{1}{3}\sum_1^3 w_i$, $u_2 = w_2 - \frac{1}{3}\sum_1^3 w_i$ and $u_3 = \frac{1}{3}\sum_1^3 w_i$. In these new coordinates $w_1 = u_1 + u_3$, $w_2 = u_2 + u_3$ and $w_3 = u_3 - u_1 - u_2$ so that $w_1 = w_2 \iff u_1 = u_2$; $w_1 = w_3 \iff 2u_1 + u_2 = 0$; $w_2 = w_3 \iff 2u_2 + u_1 = 0$ and the induced vector field (3.5) is written as:

$$\begin{aligned}\begin{bmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{u_3} \end{bmatrix} &= f_x\big(f_x + w_1 f_y + r f_z\big)\begin{bmatrix} -\big(u_1 + 2u_2\big) \\ 2u_1 + u_2 \\ 0 \end{bmatrix} \\ &+ \frac{f_w}{3}\begin{bmatrix} U_1\big(u_1, u_2, u_3\big) \\ U_2\big(u_1, u_2, u_3\big) \\ U_3\big(u_1, u_2, u_3\big) \end{bmatrix},\end{aligned} \tag{3.6}$$

where $U_1(u_1, u_2, u_3) = (2u_2 + u_1)M$, $U_2(u_1, u_2, u_3) = -(2u_1 + u_2 + 6u_3)M - 3N$, and $U_3(u_1, u_2, u_3) = -(2u_2 + u_1)M$.

Notice that in these coordinates the singular set of the vector field Y is the line $(0, 0, u_3)$.

With the following change of coordinates $v_1 = \sqrt{3}u_2$; $v_2 = -2u_1 - u_2$ and $v_3 = u_3$, we keep track of the planes: $u_1 = u_2 \iff \sqrt{3}v_1 + v_2 = 0$; $2u_1 + u_2 = 0 \iff v_2 = 0$; $u_2 = u_3 \iff \sqrt{3}v_1 - v_2 = 0$. Denoting $v = (v_1, v_2, v_3)$, the vector field (3.6) is written as:

$$\begin{bmatrix} \dot{v_1} \\ \dot{v_2} \\ \dot{v_3} \end{bmatrix} = \begin{bmatrix} -\sqrt{3}v_2 \\ \sqrt{3}v_1 \\ 0 \end{bmatrix} + \frac{f_w}{3f_x(f_x + w_1 f_y + r f_z)}\begin{bmatrix} V_1(v) \\ V_2(v) \\ V_3(v) \end{bmatrix}, \tag{3.7}$$

where $V_1(v) = (v_2 - 6v_3)M - 3N$, $V_2(v) = (-\sqrt{3}v_1 + 6v_3)M + 3N$ and $V_3(v) = \frac{1}{2}(\sqrt{3}v_1 - v_2)M$

Step 3: In order to detect the behavior of the vector field transversal to the line of singularities it is necessary to analyze the effect of the non-linear term $(V_1(v), V_2(v), V_3(v))$. For this purpose, we change to cylindrical coordinates: $v_1 = \rho\cos(\theta)$, $v_2 = \rho\sin(\theta)$, $v_3 = v_3$.

In these coordinates the distinguished planes $\sqrt{3}v_1 + v_2 = 0$, $v_2 = 0$ and $\sqrt{3}v_1 - v_2 = 0$ are respectivelly, written as follows: $\cos(\theta - \frac{\pi}{6}) = 0, \sin(\theta) = 0$, $\cos(\theta + \frac{\pi}{6}) = 0$. The vector field (3.7) is expressed as:

$$\begin{bmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{v_3} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} + \frac{f_w}{3f_x\rho(f_x + w_1 f_y + r f_z)} \times \begin{bmatrix} \rho\cos(\theta)V_1 + \rho\sin(\theta)V_2 \\ \cos(\theta)V_2 - \sin(\theta)V_1 \\ \rho V_3 \end{bmatrix}, \tag{3.8}$$

with $f_w(\rho, \theta, v_3) = \rho^2[\sqrt{3}\cos(\theta) + \sin(\theta)]\sin(\theta) = \rho^2[\sin(2\theta - \frac{\pi}{6}) + \frac{1}{2}]$. The vector field (3.8) is non-singular and its trajectories may be parametrized by the angle θ, which is an increasing function of the time. Therefore, we obtain the following expressions for the integral curve with initial conditions $\rho(0) = \rho_0, \theta(0) = 0$ and $v_3(0) = v_3^0$:

$$\begin{aligned} \rho(t, \rho_0, v_3^0)) &= \rho_0 + {\rho_0}^2 R(t, \rho_0, v_3^0)) \\ \theta(t, \rho_0, v_3^0)) &= \sqrt{3}t + \rho_0\Theta(t, \rho_0, v_3^0)) \\ v_3(t, \rho_0, v_3^0)) &= v_3^0 + \rho_0 \upsilon(t, \rho_0, v_3^0)). \end{aligned}$$

Let us recall how we have defined the return mapping ϕ_-: consider the negative trajectory of the vector field Y with initial condition at a point p in the plane $w_1 = w_3$ and let q be the point of its intersection with the plane $w_2 = w_3$. The image $\phi_-(p)$ is the reflection on the plane $w_1 = w_2$ of q, that is $\phi_-(p) = \tau_3(q)$.

In the coordinates $v = (v_1, v_2, v_3)$, we start with an initial condition at the plane $v_2 = 0$, or $\theta = 0$ and follow its trajectory $v(t), t < 0$ until it reaches a point in the plane $\sqrt{3}v_1 - v_2 = 0$, respectivelly $\theta = -\frac{2\pi}{3}$, say $v(T) = (v_1(T), v_2(T), v_3(T))$.

We take next the reflection:

$$\begin{aligned}&\hat{\tau}_3\big(v_1(T), v_2(T), v_3(T)\big)\\ &= \frac{1}{2}\Big(- v_1(T) - \sqrt{3}v_2(T), -\sqrt{3}v_1(T) + v_2(T), 2v_3(T)\Big)\\ &= \big(- v_1(T), 0, v_3(T)\big).\end{aligned}$$

In other words, T is defined implictly by the equation:

$$\sqrt{3}T + \rho_0\Theta(T, \rho_0, v_3^0)) = -\frac{2\pi}{3}.$$

The implicit function theorem gives us the smooth function $T(\rho_0, v_3^0)$ satisfying $T(0, v_3^0) = -\frac{2\pi}{3\sqrt{3}}$. Moreover, $\theta(T(\rho_0, v_3^0)) = -\frac{2\pi}{3}$. If we parametrize the trajectories by θ, using the above expression for the vector field, we may write:

$$\begin{aligned}\rho\big(\rho_0, \theta, v_3^0\big) &= \rho_0 + a_2\big(\theta, v_3^0\big)\rho_0^2 + \rho_0^3 R\big(\rho_0, \theta, v_3^0\big).\\ v_3\big(\rho_0, \theta, v_3^0\big) &= v_3^0 + b_2\big(\theta, v_3^0\big)\rho_0^2 + \rho_0^3 R_1\big(\rho_0, \theta, v_3^0\big).\end{aligned}$$

Since the line $(0, 0, v_3)$ is made of stationary points, the map $\phi_-(\rho^0, 0, v_3^0) = (\rho(-\frac{2\pi}{3}), 0, v_3(-\frac{2\pi}{3}))$ has an eigenvalue equals to 1. So, in order to verify that this line is normally atracting or repelling, it is sufficient to show that $2a_2(-\frac{2\pi}{3}, v_3^0)$ is different from zero. Actually, the sign of $V_1(0, 0, v_3^0)$ will distinguish the two cases, the repelling (positive) and the attracting (negative) one. This follows immediately by comparing the coefficients of

$$\rho'(\theta) = \frac{\rho^2\left[\sin\left(2\theta - \frac{\pi}{6}\right) + \frac{1}{2}\right]}{3f_x\big(f_x + w_1 f_y + r f_z\big)}\big[\cos(\theta)V_1 + \sin(\theta)V_2\big]. \tag{3.9}$$

Let us recall the expressions: $V_1 = (v_2 - 6v_3)M - 3N = (\rho\sin(\theta) - 6v_3)M - 3N$ and $V_2(v) = (-\sqrt{3}v_1 + 6v_3)M + 3N = (\sqrt{3}\rho\cos(\theta) + 6v_3)M + 3N$, where $M = f_x(\alpha_y w + \alpha_z r + \alpha_x)$ and $N = f_x(\beta_y w + \beta_z r + \beta_x)$. Substituting these expressions in equation (3.9), we get that ρ' is written as:

$$\begin{aligned}\rho'(\theta) &= \frac{\rho^2\left[\sin\left(2\theta - \frac{\pi}{6}\right) + \frac{1}{2}\right]}{f_x\big(f_x + w_1 f_y + r f_z\big)}\big[- \cos(\theta) + \sin(\theta)\big]\big[N + 2v_3 M\big] + O(\rho^3),\\ \rho'(\theta) &= \sqrt{2}\frac{\rho^2\left[\sin\left(2\theta - \frac{\pi}{6}\right) + \frac{1}{2}\right]\left[\sin\left(\theta - \frac{\pi}{4}\right)\right]}{\big(f_x + w_1 f_y + r f_z\big)}\\ &\quad\times\big[\big(\beta_y w_1 + \beta_z r + \beta_x\big) + 2v_3\big(\alpha_y w_1 + \alpha_z r + \alpha_x\big)\big] + O(\rho^3).\end{aligned}$$

Therefore, if $(\beta_y w_1 + \beta_z r + \beta_x)(0,0,0) \neq 0$ (condition depending on the coefficient of the two jet at zero of the initial equation), we get

$$2a_2\left(-\frac{2\pi}{3},0\right) = \int_0^{-\frac{2\pi}{3}} \sqrt{2}\,\frac{(\beta_y w_1 + \beta_z r + \beta_x)(0,0,0)}{(f_x + w_1 f_y + r f_z)(0,0,0)} \times \left[\sin\left(2\theta - \frac{\pi}{6}\right) + \frac{1}{2}\right]\left[\sin\left(\theta - \frac{\pi}{4}\right)\right] d\theta \neq 0.$$

On the other hand,

$$v_3'(\theta) = \frac{\rho^3\left[\sin\left(2\theta - \frac{\pi}{6}\right) + \frac{1}{2}\right]}{f_x(f_x + w_1 f_y + r f_z)}\left[\cos\left(\theta + \frac{\pi}{6}\right)\right] M + O(\rho^4).$$

Step 4: Given two generic systems, as in the statement of Theorem 4, associated to the flux functions H_1 and H_2, let us assume that the respective coefficients $a_2(-\frac{2\pi}{3},0)$ and $\hat{a}_2(-\frac{2\pi}{3},0)$ have the same sign. For instance, that both are negative. In this case, the corresponding return mappings, ϕ_-^1 and ϕ_-^2 are "quasi-hyperbolic contracting" to the line of fixed points. Hence by applying the methods of [6], p. 52, we obtain the conjugacy g between ϕ_-^1 and ϕ_-^2.

According to proposition 5 this conjugacy induces a conjugacy between ϕ_+^1 and ϕ_+^2.

Step 5: Conclusion of the proof of Theorem 4.

There are several ways to construct the local homeomorphism, as stated in Theorem 4, which is a topological equivalence outside the region Λ_3 that corresponds to three disctinct eigenvalues. For that, we use the previous notation and we follow the description of the properties of the vector field $\hat{X}$ of Lemma 2 associated to the implicit differential equation, as follows:

As observed before, the projection of the trajectory of the vector field X induces a one dimensional continuous foliation in the complement of Λ_3. This foliation is tranverse to the surfaces $S_f^+ = \Pi(\Sigma^+)$ and $S_f^- = \Pi(\Sigma^-)$, the components of the complement of cuspidal edge in S.

We use the projection of the trajectories of X in order to define a new foliation $\mathcal{F}$:

a) Project a trajectory of X, in the positive sense. If this trajectory does not intersect the singular set or if it passes through the cusp curve C, then its projection is a regular curve in the region Λ_1. These curves are part of the leaves of $\mathcal{F}$.

b) Otherwise, there is a first intersection point, say $p \in \Sigma^-$, where we stop.

Using the geometric properties of the projection Π we know that $\Pi(p) \in S_f^-$. Let $p' \in \Delta^-$ be the intersection of the positive trajectory of the vector field X through the point p. By construction $\Pi(p') = \Pi(\phi_-(p))$.

We define the continuation of the leaf which contains the point $q = \Pi(p)$, as the projection of the trajectory of X that starts at the point p'. This is a regular curve which enters the region Λ_1 and thus completes the foliation $\mathcal{F}$.

In other words, the leaf $\mathcal{F}_q$ that passes through a point $q = \Pi(p) \in S_f^-$ is the union of two curves $\mathcal{F}_q^L$ and $\mathcal{F}_q^R$. The first one is the projection of a negative trajectory $\gamma_t^X(p)$, $t \leq 0$, of the vector field X. The other, $\mathcal{F}_q^R$, is the projection of the positive trajectory $\gamma_t^X(p')$.

c) Observe that $\mathcal{F}$ is a continuous foliation, with C^1 leaves, except at the points belonging to S_f^-. At these point, it is topologically transverse to S_f^-. We make the same construction for both systems, obtaining the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

The homemorphism h is constructed by sending leaves of $\mathcal{F}_1$ to leaves of the corresponding foliation $\mathcal{F}_2$.

We start by using the above conjugacy $g : S_1^- \longrightarrow S_2^-$, between ϕ_-^1 and ϕ_-^2 which is part of the space of leaves of $\mathcal{F}$. Next we send $\mathcal{F}_q^L$ to $\hat{\mathcal{F}}_{h(q)}^L$ and observe that the conjugacy automatically gives that $\mathcal{F}_q^R$ is sent to $\hat{\mathcal{F}}_{h(q)}^R$. Moreover, since the restriction of g to the cuspidal edge C is the identity, it is easy to extend the homeomorphism in the whole of the space of leaves of $\mathcal{F}_1$.

Once we have a homeomorphism in the space of leaves, the homeomorphism is defined on each leaf and we only need to require that it sends S_1^- to S_2^-. Thus concluding the proof of Theorem 4. □

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