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Complete foliations of space forms by hypersurfaces

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Abstract. We study foliations of space forms by complete hypersurfaces, under some mild conditions on its higher order mean curvatures. In particular, in Euclidean space we obtain a Bernstein-type theorem for graphs whose mean and scalar curvature do not change sign but may otherwise be nonconstant. We also establish the nonexistence of foliations of the standard sphere whose leaves are complete and have constant scalar curvature, thus extending a theorem of Barbosa, Kenmotsu and Oshikiri. For the more general case of r-minimal foliations of the Euclidean space, possibly with a singular set, we are able to invoke a theorem of Ferus to give conditions under which the nonsingular leaves are foliated by hyperplanes.

Keywords: graphs, Riemannian foliations, Bernstein-type theorems, higher order mean curvatures.

Mathematical subject classification: Primary: 53C42; Secondary: 53C12, 53C40.

1 Introduction

Codimension-one foliations of Riemannian spaces have been studied, through the geometric point of view, since the beginnings of the last century, when S. Bernstein [3], proved that the only entire minimal graphs in $\mathbb{R}^3$ are planes. This result was later extended by J. Simons [12], for entire minimal graphs in $\mathbb{R}^{n+1}$ up to $n = 7$, and disproved by E. Bombieri, E. de Giorgi and E. Giusti [4] in all higher dimensions. We refer the reader to a paper of B. Nelli and M. Soret [10] for a brief account of interesting related results on Bernstein's problem, as it became known these days.

A natural extension to the problem above is to consider codimension one complete foliations of space forms, whose leaves have constant mean curva-

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ture. In this respect, J.L. Barbosa, K. Kenmotsu and G. Oshikiri [1] proved that such a foliation must have minimal leaves if the ambient space is flat, and does not exist in the sphere. Related results for graphs in products $M \times \mathbb{R}$ were also obtained by J.L. Barbosa, G.P. Bessa and J.F. Montenegro [2], by imposing some restrictions on the fundamental tone of the Laplacian on the graph.

In this paper we study foliations of space forms by complete hypersurfaces, asking that the leaves have bounded second fundamental form and two consecutive higher order mean curvatures not changing signs. For the particular case of a graph in Euclidean space whose defining function satisfies certain growth conditions, in Theorem 1 we are thus able to use a result of D. Ferus (Theorem 5.3 of [7]) to get a lower estimate on the relative nullity of the graph; we also discuss some examples that show that our hypotheses are not superfluous. As an interesting consequence, we obtain in Corollary 2 a Bernstein-type theorem for such a graph, provided its mean and scalar curvature do not change sign (but may otherwise be nonconstant).

For the case of general, transversely orientable foliations of space forms, we follow the approach of [1], computing in Proposition 2 the divergence of the vector field $P_r\overline{D}_N N$ on a leaf of the foliation; here, N is a unit vector field on the ambient space, normal to the leaves, and P_r is the r-th Newton transformation of a leaf with respect to N. We are then able to extend one of the above mentioned theorems of [1], proving the nonexistence of foliations of the standard sphere whose leaves are complete and have constant scalar curvature greater than one. We also consider a more direct generalization of the problem of Bernstein, i.e., that of the study of r-minimal foliations (possibly with a singular set) of the Euclidean space. In this setting, we are also able to rely to Ferus' theorem to prove that the nonsigular leaves are foliated by hyperplanes of a certain codimension, provided the r-th curvature of them does not vanish. We remark that problems of this kind have already been considered by the first author in the Lorentz setting [5].

Besides the formula for the divergence of $P_r\overline{D}_N N$, another central tool for our work is a further elaboration, undertaken in Proposition 1 and Corollary 1, of S.T. Yau's extension (cf. [14]) of H. Hopf's theorem on subharmonic functions on complete noncompact Riemannian manifolds.

2 Graphs in Euclidean space

In what follows, unless otherwise stated, all spaces under consideration are supposed to be connected.

In the paper [14], S.T. Yau obtained the following version of Stokes' theorem on an n-dimensional, complete noncompact Riemannian manifold M: if $\omega \in \Omega^{n-1}(M)$, an $n-1$ differential form on M, then there exists a sequence B_i of domains on M, such that $B_i \subset B_{i+1}$, $M = \cup_{i\geq 1} B_i$ and

$$\lim_{i\to+\infty} \int_{B_i} d\omega = 0.$$

By applying this result to $\omega = \iota_{\nabla f}$, where $f : M \to \mathbb{R}$ is a smooth function, ∇f denotes its gradient and $\iota_{\nabla f}$ the contraction in the direction of ∇f, Yau established the following extension of H. Hopf's theorem on a complete noncompact Riemannian manifold: a subharmonic function whose gradient has integrable norm on M must actually be harmonic.

We begin by extending the above result a little further. In what follows, we suppose M oriented by the volume element dM, and let $\mathcal{L}^1(M)$ be the space of Lebesgue integrable functions on M.

Proposition 1. *Let X be a smooth vector field on the n dimensional complete, noncompact, oriented Riemannian manifold M^n, such that* $\mathrm{div}X$ *does not change sign on M. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}X = 0$ *on M.*

Proof. Suppose, without loss of generality, that $\mathrm{div}X \geq 0$ on M. Let ω be the $(n-1)$-form in M given by $\omega = \iota_X dM$, i.e., the contraction of dM in the direction of a smooth vector field X on M. If $\{e_1, \ldots, e_n\}$ is an orthonormal frame on an open set $U \subset M$, with coframe $\{\omega_1, \ldots, \omega_n\}$, then

$$\iota_X dM = \sum_{i=1}^{n} (-1)^{i-1} \langle X, e_i \rangle \omega_1 \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \wedge \omega_n.$$

Since the $(n-1)$-forms $\omega_1 \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \wedge \omega_n$ are orthonormal in $\Omega^{n-1}(M)$, we get

$$|\omega|^2 = \sum_{i=1}^{n} \langle X, e_i \rangle^2 = |X|^2.$$

Then $|\omega| \in \mathcal{L}^1(M)$ and $d\omega = d(\iota_X dM) = (\mathrm{div}X)dM$. Letting B_i be as in the preceeding discussion, we get

$$\int_{B_i} (\mathrm{div}X) dM = \int_{B_i} d\omega \xrightarrow{i} 0.$$

But since $\mathrm{div}X \geq 0$ on M, it follows that $\mathrm{div}X = 0$ on M. □

Now, let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. If M is a complete, orientable, immersed hypersurface on $\overline{M}$, oriented by the choice of a smooth unit vector field N, we let $A : TM \to TM$ be the shape operator of M, i.e., $AX = -\overline{D}_X N$, where $\overline{D}$ stands for the Levi-Civitta connection of $\overline{M}$. For $0 \leq r \leq n$, the r-th Newton tensor P_r on M is recursively defined by

$$P_r = S_r I - A P_{r-1},$$

where $P_0 = I$, the identity operator on each tangent space of M, and S_r is the r-th elementary symmetric function of the eigenvalues of A (we also set $S_0 = 1$ and $S_r = 0$ if $r > n$). A trivial induction shows that

$$P_r = \sum_{j=0}^{r} (-1)^j S_{r-j} A^{(j)}, \tag{1}$$

where $A^{(j)}$ denotes the composition of A with itself, j times ($A^{(0)} = I$).

One step ahead, let f be a smooth function on M and $L_r f = \mathrm{tr}(P_r \mathrm{Hess} f)$. Then L_0 is the Laplacian of M and, if $\overline{M}$ has constant sectional curvature, H. Rosenberg proved in [13] that $L_r f = \mathrm{div}(P_r \nabla f)$, where div stands for the divergence on M. Concerning this setting, one gets the following consequence of Proposition 1.

Corollary 1. *Let $x : M^n \to Q^{n+1}(a)$ be a complete oriented hypersurface of a space form $Q^{n+1}(a)$, with bounded second fundamental form. If $f : M \to \mathbb{R}$ is a smooth function such that $|\nabla f| \in \mathcal{L}^1(M)$ and $L_r f$ does not change sign on M, then $L_r f = 0$ on M.*

Proof. If A is the second fundamental form of the immersion, then its eigenvalues are continuous functions on M. It thus follows from (1) that $||P_r||$ is bounded on M whenever $||A||$ is itself bounded on M. Therefore, there exists a constant $c > 0$ such that $||P_r|| \leq c$ on M, and hence

$$|P_r \nabla f| \leq ||P_r|| \, |\nabla f| \leq c |\nabla f| \in \mathcal{L}^1(M).$$

Since $L_r f = \mathrm{div}(P_r \nabla f)$ does not change sign on M, proposition 1 gives $L_r f = 0$ on M. □

We now specialize our discussion to the case of a complete oriented hypersurface $x : M^n \to \mathbb{R}^{n+1}$. If U is a parallel vector field in $\mathbb{R}^{n+1}$, we let $f, g : M \to \mathbb{R}$ be given by

$$f = \langle N, U \rangle \text{ and } g = \langle x, U \rangle, \tag{2}$$

where, as before, N is the unit normal vector field on M that gives its orientation. Letting $U^\top$ denote the orthogonal projection of U onto M, standard computations (cf. [13]) give

$$\nabla f = -A(U^\top), \ \ \nabla g = U^\top, \tag{3}$$

$$L_r f = -(S_1 S_{r+1} - (r+2)S_{r+2})f + U^\top(S_{r+1}), \tag{4}$$

$$L_r g = -(r+1)S_{r+1} f. \tag{5}$$

Specializing a little more, let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $M^n \subset \mathbb{R}^{n+1}$ be the graph of u, i.e.,

$$M^n = \big\{(x_1, \ldots, x_n, u(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1};\ (x_1, \ldots, x_n) \in \mathbb{R}^n\big\}.$$

We also make $U = (-V, 1)$ in the above discussion, where V is a parallel vector field in $\mathbb{R}^n$. Following R. Reilly [11], we can take $N = \frac{1}{W}(-\operatorname{grad} u, 1)$ as a unit normal vector field on M, where $\operatorname{grad} u$ is the gradient of u on $\mathbb{R}^n$ and $W = \sqrt{1 + |\operatorname{grad} u|^2}$. This way,

$$U^\top = \frac{1}{W^2}(\operatorname{grad} u - V + \langle \operatorname{grad} u, V\rangle \operatorname{grad} u - |\operatorname{grad} u|^2 V, \langle \operatorname{grad} u, \operatorname{grad} u - V\rangle),$$

so that $|U^\top| \leq \frac{C}{W}|\operatorname{grad} u - V|$, where $C = \sqrt{1 + 2|V|^2}$. Therefore,

$$\int_M |U^\top| dM \leq \int_{\mathbb{R}^n} \frac{C}{W}|\operatorname{grad} u - V| W dx = C \int_{\mathbb{R}^n} |\operatorname{grad} u - V| dx,$$

and this is finite if, for instance, there exist positive constants R, c and α such that $|\operatorname{grad} u(p) - V| \leq \frac{c}{|p|^{n+\alpha}}$ whenever $|p| > R$. We also point out that, in standard coordinates, the second fundamental form of M with respect to the above choice of unit normal vector field is $\frac{1}{W}\operatorname{Hess} u$, where by $\operatorname{Hess} u$ we mean the Hessian form of u on $\mathbb{R}^n$; hence, the condition that it is bounded amounts to the existence of a constant $c > 0$ for which

$$||\operatorname{Hess} u||^2 \leq c\big(1 + |\operatorname{grad} u|^2\big).$$

We can now state and prove the following

Theorem 1. *Let $M^n \subset \mathbb{R}^{n+1}$ be the graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, such that $|\operatorname{grad} u - V| \in \mathcal{L}^1(\mathbb{R}^n)$ for some $V \in \mathbb{R}^n$ and $||\operatorname{Hess} u||^2 \leq c(1 + |\operatorname{grad} u|^2)$, for some $c > 0$. If there exists $0 \leq r \leq n-1$ such that the elementary symmetric functions S_{r+1} and S_{r+2} do not change sign on M, then M has relative nullity $\nu \geq n - r$. In particular, if $S_r \neq 0$, then the graph is foliated by hyperplanes of dimension $n - r$.*

Proof. Letting f and g be as in (2), it follows from our hypotheses that both $|\nabla f|$ and $|\nabla g|$ are integrable on M. On the other hand, since M is a graph, the function f is either positive or negative on M. Since S_{r+1} doesn't change sign on M, (5) assures that the same is true of $L_r g$, and it follows from Corollary 1 that $L_r g = 0$ on M. In turn, this last information guarantees that S_{r+1} vanishes on M, so that (4) gives

$$L_r f = (r+2)S_{r+2} f.$$

By applying the same reasoning (since S_{r+2} also doesn't change sign on M), we get $L_r f = 0$ on M, and hence $S_{r+2} = 0$ on M. Finally, since $S_{r+1} = S_{r+2} = 0$, Proposition 1 of [5] gives $S_j = 0$ for all $j \geq r+1$, so that $\nu \geq n - r$.

The last claim follows from a theorem of D. Ferus (Theorem 5.3 of [7]). □

We now have immediately the following Bernstein-type result, where it is not assumed that the hypersurface has constant mean curvature.

Corollary 2. *Let $M^n \subset \mathbb{R}^{n+1}$ be the graph of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, such that $|\mathrm{grad}\, u - V| \in \mathcal{L}^1(\mathbb{R}^n)$ for some $V \in \mathbb{R}^n$ and $||\mathrm{Hess}\, u||^2 \leq c(1 + |\mathrm{grad}\, u|^2)$, for some $c > 0$. If the mean curvature of M does not change sign on it, then M is the hyperplane on $\mathbb{R}^{n+1}$ orthogonal to $(-V, 1)$.*

Proof. Letting H and R respectively denote the mean and scalar curvatures of M, just note that $S_1 = nH$ and (by Gauss' equation) $n(n-1)R = 2S_2$, so that S_1 and S_2 do not change sign on M. By the previous result, M has relative nullity n and, since it is complete, it is a hyperplane. The rest follows from our previous discussions. □

Remark 1. To see that the conditions on u are not superfluous, consider the following two examples:

1. If $u(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_r^2)(\alpha_{r+1}x_{r+1} + \cdots + \alpha_n x_n)$, where $\alpha_{r+1}, \ldots, \alpha_n$ are real constants, not all zero. If M is the graph of u, then, out of the hyperplane $\alpha_{r+1}x_{r+1} + \cdots + \alpha_n x_n = 0$, M has index of relative nullity exactly equal to $n - r$; in particular, $S_{r+1} = S_{r+2} = 0$. On the other hand, $|\mathrm{grad}\, u - V| \notin \mathcal{L}^1(\mathbb{R}^n)$ for any $V \in \mathbb{R}^n$ and there is no $c > 0$ such that $||\mathrm{Hess}\, u||^2 \leq c(1 + |\mathrm{grad}\, u|^2)$ for all $x \in \mathbb{R}^n$.

2. If $u(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ and M is the graph of u, then $S_1, S_2 > 0$ on M and $||\mathrm{Hess}\, u||^2 \leq 4n(1 + |\mathrm{grad}\, u|^2)$, although $|\mathrm{grad}\, u - V| \notin \mathcal{L}^1(\mathbb{R}^n)$ for any $V \in \mathbb{R}^n$.

3 Foliations of space forms

We now turn our attention to a more general situation, namely, we consider codimension one foliations of Riemannian manifolds and try to understand the effect of higher curvatures on the leaves. We remark that, for foliations whose leaves have constant mean curvature, this problem has been considered by Barbosa, Kenmotsu and Oshikiri in [1], and also by Bessa, Barbosa and Montenegro in [2].

As before, $\overline{M}^{n+1}$ is an $(n+1)$-dimensional orientable Riemannian manifold and $\mathcal{F}$ a smooth foliation of codimension one in $\overline{M}$. Recall (cf. [9]) that $\mathcal{F}$ is transversely orientable if we can choose a smooth unit vector field N, defined on $\overline{M}$, that is normal to the leaves of $\mathcal{F}$. If this is the case, then, for each $p \in \overline{M}$, we consider the linear operator $A : T_p\overline{M} \to T_p\overline{M}$ defined by $A(Y(p)) = -\overline{D}_{Y(p)}N$, where, as before, $\overline{D}$ denotes the Levi-Civitta connection of $\overline{M}$. It is clear that if Y is a smooth vector field on $\overline{M}$, then the same is true of $A(Y)$. Moreover, letting A_L denote the second fundamental form of a leaf L of $\mathcal{F}$, we get $A_{|L} = A_L$. Accordingly, we let $P_r : T_p\overline{M} \to T_p\overline{M}$ be the linear operator that coincides with the r-th Newton transformation on each leaf of the foliation.

Following [1], we let $X = \overline{D}_N N$, so that X is tangent to the leaves of the foliation and independent of the the choice of the field N. In what follows, we compute the divergence of $P_r(X)$ on $\overline{M}$ and on a leaf L of $\mathcal{F}$.

Proposition 2. *Let $\mathcal{F}$ be a smooth, transversely orientable foliation of codimension one of a Riemannian manifold $\overline{M}^{n+1}$, N a unit vector field on $\overline{M}$, normal to the leaves of $\mathcal{F}$ and $X = \overline{D}_N N$. If L is a leaf of $\mathcal{F}$, then*

$$\begin{aligned} \mathrm{div}_L(P_r(X)) \;=\; &\sum_{i=1}^{n}\langle \overline{R}(N, e_i)N, P_r(e_i)\rangle + \langle X, \mathrm{div}_L P_r\rangle \\ &+ \mathrm{tr}(A^2 P_r) + \langle X, P_r(X)\rangle - N(S_{r+1}), \end{aligned} \tag{6}$$

where $\overline{R}$ is the curvature tensor of $\overline{M}$, $\{e_i\}$ is an orthormal frame on L and $\mathrm{tr}(\,\cdot\,)$ *stands for the trace in L for the operator in parentheses. Moreover,*

$$\mathrm{div}_{\overline{M}} P_r(X) = \mathrm{div}_L P_r(X) - \langle P_r(X), X\rangle. \tag{7}$$

Proof. Given a point $p \in L$, choose an adapted frame field $\{e_1, \ldots, e_n, e_{n+1}\}$ defined in a neighborhood of p in $\overline{M}$, i.e., an orthonormal set of vector fields such that $e_1, \ldots, e_n$ are tangent to the leaves and $e_{n+1} = N$. Ask further that

$A(e_i(p)) = \lambda_i e_i(p)$, for all $1 \leq i \leq n$. If we call D the Levi-Civitta connection of L (and, as before, $\overline{D}$ that of $\overline{M}$), then

$$\begin{aligned}
\mathrm{div}_L P_r(X) &= \sum_{i=1}^{n} \langle D_{e_i} P_r(X), e_i \rangle \\
&= \sum_{i=1}^{n} e_i \langle P_r(X), e_i \rangle - \sum_{i=1}^{n} \langle P_r(X), D_{e_i} e_i \rangle \\
&= \sum_{i=1}^{n} e_i \langle X, P_r(e_i) \rangle - \sum_{i=1}^{n} \langle X, P_r(D_{e_i} e_i) \rangle \\
&= \sum_{i=1}^{n} e_i \langle \overline{D}_N N, P_r(e_i) \rangle - \sum_{i=1}^{n} \langle \overline{D}_N N, P_r(D_{e_i} e_i) \rangle \\
&= \sum_{i=1}^{n} \langle \overline{D}_{e_i} \overline{D}_N N, P_r(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D}_N N, D_{e_i} P_r(e_i) \rangle \\
&\quad - \sum_{i=1}^{n} \langle \overline{D}_N N, P_r(D_{e_i} e_i) \rangle \\
&= \sum_{i=1}^{n} \langle \overline{R}(N, e_i) N, P_r(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D}_N \overline{D}_{e_i} N, P_r(e_i) \rangle \\
&\quad - \sum_{i=1}^{n} \langle \overline{D}_{[N, e_i]} N, P_r(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D}_N N, D_{e_i} P_r(e_i) \rangle \\
&\quad - \sum_{i=1}^{n} \langle \overline{D}_N N, P_r(D_{e_i} e_i) \rangle \\
&= \sum_{i=1}^{n} \langle \overline{R}(N, e_i) N, P_r(e_i) \rangle - \sum_{i=1}^{n} \langle \overline{D}_N A(e_i), P_r(e_i) \rangle \\
&\quad - \sum_{i=1}^{n} \langle \overline{D}_{[N, e_i]} N, P_r(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D}_N N, D_{e_i} P_r(e_i) - P_r(D_{e_i} e_i) \rangle.
\end{aligned}$$

Now, substituting the equality

$$[N, e_i] = \sum_{j=1}^{n} \langle [N, e_i], e_j \rangle e_j + \langle [N, e_i], N \rangle N$$

into the above, we get

$$
\begin{aligned}
\mathrm{div}_L P_r(X) &= \sum_{i=1}^n \langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N\Big(\sum_{i=1}^n \langle A(e_i), P_r(e_i)\rangle\Big)\\
&\quad + \sum_{i=1}^n \langle A(e_i), \overline{D}_N P_r(e_i)\rangle - \sum_{i,j=1}^n \langle [N,e_i], e_j\rangle\langle \overline{D}_{e_j} N, P_r(e_i)\rangle\\
&\quad - \sum_{i=1}^n \langle [N,e_i], N\rangle\langle \overline{D}_N N, P_r(e_i)\rangle + \langle X, \mathrm{div}_L P_r\rangle\\
&= \sum_{i=1}^n \langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N\Big(\sum_{i=1}^n \langle e_i, AP_r(e_i)\rangle\Big)\\
&\quad + \sum_{i=1}^n \langle A(e_i), \overline{D}_N P_r(e_i)\rangle + \langle X, \mathrm{div}_L P_r\rangle\\
&\quad - \sum_{i,j=1}^n \langle \overline{D}_{e_i} N, e_j\rangle\langle A(e_j), P_r(e_i)\rangle + \sum_{i,j=1}^n \langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle\\
&\quad + \overbrace{\sum_{i=1}^n \langle \overline{D}_{e_i} N, N\rangle\langle X, P_r(e_i)\rangle}^{=0} - \sum_{i=1}^n \langle \overline{D}_N e_i, N\rangle\langle X, P_r(e_i)\rangle\\
&= \sum_{i=1}^n \langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N(\mathrm{tr} AP_r) + \langle X, \mathrm{div}_L P_r\rangle\\
&\quad + \sum_{i=1}^n \langle A(e_i), \overline{D}_N P_r(e_i)\rangle + \sum_{i,j=1}^n \langle A(e_i), e_j\rangle\langle A(e_j), P_r(e_i)\rangle\\
&\quad + \sum_{i,j=1}^n \langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle + \sum_{i=1}^n \langle e_i, \overline{D}_N N\rangle\langle X, P_r(e_i)\rangle\\
&= \sum_{i=1}^n \langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N(\mathrm{tr} AP_r) + \langle X, \mathrm{div}_L P_r\rangle\\
&\quad + \sum_{i=1}^n \langle A(e_i), \overline{D}_N P_r(e_i)\rangle + \sum_{i,j=1}^n \langle A(e_i), e_j\rangle\langle e_j, AP_r(e_i)\rangle\\
&\quad + \sum_{i,j=1}^n \langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle + \sum_{i=1}^n \langle e_i, \overline{D}_N N\rangle\langle P_r(X), e_i\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N(\mathrm{tr}AP_r) + \langle X, \mathrm{div}_L P_r\rangle \\
&\quad + \sum_{i=1}^{n}\langle A(e_i), \overline{D}_N P_r(e_i)\rangle + \sum_{i=1}^{n}\langle A(e_i), AP_r(e_i)\rangle \\
&\quad + \sum_{i,j=1}^{n}\langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle + \langle \overline{D}_N N, P_r(X)\rangle \\
&= \sum_{i=1}^{n}\langle \overline{R}(N,e_i)N, P_r(e_i)\rangle - N(\mathrm{tr}AP_r) + \langle X, \mathrm{div}_L P_r\rangle \\
&\quad + \mathrm{tr}A^2P_r + \langle X, P_r(X)\rangle + \sum_{i=1}^{n}\langle A(e_i), \overline{D}_N P_r(e_i)\rangle \\
&\quad + \sum_{i,j=1}^{n}\langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle.
\end{aligned}
$$

In order to understand the last two summands above, let $l_{ij} = \langle \overline{D}_N e_i, e_j\rangle$ and $m_{ji} = \langle A(e_j), P_r(e_i)\rangle$. It is not difficult to verify that $l_{ij} = -l_{ji}$ and $m_{ij} = m_{ji}$, so that

$$
\sum_{i,j=1}^{n}\langle \overline{D}_N e_i, e_j\rangle\langle A(e_j), P_r(e_i)\rangle = \sum_{i,j=1}^{n} l_{ij}m_{ji} = 0.
$$

On the other hand,

$$
\begin{aligned}
\sum_{i=1}^{n}\langle A(e_i), \overline{D}_N P_r(e_i)\rangle &= \sum_{i,j=1}^{n}\langle A(e_i), e_j\rangle\langle \overline{D}_N P_r(e_i), e_j\rangle \\
&= \sum_{i,j=1}^{n}\langle A(e_i), e_j\rangle N\Big(\langle P_r(e_i), e_j\rangle\Big) \\
&\quad - \sum_{i,j=1}^{n}\langle A(e_i), e_j\rangle\langle P_r(e_i), \overline{D}_N e_j\rangle \\
&= \sum_{i,j=1}^{n}\langle A(e_i), e_j\rangle N\Big(\langle P_r(e_i), e_j\rangle\Big) \\
&\quad - \sum_{i,j,k=1}^{n}\langle A(e_i), e_j\rangle\langle P_r(e_i), e_k\rangle\langle e_k, \overline{D}_N e_j\rangle.
\end{aligned}
$$

Letting $h_{ij} = \langle A(e_i), e_j\rangle$ and $t_{ik} = \langle P_r(e_i), e_k\rangle$, we get $h_{ij} = h_{ji}$ and $t_{ik} = t_{ki}$, and hence

$$\sum_{i,j,k=1}^{n} \langle A(e_i), e_j\rangle\langle P_r(e_i), e_k\rangle\langle e_k, \overline{D}_N e_j\rangle = \sum_{i,j,k=1}^{n} h_{ij}t_{ik}l_{jk} = 0.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\langle A(e_i), \overline{D}_N P_r(e_i)\rangle &= \sum_{i,j=1}^{n}\langle A(e_i), e_j\rangle N\Big(\langle P_r(e_i), e_j\rangle\Big)\\ &= \sum_{i,j=1}^{n} h_{ij}N(t_{ij})\\ &= N\Big(\sum_{i,j=1}^{n} h_{ij}t_{ij}\Big) - \sum_{i,j=1}^{n} N(h_{ij})t_{ij}\\ &= N(\mathrm{tr}(AP_r)) - \sum_{i,j=1}^{n} N(h_{ij})t_{ij}.\end{aligned}$$

Now, by means of computations analogous to those leading to (17), on page 193 of [5], we conclude that $\sum_{i,j=1}^{n} N(h_{ij})t_{ij} = N(S_{r+1})$ at p, and this concludes the proof of (6).

It is now an easy matter to get (7):

$$\begin{aligned}\mathrm{div}_{\overline{M}} P_r(X) &= \sum_{i=1}^{n}\langle\overline{D}_{e_i}P_r(X), e_i\rangle + \langle\overline{D}_N P_r(X), N\rangle\\ &= \sum_{i=1}^{n}\langle\overline{D}_{e_i}P_r(X), e_i\rangle - \langle P_r(X), \overline{D}_N N\rangle\\ &= \mathrm{div}_L P_r(X) - \langle P_r(X), X\rangle.\end{aligned}$$

□

Remark 2. Concerning the above computations, if $\overline{M}^{n+1}$ has constant sectional curvature, then Rosenberg proved in [13] that $\mathrm{div}_L P_r = 0$, thus simplifying (6). We shall use this fact twice in what follows.

We now study codimension-one foliations of $\mathbb{S}^{n+1}$ whose leaves have constant scalar curvature, thus extending Corollary 3.5 of [1][1].

[1] As is the case of [1] (since even-dimensional spheres cannot have transversely orientable foliations), the interesting case is that of odd-dimensional spheres. However, since the proof does not distinguish between odd and even, we present it in general form.

Theorem 2. *There is no smooth, transversely orientable foliation of codimension one of the Euclidean sphere $\mathbb{S}^{n+1}$, whose leaves are complete and have constant scalar curvature greater than one.*

Proof. Suppose there exists a foliation $\mathcal{F}$ of $\mathbb{S}^{n+1}$ with the properties above, let N be a unit vector field on $\mathbb{S}^{n+1}$ normal to the leaves and $A_L(\cdot) = -\overline{D}_{(\cdot)}N$ be the shape operator of a leaf L with respect to N. If R_L denotes the constant value of the scalar curvature of the leaf L of $\mathcal{F}$, it follows from Gauss' equation that $2S_2 = n(n-1)(R_L - 1)$, so that S_2 is a positive constant.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A_L, then

$$S_1^2 = |A|^2 + 2S_2 > |A|^2 \geq \lambda_i^2.$$

Choosing the orientation in such a way that $S_1 > 0$, it follows from the above inequalities that $S_1 - \lambda_i > 0$. This says that P_1 is positive definite on L.

Since the scalar curvature function $R : \mathbb{S}^{n+1} \to \mathbb{R}$, that associates to each point the value of the scalar curvature of the leaf of $\mathcal{F}$ through that point, is constant on the leaves, Proposition 2.31 of [1] gives that either R is constant on $\mathbb{S}^{n+1}$, or there exists a compact leaf L of $\mathcal{F}$ having the property that

$$R_L = \max_{p \in \mathbb{S}^{n+1}} R(p).$$

Assume first that R is nonconstant on $\mathbb{S}^{n+1}$, and let L be the compact leaf of $\mathcal{F}$ with maximal scalar curvature, so that $N(S_2) = 0$ along L. The curvature operator of the sphere, together with Remark 2 and (6), now give

$$\mathrm{div}_L P_1(X) = \mathrm{tr}(P_1) + \mathrm{tr}(A^2 P_1) + \langle X, P_1(X) \rangle > 0.$$

On the other hand, since L is compact, divergence theorem applied to L gives $\mathrm{div}_L P_1(X) = 0$, which is a contradiction.

Now, assume that R is constant on $\mathbb{S}^{n+1}$. Then $N(S_2) = 0$, and (6) and (7) give

$$\mathrm{div} P_1(X) = \mathrm{tr}(P_1) + \mathrm{tr}(A^2 P_1) > 0.$$

However, integration over $\mathbb{S}^{n+1}$ yields $\mathrm{tr}(P_1) = \mathrm{tr}(A^2 P_1) = 0$, which contradics the positive definiteness of P_1. This concludes the proof of the theorem. □

Remark 3. We point out that there are several families of compact tori in $\mathbb{S}^{n+1}$ with constant scalar curvature greater than one, and refer the reader to Example 4.4 of [6] for the details. Of course, none of them constitutes a foliation of $\mathbb{S}^{n+1}$.

We finish this paper with a generalization of Theorem 1 to a singular foliation of $\mathbb{R}^{n+1}$, by which we mean a foliation $\mathcal{F}$ of $\mathbb{R}^{n+1} \setminus S$, where $S \subset \mathbb{R}^{n+1}$ is a set of Lebesgue measure zero. In order to state the result, if $\mathcal{F}$ is a transversely orientable such foliation of $\mathbb{R}^{n+1}$, with unit normal vector field N normal to the leaves, then (as before) we let $X = \overline{D}_N N$, where $\overline{D}$ is the Levi-Civitta connection of $\mathbb{R}^{n+1}$. We also recall the reader that an isometric immersion $x : M^n \to \overline{M}^{n+1}$ is said to be r-minimal if $S_{r+1} = 0$ on M.

Theorem 3. *Let $\mathcal{F}$ be a smooth, transversely orientable singular foliation of codimension one of $\mathbb{R}^{n+1}$, whose leaves are complete, r-minimal and such that S_r doesn't change sign on them. If $|X| \in \mathcal{L}^1$ and $|A|$ is bounded along each leaf, then the relative nullity of each leaf is at least $n - r$. In particular, if $S_r \neq 0$ on a leaf, then this leaf is foliated by hyperplanes of dimension $n - r$.*

Proof. Let L be a leaf of $\mathcal{F}$. Since S_r doesn't change sign on L, we again have P_r semi-definite by a result of J. Hounie and M. L. Leite [8], so that $\mathrm{tr}(A^2 P_r)$ and $\langle X, P_r(X)\rangle$ are both nonnegative or both nonpositive on L. Therefore, by applying (6) and Remark 2 again, we get

$$\mathrm{div}_L(P_r(X)) = \mathrm{tr}(A^2 P_r) + \langle X, P_r(X)\rangle,$$

which is either greater than or less than zero on L. It thus follows from Proposition 1 that $\mathrm{div}_L P_r(X) = 0$, and, since $S_{r+1} = 0$ on L, we get

$$\mathrm{tr}(A^2 P_r) = -(r+2)S_{r+2} = 0.$$

This way, as before we get $S_k = 0$ for all $k \geq r + 1$, and it suffices to reason as in the end of the proof of Theorem 1, invoking Ferus' theorem. □

Remark 4. As an example of the situation described in the theorem above, one has the singular foliation of $\mathbb{R}^{n+1}$ by the concentric cylinders $\mathbb{S}^r_R \times \mathbb{R}^{n-r}$. Here, $\mathbb{S}^r_R \subset \mathbb{R}^{r+1}$ denotes the sphere with center $0 \in \mathbb{R}^r$ aprop:first corollary of Yau 76nd radius $R > 0$; the singular set of the foliation is the $(n - r)$-hyperplane $\{0\} \times \mathbb{R}^{n-r}$ in $\mathbb{R}^{n+1}$.

References

[1] J.L.M. Barbosa, K. Kenmotsu and G. Oshikiri. *Foliations by hypersurfaces with constant mean curvature*. Math. Zeit., **207** (1991), 97–108.

[2] J.L. Barbosa, G.P. Bessa and J.F. Montenegro. *On Bernstein-Heinz-Chern Flanders inequalities*. Mathematical Proceedings of the Cambridge Philosophical Society, **144** (2008), 457–464.

[3] S. Bernstein. *Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique*. Comm. de la Soc. Math. de Kharkov (2éme sér.), **15** (1915), 38–45.

[4] E. Bombieri, E de Giorgi and E. Giusti. *Minimal Cones and the Bernstein Problem*. Inv. Math., **82** (1968), 243–269.

[5] A. Caminha. *On spacelike hypersurfaces of constant sectional curvature Lorentz manifolds*. J. of Geom. and Physics, **56** (2006), 1144–1174.

[6] A. Caminha. *On hypersurfaces into Riemannian spaces of constant sectional curvature*. Kodai Mathematical Journal, **29** (2006), 185–210.

[7] M. Dajczer et al. *Submanifolds and Isometric Immersions*. Publish or Perish, Houston (1990).

[8] J. Hounie and M.L. Leite. *Two-Ended Hypersurfaces with Zero Scalar Curvature.* Indiana Univ. Math. J., **48** (1999), 867–882.

[9] A. Lins Neto and C. Camacho. *Geometric Theory of Foliations*. Birkhauser, Boston (1984).

[10] B. Nelli and M. Soret. *The State of the Art of Bernstein's Problem*. Mat. Contemp., **29** (2005), 69–77.

[11] R. Reilly. *On the Hessian of a Function and the Curvatures of its Graph*. Michigan Math. J., **20** (1973), 373–383.

[12] J. Simons. *Minimal Varieties of Riemannian Manifolds*. Ann. of Math., **88** (1968), 62–105.

[13] H. Rosenberg. *Hypersurfaces of Constant Curvature in Space Forms*. Bull. Sc. Math., **117** (1993), 217–239.

[14] S.T. Yau. *Some Function-Theoretic Properties of Complete Riemannian Manifolds and their Applications to Geometry*. Indiana Univ. Math. J., **25** (1976), 659–670.

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