Bull Braz Math Soc, New Series 41(3), 355-387

On a characterization of analytic compactifications for $\mathbb{C}^* \times \mathbb{C}^*$

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Abstract. Let $\mathcal{M}$ be a minimal compact surface, let $\Gamma \subset \mathcal{M}$ be a compact analytic subvariety. Assume that $X := \mathcal{M} \setminus \Gamma$ is Stein. Then we will show that X admits algebraic compactifications $\mathbf{M}_i$ (resp. non algebraic compactifications $\mathbb{M}_i$) which are not birationally equivalent (resp. not bimeromorphically equivalent) iff X is biholomorphic to $\check{\mathbb{T}} := \mathbb{C}^* \times \mathbb{C}^*$, a toric surface. However in contrast with $\check{\mathbb{T}}$, we shall show that there exist compactifiable Stein surfaces which do not admit any affine structure. Also as applications, we shall characterize the algebraic structures of arbitrary compactifiable surfaces X according to the topological type of Γ.

Keywords: affine C-bundle, nonelliptic Hopf surface, Kodaira dimension.

Mathematical subject classification: Primary: 14J26, 32E10, 32J05; Secondary: 32J15.

1 Introduction

Unless the contrary is explicitly stated, all C-analytic spaces are assumed to be *non compact* and algebraic varieties are defined over $\mathbb{C}$, irreducible and non complete. Also 2-dimensional, connected C-analytic manifolds will be referred to as *surfaces*. Since our investigations rely entirely on Kodaira classification of compact surfaces, unless otherwise specified, all compact surfaces are assumed to be *minimal* i.e. free from *exceptional curves of the first kind*; although some results mentioned here also hold for arbitrary compact surfaces. For a given compact surface M, let us denote by $a(M)$:= the transcendence degree of the field of global meromorphic functions on M over $\mathbb{C}$. Also 1-dimensional C-analytic spaces will be referred to simply as *curves*.

Received 24 November 2009.

Definition 1.1.

(1) A compact surface M is said to be an *analytic compactification* of a given surface X if there are given:

   (a) a compact C-analytic subvariety $\Gamma \subset M$, and
   (b) a biholomorphism $X \cong M \backslash \Gamma$.

(2) A surface X is said to be *compactifiable* if it admits an analytic compactification M.

(3) A compactifiable surface X is said to admit an *algebraic* (resp. a *non algebraic*) compactification if M is a projective algebraic (resp. a non algebraic) variety. Then, by abuse of language, we shall say that X admits an algebraic (resp. a non algebraic) structure.

(4) A surface X is said to admit an *affine* structure if there exists an affine variety $\mathcal{X}$ such that $X \cong \mathcal{X}_h$, where $\mathcal{X}_h$ is the underlying C-analytic space associated to $\mathcal{X}$.

(5) Finally, to simplify our notation, from now on, we shall refer to any surface $X \cong \mathbb{C}^* \times \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C} \setminus 0$, as a *toric surface* and will be denoted from now on by $\check{\mathbb{T}}$.

Our main concern here is the following:

Problem 1.2. To classify all the compactifiable Stein surfaces?

In [30], it was shown that all compactifiable Stein surfaces are *quasiprojective*; in particular, they admit algebraic structures, namely, one has

Theorem 1.3. *Let X be a compactifiable Stein surface. Then there exists an algebraic variety $\mathcal{X}$ such that $X \cong \mathcal{X}_h$.*

Furthermore, one has:

Theorem 1.4. [29] (Theorem 6) *Let X be a compactifiable Stein surface. Then all non algebraic compactifications of X are bimeromorphically equivalent, provided $X \ncong \check{\mathbb{T}}$.*

Furthermore, a complete classification of Stein surfaces in Theorem 1.4 is also established, namely

Theorem 1.5. [30] (Theorem 3) *Let X be a given Stein surface.*

Then X admits a non algebraic compactification iff $X \cong \mathbb{A}^{\nu}_{\alpha}$.

Notation 1.6. [1] Let $0 < |\alpha| < 1$. Then from now on, affine C-bundles of degree $\nu \leq 0$, without global holomorphic sections, over the elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$ will be denoted by $\mathbb{A}_\alpha^\nu$.

Notice that it is a serendipity that, for any α,

$$\check{\mathbb{T}} \cong \mathbb{A}_\alpha^0 \tag{1.6.1}$$

Confronted with this state of affairs, it is natural to consider the following:

Problem 1.7. Let $\Gamma_i \subset M_i$ with $i = 1$ or 2 be compact analytic subvarieties in the compact algebraic surfaces M_i. Assume that $X_i := M_i \setminus \Gamma_i$ are biholomorphic Stein surfaces.

Is it true that M_i are always birationally equivalent?

In analogy with Theorem 1.4, our Main Theorem which is even valid for *non minimal* compact surfaces, will tell us that $X_i \ncong \check{\mathbb{T}}$ is the necessary and sufficient condition for the birational equivalence of the M_i's. As pointed out to us by the referee, Problem 1.7 is a triviality ([8] Corollary I.4.5), if one replaces the assumption "biholomorphic Stein surfaces" by "algebraically isomorphic" with M_i being arbitrary complete algebraic varieties of any dimension. Indeed, in order to explore a peculiar aspect of the Main Theorem, let us consider the following:

Example 1.8. Let $\mathfrak{A}$ be a fixed non singular affine curve of genus $g \geq 1$ and let $\mathcal{X}_i$ with $i = 1$ or 2, be the total spaces of 2 distinct *non trivial* algebraic line bundles on $\mathfrak{A}$. Then one can check that:

(1) $\mathcal{X}_i$ are affine varieties;

(2) their underlying analytic spaces X_i are biholomorphic to $\mathfrak{A}_h \times \mathbb{C}$ since $\mathfrak{A}_h$ is an open Riemann surface.

In view of our Main Theorem, the Stein surfaces X_i only admit birational algebraic compactifications; yet, one can check that ([25] Proposition 3.1) $\mathcal{X}_i$ are not algebraically isomorphic.

This shows a marked difference between this setting in algebraic geometry and its counterpart in the analytic category. Notice that all known examples of compactifiable Stein surfaces, which are not affine do indeed admit some affine structure; more generally, one would like to know whether the algebraic variety $\mathcal{X}$ in Theorem 1.3 can always be chosen to be affine. Precisely, here our fundamental issue would be ([30] Problem 2').

Problem 1.9. (Enoki) Do compactifiable Stein surfaces always admit some affine structure?

Finally there was the following

Problem 1.10. (Hartshorne) To classify compactifiable Stein surfaces which are not affine?

As an application of the above result, let us consider the following general setting:

Problem 1.11. Let $\Gamma := \cup_i \Gamma_i$ (resp. $\Gamma' := \cup_i \Gamma'_i$) be a connected compact curve in a given compact surface M(resp. M'), where Γ_i (resp. Γ'_i) are the irreducible components of Γ (resp. Γ'). Let $a_{i,j} := \Gamma_i \cdot \Gamma_j$ (resp. $a'_{i,j} := \Gamma'_i \cdot \Gamma'_j$) and let $\mathsf{A} := (a_{ij})$ (resp. $\mathsf{A}' := (a'_{ij})$) be the intersection matrix. Assume that $X := M \setminus \Gamma$ and $X' := M' \setminus \Gamma'$ are biholomorphic. Does the bimeromorphy of M and M' determine by the eigenvalues of A and A'?

This paper is the continuation of [29, 30, 31]; in particular, it will provide an affirmative answer (resp. counterexamples) to Problem 1.7 (resp. Problem 1.9). Also it seeks to rectify and strengthen some results there. So the organization will be as follows: In section 2, we shall briefly review the intrinsic character of $\check{\mathbb{T}}$. In section 3, we shall classify the non algebraic structures of compactifiable Stein surfaces. In section 4, the uniqueness issue of compactifiable Stein surfaces will be taken up. Section 5 will be devoted to the proof of the Main Theorem which provides an affirmative answer to Problem 1.7. The affine structure of compactifiable Stein surfaces will be explored in section 6 in which counterexamples to Problem 1.9 will be exhibited. Also detailed discussions of Problem 1.10 will be carried out. Finally in section 7, we shall tackle Problem 1.11.

2 The toric surface

2.1. This venture, as well as many others, was inspired by the groundbreaking paper [9] and the following pioneering observation: [23] (p. 108)

"For any non singular elliptic curve $\tilde{\tau} := \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ viewed as a Lie group, there exists a unique algebraic group $\mathcal{G}$ which is a non trivial extension

$$0 \longrightarrow \mathbb{G}_a \longrightarrow \mathcal{G} \longrightarrow \tilde{\tau} \longrightarrow 0 \tag{2.1.1}$$

where $\mathbb{G}_a$ is the 1-dimensional additive group. Consequently, one can check [7] that

(a) the *algebraic* cohomology group $\mathcal{H}^0(\mathcal{G}, \mathbf{O}_\mathcal{G}) = \mathbb{C}$ where $\mathbf{O}_\mathcal{G}$ is the *algebraic* structure sheaf of $\mathcal{G}$; but

(b) *analytically* $\mathcal{G}$ is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m =: \mathcal{G}_{an}$ where $\mathbb{G}_m$ is the 1-dimensional multiplicative group. Thus the *analytic* cohomology group $H^0(\mathcal{G}_{an}, \mathbf{O}_{an})$ has infinite dimension, where $\mathbf{O}_{an}$ is the analytic structure sheaf of $\mathcal{G}_{an}$".

2.2. We infer readily that $\check{\mathbb{T}}$ admits both affine algebraic and non affine algebraic structures. In other words, $\check{\mathbb{T}}$ admits two distinct families of algebraic compactifications:

(1) the non rational ones, namely the elliptic ruled surfaces $\pi : \mathcal{E} \to \tilde{\tau}$.

(2) the rational ones, namely Hirzebruch surfaces, $\mathbb{F}_n$ for any $n \geq 0$ and $n \neq 1$.

Remark 2.3.

(1) It was shown (see e.g. [24, 25, 27]) that those 2 families are the only algebraic compactifications of $\check{\mathbb{T}}$.

(2) Although all rational structures of $\check{\mathbb{T}}$ are birationally equivalent, the novelty of (2.1.1) stems from the fact that it inherits $\check{\mathbb{T}}$ with *infinitely* many different (i.e. non birationally equivalent) algebraic structures.

2.4. This phenomenon shows a sharp contrast with the case when $\dim_C X = 1$ [25] or when X is a compact C-analytic space which admits, in view of GAGA principle [8], at most one algebraic structure. An alternate approach to (2.1.1) was also established in [18] (p. 145) as follows:

2.5. Let $\mathcal{G}$ be the rank 2 group of (2×2) diagonal matrices with complex entries. Hence $\mathcal{G} \cong \mathbb{C}^* \times \mathbb{C}^*$. Let $\mathcal{A}$ be the subgroup of $\mathcal{G}$, consisting of those matrices of the form

$$\begin{pmatrix} \exp z & 0 \\ 0 & \exp iz \end{pmatrix}$$

with $z \in \mathbb{C}$. Obviously $\mathcal{A}$ is a closed subgroup of $\mathcal{G}$. Since $\mathcal{A} \cong \mathbb{C}$, we infer that $\mathcal{G}$ is a topologically trivial principal bundle over $\mathcal{G}/\mathcal{A}$ with structural group $\mathcal{A}$. Since $\mathbb{C}$ is contractible, one has an isomorphism of fundamental groups $\pi_1(\mathcal{G}) \cong \pi_1(\mathcal{G}/\mathcal{A})$. Since $\mathcal{G}/\mathcal{A}$ is 1-dimensional and its fundamental group is abelian with 2 generators, it follows readily that $\mathcal{G}/\mathcal{A}$ is an elliptic curve.

2.6. This result tells us that the toric surface $\check{\mathbb{T}}$ admits a structure of an *affine principal line bundle* of *degree zero* over an elliptic curve.

2.7. Also notice that few years earlier, the structure of such a bundle, also known as A-bundle of degree zero, was thoroughly investigated in [1]. Apparently, it was not aware of the important isomorphism (1.6.1), until [23] and [18] came along.

3 The existence of an algebraic structure

Our main goal here is the following

Problem 3.1. To classify the Stein surfaces X which admit non algebraic compactifications?

Since, as we'll see later, affine C-bundles of degree $\nu \leq 0$ over some elliptic curve, are answers to Problem 3.1, so let us recall, for the sake of completeness, some fundamental constructions:

3.2. [4] Let $k \geq 1$ be some fixed integer. Let $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$, let $\mathbf{t} := (t_1, \ldots, t_k) \in \mathbb{C}^k$ and let $\nu \in \mathbb{C}^*$. Now let

$$\tau := \sum_{j=1}^{k} t_j \nu^{j-1}$$

and let us define a holomorphic automorphism

$$\begin{aligned} g_{k,\alpha,\tau} \quad : \quad & \mathbb{C} \times \mathbb{C}^* \longrightarrow \mathbb{C} \times \mathbb{C}^*, \quad \text{by} \\ & (u, v) \mapsto (v^k u + \tau, \alpha v). \end{aligned}$$

3.3. Let $\mathcal{A}_{k,\alpha,\mathbf{t}}$ be the quotient surface $\mathbb{C} \times \mathbb{C}^* / \langle g_{k,\alpha,\tau} \rangle$. Then one can check that:

(1) $\mathcal{A}_{k,\alpha,\mathbf{t}}$ is a bundle of affine lines over the elliptic curve $C_\alpha := \mathbb{C}^*/\langle \alpha \rangle$, with structural group, the affine group. Its linear part L is a holomorphic line bundle over C_α such that $c_1(\mathsf{L}) = -k$.

(2) In correlation with notation 1.6, notice that, in the case where $\mathbf{t} \neq 0$, one has, with $\nu = -k$,

$$\mathcal{A}_{k,\alpha,\mathbf{t}} = \mathbb{A}_\alpha^\nu$$

Hence, from now on, $\mathcal{A}_{k,\alpha,\mathbf{t}}$ will be referred to, as affine C-bundles of degree $-k$ over C_α.

(3) As mentioned in 2.6, $\check{\mathbb{T}}$ can be realized as an affine C-bundle of degree zero over some elliptic curve. However, in contrast with $\check{\mathbb{T}}$, each $\mathcal{A}_{k,\alpha,\mathbf{t}}$ admits (up to birational equivalence) unique algebraic structures (see e.g. Theorem 3.11 below). In particular, the latter does not admit rational compactifications.

Now let us mention some intrinsic properties of affine C-bundles $\mathcal{A}_{k,\alpha,\mathbf{t}}$.

Theorem 3.4. [4] *Let $\mathcal{A}$ be an affine C-bundle of degree $-k < 0$, over some elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$. Then $\mathcal{A}$ is equivalent as an affine C-bundle to some $\mathcal{A}_{k,\alpha,\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{C}^k$.*

Lemma 3.5. [30]

(1) *When $\mathbf{t} \neq 0$, $\mathcal{A}_{k,\alpha,\mathbf{t}}$ are free of compact curves.*

(2) *Meanwhile, $\mathcal{A}_{k,\alpha,0}$ is the total space of a line bundle L on $\mathbb{C}^*/\langle\alpha\rangle$ such that $c_1(\mathsf{L}) = -k$.*

Notation 3.6. For any geometric ruled surface $\pi : M \to \mathfrak{C}_g$ where $\mathfrak{C}_g$ is a compact non singular curve of genus $g \geq 0$, there exists a rank 2 vector bundle $\mathfrak{V}_g$ on $\mathfrak{C}_g$ such that $M \simeq \mathbb{P}(\mathfrak{V}_g)$. Furthermore, we assume that $\mathfrak{V}_g$ is *normalized* in the sense of Hartshorne [8] (V.2.8.1) and the number $e := -c_1(\det \mathfrak{V}_g)$ will be referred to as an *invariant* of M. Also let Ξ be the *canonical* section of M with $\mathbf{O}_M(\Xi) \simeq \mathcal{O}(1)$, where $\mathcal{O}$ is the structural sheaf of $\mathbf{P}(\mathfrak{V}_g)$. Then $\Xi^2 = -e$. Also, let $\mathbb{F} := \pi^{-1}(x)$ for any $x \in \mathfrak{C}_g$ be the fibres of M.

Convention 3.7. [26] For any fixed $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$, let $\mathfrak{V}_1$ be a rank 2 vector bundle on the elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$ and let $\mathcal{R} := \mathbb{P}(\mathfrak{V}_1)$ be the corresponding ruled surface.

(1) If $\mathfrak{V}_1$ is *indecomposable* with invariant $e = 0$, then let us denote the ruled surface $\mathcal{R}$ by R_0^α. The latter contains a unique section, say Ξ_0 such that $\Xi_0^2 = 0$.

(2) If $\mathfrak{V}_1$ is *indecomposable* with invariant $e = -1$, then let us denote the ruled surface $\mathcal{R}$ by R_{-1}^α. The latter contains *infinitely* many sections, say Ξ_{-1} such that $\Xi_{-1}^2 = 1$.

(3) If $\mathfrak{V}_1$ is *decomposable* with invariant $e \geq 0$, then let us denote the ruled surface $\mathcal{R}$ by S_e^α. The latter contains a *canonical* section, denoted by Ξ such that $\Xi^2 = -e$ and another section, say Ξ_∞ such that $\Xi \cap \Xi_\infty = \emptyset$ and that $\Xi_\infty^2 = e$.

Definition 3.8. (see e.g. [28]) A surface X is said to be *strongly pseudoconvex* (or *1-convex* for short) if there exist

(1) a 2-dimensional Stein space Y with only finitely many isolated normal singularities, say p_i and

(2) a proper and surjective morphism $\pi : X \to Y$, inducing a biholomorphism

$$X \setminus S \simeq Y \setminus \bigcup_i \{p_i\}$$

where $S := \cup_i \pi^{-1}(p_i)$ will be referred to as the *exceptional set* of X.

Remark 3.9. Obviously any Stein surface is 1-convex (with $S = \emptyset$). So from now on, 1-convex surfaces which are not Stein (i.e. $\dim_C S > 0$) will be referred to as *properly* 1-convex surfaces.

Lemma 3.10.

(1) *The affine C-bundle $\mathcal{A}_{k,\alpha,\mathbf{t}}$(with $\mathbf{t} \neq 0$) are compactifiable Stein surfaces which also admit affine structure.*

(2) *Meanwhile $\mathcal{A}_{k,\alpha,0}$ are compactifiable properly 1-convex surfaces with exceptional set, an elliptic curve Ξ, such that $\Xi^2 = -k < 0$.*

Proof. By definition, each such $\mathcal{A}_{k,\alpha,\mathbf{t}}$ admits an elliptic ruled surface $\pi : \mathbb{E}_\alpha := \mathbb{P}(\mathfrak{V}_1) \to \mathfrak{C}_\alpha$ as its compactification. In particular, one can find a section $\Theta \subset \mathbb{E}_\alpha$ such that $\mathcal{A}_{k,\alpha,\mathbf{t}} \simeq \mathbb{E}_\alpha \setminus \Theta := \mathbb{X}$. Now one will have the following 3 alternatives:

- If $\Theta^2 < 0$, then $\mathfrak{V}_1$ is necessarily decomposable and $\Theta = \Xi$ the canonical section; hence there exists a section at "infinity", say $\Lambda \subset \mathbb{E}_\alpha$ such that $\Lambda^2 > 0$ and $\Theta \cdot \Lambda = 0$ i.e. $\Lambda \subset \mathbb{X} \simeq \mathcal{A}_{k,\alpha,\mathbf{t}}$ which is not possible in view of Lemma 3.5.

- If $\Theta^2 = 0$, then one has 2 possibilities:

  (a) If $\mathfrak{V}_1$ is decomposable, then $\mathcal{A}_{k,\alpha,\mathbf{t}}$ will contain at least one compact curve; but that will contradict Lemma 3.5.

  (b) On the other hand, if $\mathfrak{V}_1$ is indecomposable, it means that $\mathcal{A}_{k,\alpha,\mathbf{t}}$ is an affine bundle of degree zero, but this is not possible, since $k \neq 0$.

- Therefore, $\Theta^2 > 0$. We infer readily that $\mathcal{A}_{k,\alpha,\mathbf{t}}$ are 1-convex, see e.g. [29].

(1) Now, as far as affine C-bundle $\mathcal{A}_{k,\alpha,\mathbf{t}}$ are concerned, as previously observed (lemma 3.5) since $\mathbf{t} \neq 0$, $\mathcal{A}_{k,\alpha,\mathbf{t}}$ are free of compact curves. Hence Θ is actually an ample divisor. Thus $\mathbb{X}$ is affine. In particular $\mathcal{A}_{k,\alpha,\mathbf{t}}$ is Stein.

(2) On the other hand, as noticed earlier, if $\Theta := \mathbb{E}_\alpha \setminus \mathcal{A}_{k,\alpha,0}$, then $\Theta^2 = k$; in particular, $\mathcal{A}_{k,\alpha,0}$ is properly 1-convex and admits $\Xi :=$ the canonical section of L as exceptional set with $\Xi^2 = -k < 0$. $\square$

We are now in a position to provide a complete structure of elliptic surfaces M which are compactifications of $\mathcal{A}_{k,\alpha,\mathbf{t}}$.

Theorem 3.11. *Let M be an algebraic compactification of $\mathcal{A}_{k,\alpha,\mathbf{t}}$ and let $\Gamma := M \backslash \mathcal{A}_{k,\alpha,\mathbf{t}}$. Then*

(1) $\Gamma^2 = k \geq 1$.

(2) $M \simeq \mathsf{R}_0^\alpha$ *(resp. S_e^α) iff $\mathbf{t} \neq 0$ and k (resp. e) is even.*

(3) $M \simeq \mathsf{R}_{-1}^\alpha$ *(resp. S_e^α) iff $\mathbf{t} \neq 0$ and k (resp. e) is odd.*

(4) $M \simeq \mathsf{S}_e^\alpha$ *with $e = k$ iff $\mathbf{t} = 0$.*

Proof. In view of Lemma 3.10, $M \simeq \mathbb{P}(\mathfrak{V}_1)$ is an elliptic ruled surface. Also it follows from [4] (Proposition 7.1) that $\Gamma^2 = k$.

- Assume that $\mathbf{t} \neq 0$. Then $\mathcal{A}_{k,\alpha,\mathbf{t}}$ is Stein. Consequently Γ is an ample divisor.

  (1) If $k = 2p$ with $p \geq 1$, then it is easy to see that $M \simeq \mathsf{R}_0^\alpha$, provided that $\mathfrak{V}_1$ is indecomposable. Then one can check that

  $$\Gamma \equiv \Xi_0 + p\mathbb{F}$$

  where $\equiv$ stands for numerical equivalence.
  On the other hand if $\mathfrak{V}_1$ is decomposable, then $M \simeq \mathsf{S}_e^\alpha$ with e even. Then one has ([8] Proposition V.2.6), in view of the ampleness of Γ

  $$\Gamma \equiv \Xi + r\mathbb{F}$$

  where $r = p + \frac{e}{2}$ and $p > \frac{e}{2}$.

(2) If $k = 2p - 1$ with $p \geq 1$, then $M \simeq \mathsf{R}^{\alpha}_{-1}$ provided $\mathfrak{V}_1$ is indecomposable. Then one has

$$\Gamma \equiv \Xi_{-1} + (p-1)\mathbb{F}.$$

On the other hand if $\mathfrak{V}_1$ is decomposable then $M \simeq \mathsf{S}^{\alpha}_{e}$ with e odd. Then one can check that ([8] loc.cit.)

$$\Gamma \equiv \Xi + s\mathbb{F}$$

where $s = p + \frac{(e-1)}{2}$ and $p > \frac{(e+1)}{2}$.

- Assume that $\mathbf{t} = 0$. Then $\mathcal{A}_{k,\alpha,0}$ is 1-convex. Consequently, $M \simeq \mathsf{S}^{\alpha}_{e}$ with $e = k$. $\square$

Corollary 3.12. *Let M be an algebraic compactification of $\mathcal{A}_{1,\alpha}, \mathbf{t}$ with $\mathbf{t} \neq 0$. Then necessarily $M \simeq \mathsf{R}^{\alpha}_{-1}$.*

Therefore, for any $0 \neq \mathbf{t} \neq \mathbf{t}' \neq 0$, one has

$$\mathcal{A}_{1,\alpha,\mathbf{t}} \simeq \mathcal{A}_{1,\alpha,\mathbf{t}'}$$

On the other hand, one has, for any α and β,

$$\mathsf{R}^{\alpha}_{0} \setminus \Xi_0 \simeq \check{\mathbb{T}} \simeq \mathsf{R}^{\beta}_{0} \setminus \Xi_0$$

Definition 3.13. [16] Let $t, \alpha, \beta \in \mathbb{C}$ with $0 < |\alpha| \leq |\beta| < 1$, let $U := \mathbb{C}^2 \setminus (0,0)$, let $m \geq 1$ and let $g \colon U \to U$ be an automorphism of U defined by

$$g(z,w) := (\alpha z + t w^m, \beta w)$$

such that

$$(\alpha^m - \beta)t = 0$$

Now one can check [16] (p. 695) that the cyclic group $\langle g \rangle$ is properly discontinuous and the quotient space $U/\langle g \rangle$ is a compact surface.

(1) Assume that $t \neq 0$. Then

$$g(z,w) = (\alpha z + t w^m, \alpha^m w)$$

Now let $\mathsf{H}_{\alpha,t,m} := U/\langle g \rangle$. Then one can check that

$$\begin{aligned} b_1(\mathsf{H}_{\alpha,t,m}) &= 1 \quad \text{and} \\ b_2(\mathsf{H}_{\alpha,t,m}) &= a(\mathsf{H}_{\alpha,t,m}) = 0 \end{aligned}$$

where $b_i()$ are the ith Betti numbers. Furthermore the puncture line $U \cap \{w = 0\}$ is invariant under g, so it is mapped by the projection $\pi : U \to \mathsf{H}_{\alpha,t,m}$ onto a non singular compact elliptic curve $\gamma_\alpha := \mathbb{C}^*/\langle\alpha\rangle$ which is the only compact curve in $\mathsf{H}_{\alpha,t,m}$. Also we have $\gamma_\alpha^2 = 0$.

From now on $\mathsf{H}_{\alpha,t,m}$ will be referred to as the *non elliptic Hopf surface of Type (I)*.

(2) Assume that $t = 0$ and $\alpha^p \neq \beta^q$ for any $p, q \in \mathbb{Z}^+$. Now let $\mathbb{H}_{\alpha,\beta} := U/\langle g\rangle$. Then one can check that

$$\begin{aligned} b_1(\mathbb{H}_{\alpha,\beta}) &= 1 \quad \text{and} \\ b_2(\mathbb{H}_{\alpha,\beta}) &= a(\mathbb{H}_{\alpha,\beta}) = 0 \end{aligned}$$

In this case, the puncture line $U \cap \{z = 0\}$ which is also invariant under g is mapped by π onto a non singular compact elliptic curve $\gamma_\beta := \mathbb{C}^*/\langle\beta\rangle$. Also one can check that γ_α and γ_β are the only compact curves in $\mathbb{H}_{\alpha,\beta}$. Furthermore, one has $\gamma_\beta^2 = 0$. From now on, $\mathbb{H}_{\alpha,\beta}$ will be referred to as the *non elliptic Hopf surface of Type (II).*

3.14. It was first shown in [9] that, for any $\alpha \in \mathbb{C}^*$, $m \geq 1$ and $t \in \mathbb{C}^*$ one has

$$\mathsf{H}_{\alpha,t,m} \backslash \gamma_\alpha \simeq \check{\mathbb{T}}$$

3.15. By following the program which was initiated by Kodaira, remarkable constructions of non algebraic surfaces M of class VII_0 i.e. compact surfaces with $b_1(M) = 1$, besides non elliptic Hopf surfaces, as mentioned in Definition 3.13, were explicitly exhibited in [4].

Indeed, based on special constructions by Inoue [14] and Kato [15], for any fixed $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$, $1 \leq k \in \mathbb{Z}$ and $\mathbf{t} \in \mathbb{C}^k$, Enoki [4] exhibited a series of compact surfaces, denoted by $\mathcal{M}_{\alpha,k,\mathbf{t}}$ which have the following crucial properties:

$$b_1(\mathcal{M}_{\alpha,k,\mathbf{t}}) = 1 \leq k = b_2(\mathcal{M}_{\alpha,k,\mathbf{t}}). \tag{3.15.1}$$

It follows from (3.15.1) that, automatically, one has

$$a(\mathcal{M}_{\alpha,k,\mathbf{t}}) = 0$$

Furthermore $\mathcal{M}_{\alpha,k,\mathbf{t}}$ contains a connected compact curve, say $\mathcal{D}_k$, with ${\mathcal{D}_k}^2 = 0$ such that its configuration is as follows:

(1) $\mathcal{D}_1$ is a rational curve with a single node.

(2) $\mathcal{D}_2 = \gamma_1 \cup \gamma_2$ such that, for $i = 1$ or 2:

- $\gamma_i^2 = -2$;
- $\gamma_i \simeq \mathbb{P}_1$;
- The γ_i intersect transversally at exactly 2 points.

(3) If $k \geq 3$, then $\mathcal{D}_k = \cup_{1 \leq j \leq k} \gamma_j$ where $\gamma_i \simeq \mathbb{P}_1$, $\gamma_i^2 = -2$ for any i with $1 \leq i \leq k$, and

$$\gamma_i . \gamma_j = \begin{cases} -2 & \text{if} \quad i = j \\ 1 & \text{if} \quad |i - j| = 1 \quad \text{or} \quad k - 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.16. [4] *If* $\mathbf{t} \neq 0$ *then* $\mathcal{D}_k$ *is the only compact curve in* $\mathcal{M}_{\alpha,k,\mathbf{t}}$. *On the other hand,* $\mathcal{M}_{\alpha,k,0}$ *contains only 2 compact curves:* $\mathcal{D}_k$ *and an elliptic curve* $\tilde{\tau}$ *such that* $\tilde{\tau}^2 = -k$.

Remark 3.17. In the literature, $\mathcal{M}_{\alpha,k,0}$ are referred to as *Parabolic Inoue surfaces*.

In parallel with Theorem 3.11 and Corollary 3.12, one has:

Theorem 3.18. [4] *Let* M *be an non algebraic compactification of* $\mathcal{A}_{k,\alpha,\mathbf{t}}$ *and let* $\Gamma := M \backslash \mathcal{A}_{k,\alpha,\mathbf{t}}$. *Then*

(1) $\Gamma^2 = 0$.

(2) $M \simeq \mathcal{M}_{k,\alpha,\mathbf{t}}$ *if* $\mathbf{t} \neq 0$.

(3) $M \simeq \mathcal{M}_{k,\alpha,0}$ *if* $\mathbf{t} = 0$.

Corollary 3.19. [15] *For any* $0 \neq \mathbf{t} \neq \mathbf{t}' \neq 0$, *one has*

$$\mathcal{M}_{1,\alpha,\mathbf{t}} \simeq \mathcal{M}_{1,\alpha,\mathbf{t}'}$$

From these results, we infer that Problem 3.1 is completely settled by the following: (see also [30] Theorem 3).

Theorem 3.20. *Let* M *be a non algebraic compact surface, let* $\Gamma \subset M$ *be a compact analytic subvariety and let* $X := M \backslash \Gamma$. *Then the following conditions are equivalent:*

(1) X *is Stein.*

(2) $X \simeq \check{\mathbb{T}}$ *(resp.* $X \simeq \mathcal{A}_{k,\alpha,\mathbf{t}}$ *for some* k, α *and* $\mathbf{t} \neq 0$*) if* $b_2(M) = 0$ *(resp.* $b_2(M) = k$*).*

(3) *X admits affine structure.*

Corollary 3.21. *Any compactifiable Stein surface is quasi-projective.*

Corollary 3.22. *Any compactification M of a Stein surface X is projective algebraic, provided $X \not\simeq \mathbb{A}^{\nu}_{\alpha}$ for some $\nu \leq 0$ and $0 < |\alpha| < 1$.*

Remark 3.23. Theorem 3.20 does hold for (non minimal) non algebraic compact surfaces. Indeed, let M, Γ and X be as in Theorem 3.20 with M being an arbitrary non algebraic compact surface.

Claim: If $\gamma \subset M$ is an exceptional compact curve of the first kind, then necessarily $\gamma \subset \Gamma$.

Assume the contrary. Let $\Gamma_0 \subset M$ be an exceptional curve of the first kind such that $\Gamma_0 \not\subset \Gamma$. Let $\cup_i \Gamma_i$ with $i \geq 1$ be an irreducible decomposition of Γ. Let $\Gamma' := \Gamma \cup \Gamma_0$ and let $X' := M \setminus \Gamma'$. Then certainly $X' \simeq X \setminus (\Gamma_0 \cap X)$ is Stein, since X is. On the other hand, since M is non algebraic, it follows readily that the intersection matrix $A' := (a_{ij})$ where $a_{ij} := \Gamma_i.\Gamma_j$, with i and j ≥ 0, is negative semi-definite. Consequently $A := (a_{ij})$ with i and j ≥ 1 (see e.g. [4] (Lemma 1.1)) is negative definite. In view of Hartog's Theorem, this will contradict the fact that X is Stein. Hence our claim is proved.

4 The uniqueness issue

4.1. As noticed earlier, one has, for any α and β:

$$\mathsf{H}_{\alpha,t,m} \setminus \gamma_\alpha \simeq \check{\mathbb{T}} \simeq \mathsf{H}_{\beta,s,m} \setminus \gamma_\beta \tag{4.1.1}$$

On the other hand, one can check that

$$\mathsf{H}_{\alpha,t,m} \text{ is bimeromorphic to } \mathsf{H}_{\beta,t,m} \text{ iff } \alpha = \beta \tag{4.1.2}$$

Hence complementing Remark 2.3(2), we infer from (4.1.1) and (4.1.2) that $\check{\mathbb{T}}$ admits *infinitely* many different (i.e. non bimeromorphically equivalent) non algebraic structures. Hence one would like to raise the following

Problem 4.2. Up to biholomorphism, is $\check{\mathbb{T}}$ the only compactifiable Stein surface which admits non algebraic (resp. algebraic) compactifications which are not bimeromorphically equivalent?

Our main purpose here is, on the basis of Theorem 1.4, to provide an affirmative answer to this Problem, namely

Theorem 4.3. [Main Theorem] *Let X be a given compactifiable Stein surface.*

Then all algebraic compactifications of X are birationally equivalent provided $X \not\cong \check{\mathbb{T}}$.

The proof of this result will be given in the next section. We would like to exhibit here a special but practical version of Theorem 4.3 as an illustration which has special interest in its own right. But first of all, few basic ingredients are in order.

Definition 4.4. [13] *Let $\mathcal{D}$ be a non singular algebraic curve and let C be its non singular compactification (which exists and is unique). Hence there exists finitely many points $\{q_j\} \in C$ such that $\mathcal{D} \simeq C \setminus \cup_i q_i$. Now let $g :=$ genus of C and $n :=$ card $|q_j|$. Then we say that $\mathcal{D}$ is of type (g, n).*

Theorem 4.5. [13] (Theorem 5) *Let X be a Stein surface. Assume that there exist a non singular algebraic curve $\mathcal{R}$ and a surjective mapping $\pi : X \to \mathcal{R}$. Assume that*

(1) *π is of maximal rank for any $x \in X$.*
(2) *each fibre $\mathcal{D} := \pi^{-1}(t)$ for any $t \in \mathcal{R}$ is a non singular algebraic curve of type (g, n) such that*
$$2g + n > 2$$

Then X only admits algebraic compactifications which are birationally equivalent.

We are now ready to state a special case of Theorem 4.3.

Proposition 4.6. *Theorem 4.3 holds if one assumes that X is a product of two non singular algebraic curves, namely $X \simeq \mathsf{A}_1 \times \mathsf{A}_2$.*

Proof. Notice that any non singular algebraic curve $\mathcal{D}$ does satisfy the assumption (2) in Theorem 4.5 with 2 exceptions, namely: $\mathbb{C}$ and $\mathbb{C}^*$. Consequently, it follows readily that our Proposition does hold, with possibly 3 exceptions:

(1) $\mathbb{C}^2$ or
(2) $\mathbb{C} \times \mathbb{C}^*$ or
(3) $\mathbb{C}^* \times \mathbb{C}^*$.

However, we infer from results in [17] (resp. in [27]) that the only compactifications of $\mathbb{C}^2$ (resp. $\mathbb{C} \times \mathbb{C}^*$) are rational compact surfaces. Hence our proof is complete. $\square$

5 The intrinsic behavior of $\check{\mathbb{T}}$

The main purpose of this section is to devote to a complete proof of Theorem 4.3 above. But first of all, few basic ingredients are in order:

Definition 5.1. Let M be an analytic compactification of a C-analytic manifold X, let $\Gamma := M \setminus X$ and let $\mathcal{K}_M$ be the canonical bundle of M. From the vector space $\mathbf{V} := H^0(M, \mathcal{O}(m\mathcal{K}_M + (m-1)\Gamma)$, let us consider a basis $\{\phi_1, \ldots, \phi_N\}$ which gives rise to a well defined *meromorphic* map:

$$\begin{aligned} \Phi_m \quad : \quad & M \longrightarrow \mathbb{P}_N, \\ & z \mapsto \Phi_m(z) := [\phi_0, \ldots, \phi_N]. \end{aligned}$$

where $N := \dim \mathbf{V} - 1$.

Following [21], let $N(X) := \{m > 0 | \dim \mathbf{V} > 0\}$ and let us define

$$k_a(X) = \begin{cases} \max_m \left\{ \dim \Phi_m(M) \right\} & \text{if } N(X) \neq \emptyset \\ -\infty & \text{if } N(X) = \emptyset \end{cases} \tag{5.1.1}$$

Notice that $k_a(X)$ which will be referred to as the *analytic Kodaira dimension* of X, is a bimeromorphically invariant. On the other hand, we have

Definition 5.2. [11] From the above formula (5.1.1), if one replaces the vector space $\mathbf{V}$ by $\mathbf{W} := H^0(M, \mathcal{O}(m(\mathcal{K}_M + \Gamma))$, then one obtains the so called *logarithmic Kodaira dimension* of X which we shall denote from now on by $k_l(X)$.

Remark 5.3. Although $k_l(X)$ is a birational invariant, it is, unlike $k_a(X)$, not even biholomorphically invariant (see e.g. Example 5.4 below), an aspect which will be fully exploited later on in our strategy.

Also, it is obvious from the definition that, in the special case where $\Gamma = \emptyset$, $k_a(X)$ and $k_l(X)$ will coincide with the standard notion of Kodaira dimension $k(M)$ for compact manifold M [12]. Now in general, if M is a compactification of a C-analytic manifold X, then one has [21]:

$$-\infty \leq k(M) \leq k_a(X) \leq k_l(X) \leq \dim_C M$$

Example 5.4. Let $X_1 = \mathbb{P}_2 \setminus \Gamma$ where Γ consists of 3 lines in general position. By using the same convention as in 3.7, let $X_2 := \mathsf{R}_0^\alpha \setminus \Xi_0$. Then one can check that:

(1) $X_1 \simeq X_2 \simeq \check{\mathbb{T}}$;

(2) $k_l(X_1) = 0$ and $k_l(X_2) = -\infty$;

(3) $k_a(X_1) = k_a(X_2) = -\infty$.

Proposition 5.5. [21] (Proposition 2.2) *One has*

$$k_a(X) = \dim_C X \text{ iff } k_l(X) = \dim_C X.$$

Proposition 5.6. [21] *Let Γ be a compact curve of degree d in $\mathbb{P}_2$ and let $X := \mathbb{P}_2 \setminus \Gamma$. Then*

$$k_a(X) = \begin{cases} 2 & \text{if } d > 3 \\ -\infty & \text{otherwise} \end{cases}$$

As far as Stein surfaces X with $k_a(X) < 2$ are concerned, we have the following crucial result:

Theorem 5.7. [29] (Lemma 2) *Let M be an algebraic compactification of a Stein surface X.*

Assume that $k_a(X) < 2$. Then M is a ruled surface.

Complementing Theorem 5.7, a main result in [22] Theorem 3.4 (which does hold even for non minimal algebraic compactifications), provides us the following:

Theorem 5.8. *Let $\pi : \mathcal{R}_g \to C_g$ with $g \geq 1$ be an irrational ruled surface and let $\Gamma \subset \mathcal{R}_g$ be a connected compact curve.*

Assume that $X := \mathcal{R}_g \setminus \Gamma$ is Stein and admits non birationally equivalent algebraic compactifications. Then

- $g = 1$ *i.e. $\pi : \mathcal{R}_1 \to C_1$ is necessarily an elliptic ruled surface. Furthermore*
- Γ *is either:*

  (1) *a section, or*

  (2) *an irreducible 2-section, or*

  (3) *a reducible 2-section, $C \cup \mathcal{D}$ where $C(resp.\mathcal{D})$ is a section.*

Notation 5.9. From now on, the surface $X := \mathcal{R}_1 \setminus \Gamma$ in Theorem 5.8 will be denoted by $\mathfrak{X}_I$ (resp. $\mathfrak{X}_{II}$, resp. $\mathfrak{X}_{III}$) if Γ is of type (1) (resp. (2), resp (3)) of Theorem 5.8. Also from now on, let us adhere to the following notation: $\mathcal{R}_1 = \mathbb{P}(\mathfrak{V})$ for some rank 2 bundle $\mathfrak{V}$ over C_1 with invariant e.

Definition 5.10. [5] Let $\pi : M \to C$ be a ruled surface and let $\Gamma \subset M$ be a compact curve. Then $X := M \backslash \Gamma$ is said to admit a $\mathbb{C}^*$-fibration if $\Gamma \cdot \mathbb{F} = 2$ for generic fibre $\mathbb{F}$ in M. From now on a $\mathbb{C}^*$-fibration structure on X will be denoted by $f : X \to C$ where $f := \pi | X$.

By taking Proposition 5.5 into account, we have the following alternatives:

Theorem 5.11. [21] *Let X be a quasi-projective surface. Assume that $\mathsf{k}_l(X) =$ 2. Then all algebraic compactifications of X are bimeromorphically equivalent.*

Theorem 5.12. [6] *(see also Sakai, F. Math. ann. 254, (1980) Theorem* 3.15 *(i) p. 104).*

Let X be an affine surface. Assume that $\mathsf{k}_l(X) = 1$.

Then X admits a structure of a $\mathbb{C}^*$-fibration $f : X \to C$ which is uniquely determined.

Precisely, if $g = X \to C'$ is another $\mathbb{C}^*$-fibration, then there exists an isomorphism $\sigma : C \simeq C'$ such that $g = \sigma \circ f$.

Theorem 5.13. *Let M be a compactification of a Stein surface X. Assume that $\mathsf{k}_l(X) = 0$. Then M is a rational surface provided $X \not\cong \check{\mathbb{T}}$.*

Proof. In view of Theorem 5.7, M is a ruled surface. Assume that $M = \mathcal{R}_g$ is an irrational surface with $g \geq 1$. It follows from our assumption that,

$$\mathcal{K} + \Gamma \equiv 0 \tag{5.13.1}$$

where $\mathcal{K}$ is the canonical bundle of $\mathcal{R}_g$ and $\Gamma = \mathcal{R}_g \setminus X$. In view of Theorem 5.8, one can assume that Γ is free of fibre components.

(1) Assume that Γ is irreducible. Hence from (5.13.1) and [8] (Corollary V.2.11) we have that

$$\begin{aligned} \Gamma^2 &= \mathcal{K}^2 \\ &= 8(1-g) \leq 0 \end{aligned}$$

Since X is Stein, so necessarily

$$\Gamma^2 = 0 \quad \text{i.e. } g = 1 \tag{5.13.2}$$

In this situation, one has 3 alternatives for the elliptic ruled surface $\mathcal{R}_1 := \mathbf{P}(\mathfrak{V})$.

(a) $\mathfrak{V}$ is a decomposable rank 2 bundle with $e = 0$.

In this case, its canonical section Γ will satisfy (5.13.2). From the decomposability assumption of $\mathfrak{V}$, it follows that $\mathcal{R}_1$ also contains compact curves Θ with $\Theta \cdot \Gamma = 0$, i.e. $\Theta \subset X$, contradicting the fact that X is Stein.

(b) $\mathfrak{V}$ is an indecomposable rank 2 bundle with invariant $e = -1$. Then the curve

$$\Gamma \equiv 2\Xi - \mathbb{F} \tag{5.13.3}$$

which is an elliptic curve, will satisfy (5.13.2). But it was shown [20] (Lemma 6.8) that in this situation, X also contains compact curves. That will contradict the Steiness of X. In fact, in [26], by realizing $\mathcal{R}_1$ as an hyperelliptic surface over $\mathbf{P}_1$, (at least) 3 disjoint compact curves of type (5.13.3) in X were explicitly exhibited. Consequently, this case can not occur.

(c) $\mathfrak{V}$ is an indecomposable rank 2 bundle with invariant $e = 0$.

In this case, we infer readily from Example 5.4 that $\mathcal{R}_1 \simeq \mathsf{R}_0^{\alpha}$ and $\Gamma = \Xi_0$. Consequently, $X \simeq \mathsf{R}_0^{\alpha} \setminus \Xi_0 \simeq \check{\mathbb{T}}$.

(2) Assume that $\Gamma := \mathcal{R}_1 \setminus X$ is reducible. Then, let $\Lambda \subset \Gamma$ be an irreducible component. As mentioned earlier, one can assume that Γ is free of fibre components; hence $\Lambda \cdot \mathbf{F} > 0$ for any fibre $\mathbf{F} \subset \mathcal{R}_1$. Hence

$$\begin{aligned} 0 &= \Lambda \cdot (\mathcal{K} + \Gamma) = \Lambda^2 + \mathcal{K} \cdot \Lambda + \Lambda \cdot (\Gamma \setminus \Lambda) \\ &= 2g(\Lambda) - 2 + \Lambda \cdot (\Gamma \setminus \Lambda). \end{aligned}$$

Hence $g(\Lambda) = 1$ and $\Lambda \cdot (\Gamma \setminus \Lambda) = 0$ i.e. Λ is isolated in Γ. But X is Stein, so $\Lambda = \Gamma$ and the same argument as above will apply. □

Remark 5.14.

- It follows from the arguments in (1) (c) of the proof of Theorem 5.13, we infer that the surfaces $X := \mathfrak{X}_m$ where $m = II$ or III are *affine*.

- A complete list of rational surfaces satisfying Theorem 5.13 can be found in [10] (proposition 6 and 16).

We are now in a position to provide a complete proof of the Main Theorem.

Proof. We are going to show that all algebraic compactifications of $X = \mathfrak{X}_m$ where $m = I$ or II or III, are birationally equivalent, unless $X \simeq \check{\mathbb{T}}$.

Step 1: Assume that

$$X = \mathfrak{X}_m \quad \text{with} \quad m = I \quad \text{or} \quad II. \tag{5.14.1}$$

Let us assume that X admits a rational compactification say M. So let us consider the following exact sequence of homology groups with $\mathbb{C}$-coefficients

$$\begin{aligned} 0 \ = H_3(\mathsf{M}) &\to H_3(\mathsf{M}, \Lambda) \to H_2(\Lambda) \to H_2(\mathsf{M}) \\ &\to H_2(\mathsf{M}, \Lambda) \to H_1(\Lambda) \to H_1(\mathsf{M}) = 0 \end{aligned} \tag{5.14.2}$$

where $\Lambda := \mathsf{M} \setminus X$.

On the one hand, in view of the hypothesis (5.14.1), one can check that:

$$b_1(X) = 2 \quad \text{and} \quad b_2(X) = 1. \tag{5.14.3}$$

On the other hand, by duality, we have:

$$H_3(\mathsf{M}, \Lambda) = H^1(X) \quad \text{and} \quad H_2(\mathsf{M}, \Lambda) = H^2(X). \tag{5.14.4}$$

Since $\dim_C H^2(\Lambda) =: \mu(\Lambda)$ is equal to the number of irreducible components of Λ, by combining (5.14.2), (5.14.3) and (5.14.4), one can check that:

(1) If $\mathsf{M} = \mathbf{F}_n$ then $\mu(\Lambda) = 4$, Λ consists of 2 sections and 2 fibres and $X \simeq \check{\mathbb{T}}$;

(2) If $\mathsf{M} = \mathbb{P}_2$, then $\mu(\Lambda) = 3$, Λ consists of 3 lines in general position and $X \simeq \check{\mathbb{T}}$.

Step 2: As notice earlier, $\check{\mathbb{T}}$ has a structure of an affine C-bundle of degree 0 over an elliptic curve; so let $\mathcal{R}_1$ be a compactification of $\check{\mathbb{T}}$. Then it follows readily that $\Gamma := \mathcal{R}_1 \setminus \check{\mathbb{T}}$ is a section with $\Gamma^2 = 0$. We infer readily from Step 1, that $\mathfrak{X}_{II}$ does not admit any rational compactification; meanwhile $\mathfrak{X}_I$ admits a rational compactification iff $\mathfrak{X}_I \simeq \check{\mathbb{T}}$.

Step 3: In step 1, assume that

$$X = \mathfrak{X}_{III} \tag{5.14.5}$$

Since C and $\mathcal{D}$ are sections and since $b_1(X) = 0$ for $i \geq 3$, the topological Euler number $\chi(X)$ of X can be expressed as follows:

$$\begin{aligned} \chi(X) &= b_0(X) - b_1(X) + b_2(X) \\ &= \chi(\mathcal{R}_I) - \chi(\Lambda) \\ &= \nu > 0 \end{aligned} \tag{5.14.6}$$

where $\nu := \mathrm{Card}(C \cup \mathcal{D})$. In view of [5] (Lemma 7.10), one has

$$b_1(X) = 2 \quad \text{or} \quad 3 \tag{5.14.7}$$

Now let us assume that X admits a rational compactification, say M with $\Lambda := \mathsf{M} \setminus X$.

(1) Assume that all the components of Λ are projective lines. By excluding the case where $X \simeq \check{\mathbb{T}}$, one has the following alternatives:

   (a) If $\mathsf{M} = \mathbf{P}_2$, then one can check that [11] (Example 3, on p. 176) M is either the product of 2 affine curves or $\mathsf{k}_a(X) = 2$. Therefore Theorem 4.5 or Theorem 5.11 will exclude this possibility for X.

   (b) On the other hand, let us assume that $\mathsf{M} = \mathbf{F}_n$. Then some careful calculations (see e.g. [5] Lemma 7.9) shows us that, in view of (5.14.2) and under the constraints (5.14.7), Λ must consist of 1 section and 3 fibres (resp. 1 section and 4 fibres). Now one has

$$\begin{aligned} \chi(X) &= \chi(\mathsf{F}_n) - \chi(\Lambda) \\ &= 4 - (1 - 0 + \mu(\Lambda)) \\ &= -1 (resp. -2) \end{aligned}$$

   which in either case, will contradict (5.14.6). Hence a rational compactification M of $\mathfrak{X}_{III}$ is not possible.

(2) Otherwise, at least one irreducible component of Λ say θ is either:

   - an irrational curve (with possible singularities), or
   - a rational curve with one node.

   Also, one can assume that $\Lambda \setminus \theta=$ union of fibres of M.

   (a) If $\mathsf{M} = \mathbf{P}_2$, Proposition 5.6 tells us that $\mathsf{k}_a(X) = 2$. Hence Theorem 5.11 tells us that such a rational compactification M cannot occur.

   (b) On the other hand if $\mathsf{M} = \mathbf{F}_n$ with $n = e \geq 0$, it follows from [8] that

$$\theta = a\Xi + b\mathbb{F} \quad \text{with} \quad a \geq 1 \quad \text{and} \quad b \geq ae \tag{5.14.8}$$

   where Ξ (resp. $\mathbb{F}$) is a canonical section (resp. a fibre) of M with $\Xi^2 = -e$.

   In view of the argument in Step 2 and the minimality of M, one can assume that both a and $n > 1$.

(i) If $a = 2$, clearly X is affine and $\mathsf{k}_l(X) = 0$ or 1. Then Theorem 5.13 (resp. Theorem 5.12) tells us that this is not possible, in view of the existence of the elliptic ruled surface $\mathcal{R}_1$ which is a compactification of $\mathfrak{X}_{III}$.

(ii) Assume that $a > 2$. Since $\mathcal{K}_{\mathsf{M}} \equiv -2\Xi + (-2-e)\mathbb{F}$ [8] where $\mathcal{K}_{\mathsf{M}}$ is the canonical bundle of M, we have:

$$\mathcal{K}_{\mathsf{M}} + \Lambda > \mathcal{K}_{\mathsf{M}} + \theta \equiv (a-2)\Xi + (b-2-e)\mathbb{F}$$

We infer from equation (5.14.8) that $\mathcal{K}_{\mathsf{M}} + \Lambda$ is an ample divisor [8] (Corollary V.2.18). Thus $\mathsf{k}_l(X) = 2$.
Consequently such a rational compactification for M cannot occur.

Step 4: Let us use the same conventions as in 5.7. Assume that $X = \mathfrak{X}_I$ admits another elliptic ruled surface, say $\mathcal{E}_1 \to \mathcal{D}_1$ as its compactification, i.e. there exists a compact curve $\theta \subset \mathcal{E}_1$ such that

$$\sigma : X \simeq X' := \mathcal{E}_1 \backslash \theta \tag{5.14.9}$$

where $\mathcal{E}_1 = \mathbb{P}(\mathfrak{W})$ for some rank 2 bundle $\mathfrak{W}$ on $\mathcal{D}_1$ with invariant e'.

Since one excludes the case where $X \simeq \check{\mathbb{T}}$, one can assume that:

$$\Gamma^2 = k > 0 \tag{5.14.10}$$

Consequently X is an affine surface, where $\Gamma := \mathcal{R}_1 \setminus X$. In view of Theorem 5.8, we can assume that Θ has no fibre components. Now by using, Mumford-Ramanujam theory (see e.g. [4]), one can show (see e.g. Lemma 1.5 [4]), in view of the isomorphism (5.14.9), that ∂U is *homotopically equivalent* to ∂V where U (resp. V) is some tubular neighborhood of Γ (resp. Θ); we infer from (5.14.10) that $\theta^2 = k$.

Case 1: Assume that $\mathfrak{V}$ is decomposable. Hence $\mathcal{R}_1$ admits a canonical section Ξ such that $\Xi^2 = -\epsilon < 0$. From the isomorphism (5.14.9), it follows readily that $\sigma(\Xi \setminus (\Xi \cup \Gamma)) =: \mathfrak{E} \subset \mathcal{E}_1$ is an algebraic curve. Since $\dim_C \mathfrak{E} = 1$, it admits a unique algebraic structure [25]. So let Ξ' be the compactification of $\mathfrak{E}$. Then one deduces from (5.14.9), that $\Xi \simeq \Xi'$ and $\Xi'^2 = \Xi^2 = -\epsilon < 0$. Since $\mathcal{E}_1$ is an elliptic ruled surface, this will imply [8] (V.2) that $\mathfrak{W}$ is also decomposable rank 2 bundle. Consequently $\mathcal{E}_1$ will admit Ξ' as its canonical section. Now by identifying the base curve of the elliptic ruled surface with its canonical section, we infer readily that $C_1 \simeq \mathcal{D}_1$ and $e = e' = \epsilon > 0$. Consequently $\mathcal{R}_1 \simeq \mathcal{E}_1$.

Case 2: Assume that $\mathfrak{V}$ is indecomposable. Then the same argument as above will tell us that $\mathfrak{W}$ must be also indecomposable. Hence (5.14.9) is actually an isomorphism of affine C-bundle of degree $\neq 0$. As noted earlier, Theorem 3.1 (see also [15]), each affine C-bundle A of degree $-k < 0$, over an elliptic curve $\tilde{\tau}$, is completely determined by its linear part L which is a line bundle over $\tilde{\tau}$ such that $c_1(\mathsf{L}) = -k$; however from Lemma 3.10, the total space of L is in fact a properly 1-convex surface which is completely determined by its exceptional set S which in turn is the canonical section of the elliptic ruled surface. Again, by identifying the base curve with its canonical section, we infer readilly that $C_1 \simeq \mathcal{D}_1$; hence $\mathcal{R}_1 \simeq \mathcal{E}_1$.

5.14.1. Remark. This result is an *analogue* of Theorem 5.12 for affine surfaces X which admit a C-fibration structure over some elliptic curve $\tilde{\tau}$ and which satisfy the condition $\mathsf{k}_l(X) = -\infty$.

Step 5: We stay in the same situation as in Step 4 but with

$$X = \mathfrak{X}_{II} \quad \text{or} \quad \mathfrak{X}_{III} .$$

Assume that $\mathsf{k}_l(X') = -\infty$. Since X' is Stein then θ got to be a section without any fibre components. The argument in Step 4 shows us that this is not possible. Otherwise, one can check that $\mathsf{k}_l(X') \geq 0$. Then we infer from Theorem 5.13 that $\mathsf{k}_l(X') > 0$. On the other hand if $\mathsf{k}_l(X') = 1$ (resp. $= 2$) we infer from Theorem 5.12 (resp. Theorem 5.11) that all algebraic compactifications of X are birationally equivalent.

Step 6: Assume that $\mathfrak{X}_m$ with $m = I$ or II or III admits an irrational ruled surface $\mathcal{R}_g$ with $g > 1$ as its compactification. Clearly this case cannot occur in view of Theorem 5.8. □

From theorem A in [28] and Theorem 4 in [31], we derive the following

Corollary 5.15. *Let X be a compactifiable 1-convex surface.*

Then the algebraic compactifications of X are birationally equivalent iff $X \not\cong \check{\mathbb{T}}$.

Corollary 5.16. *Let X be a compactifiable Stein surface.*

Then the following conditions are equivalent:

(1) *$X \not\cong \mathbb{A}^{\nu}_{\alpha}$, with $\nu \leq 0$.*

(2) *X only admits algebraic compactifications which are birationally equivalent.*

6 The affine structures

As far as affine surfaces are concerned, we have the following:

Proposition 6.1. [30, 31] *Let X be an affine surface.*

Then $\mathsf{k}_a(X) = -\infty$ or 2.

In parallel to Proposition 6.1, we have

Theorem 6.2. *Let X be a compactifiable Stein surface.*

Then $\mathsf{k}_a(X) = -\infty$ or 2.

Proof. Let M be a compactification of X and let $\Gamma := M \setminus X$.

Case 1: Assume that $M = \mathbf{P}_2$. Then our conclusion will follow from Proposition 5.6.

Case 2: Assume that M is a ruled surface. Then a main result in [32] tells us that X is biholomorphic to, either

(a) an affine surface, or

(b) $\mathbb{P}(\mathfrak{V}) \setminus \Gamma$ where $\mathfrak{V}$ is an indecomposable rank 2 bundle over a compact curve C_g with $g > 0$ with invariant $e < 0$ and where Γ is a section with $\Gamma^2 = 0$.

For (a) Proposition 6.1 will apply. As far as (b) is concerned, one has:

Claim: If $\Gamma \subset \mathbf{P}(\mathfrak{V})$ is a section, then $\mathsf{k}_a(X) = -\infty$ where $X := \mathbf{P}(\mathfrak{V}) \setminus \Gamma$.

Proof of the claim: Assume that $\mathsf{k}_l(X) \geq 0$. Let us consider the linear system $|m(\mathcal{K}_M + \Gamma|$ for any integer $m > 0$ where $\mathcal{K}_M$ is the canonical bundle of M. Since Γ is a section, $\mathcal{K}_M + \Gamma \equiv -\Xi + k\mathbb{F}$ for some integer k. Hence one can find at least one effective element $\mathsf{D} \in |m(\mathcal{K}_M + \Gamma)|$. But $\mathsf{D} \cdot \mathbb{F} = -m < 0$. Contradiction. Hence our claim is proved.

Consequently $\mathsf{k}_l(X)$ and a fortiori, $\mathsf{k}_a(X)$ is equal to $-\infty$.

Case 3: Assume that M is algebraic and $\mathsf{k}(M) \geq 0$. Then it follows from [29] that $\mathsf{k}_a(X) = 2$.

Case 4: Assume that M is a non algebraic surface. Then it follows from Theorem 3.20 that X is biholomorphic to either:

(1) $\check{\mathbb{T}}$ which, as an affine line bundle of degree 0, admits an elliptic ruled surface as its compactification such that Γ is a section with $\Gamma^2 = 0$. Thus, from the above Claim, one has $\mathsf{k}_a(X) = -\infty$, or

(2) $\mathcal{A}_{k,\alpha,\mathbf{t}}$ which, as affine C-bundle of degree $-k$, will admit an elliptic ruled surface as its compactification such that Γ is a section with $\Gamma^2 = k$. Hence again $\mathsf{k}_a(X) = -\infty$. □

Remark 6.3.

(1) In [30] a proof of Theorem 6.2 was also given. However, it was incomplete.

(2) As notice in Remark 2.3 (resp. Lemma 3.10), the toric surface $\check{\mathbb{T}}$ (resp. the affine C-bundles $\mathcal{A}_{k,\alpha,\mathbf{t}}$ with $\mathbf{t} \neq 0$) admits affine structures. On the other hand, as previously noticed, the analytic Kodaira dimension is bimeromorphically invariant; hence the above results naturally lead us to the following:

Problem 6.4. Do compactifiable Stein surfaces X always admit some affine structure?

6.5. Counterexample. Let $\mathfrak{V}_g$ be an indecomposable rank 2 bundle on some non singular compact curve of genus $g \geq 1$ with invariant $e = 0$. Then one can check that the ruled surface $\mathcal{R}_g := \mathbf{P}(\mathfrak{V}_g)$ carries a section Γ such that

$$\Gamma^2 = -e = 0 \tag{6.5.1}$$

On the other hand, it was shown in [32] that $\mathfrak{X}_g := \mathcal{R}_g \setminus \Gamma$ is indeed Stein. Notice that $\mathfrak{X}_1 \simeq \check{\mathbb{T}}$. We deduce from (6.5.1) that $\mathfrak{X}_g$ are not affine. Then it follows readily from Theorem 4.3 that, in contrast to $\mathfrak{X}_1$, all compactifications of $\mathfrak{X}_g$ for $g > 1$, are birationally equivalent: therefore $\mathfrak{X}_g$ do not admit any affine structure if $g > 1$.

6.6. Notice that $\mathsf{k}_a(\mathfrak{X}_g) = -\infty$. In [31] it was anticipated that Problem 6.4 might have an affirmative answer, provided $\mathsf{k}_a(X) = 2$; unfortunately, it was merely a wishful thinking, as shown by the following:

6.7. Counterexample. Step 1: Here we follow closely an idea in [2]. For any $g \geq 1$, let us select a sufficiently ample divisor, say $\delta \subset \mathcal{R}_g$ such that:

(1) $\mathcal{K}_{\mathcal{R}_g} + \delta$ is very ample, where $\mathcal{K}_{\mathcal{R}_g}$ is the canonical bundle of $\mathcal{R}_g$, and

(2) $|2\delta|$ contains an effective smooth divisor, say Δ.

Let $\pi : \mathcal{M}_g \to \mathcal{R}_g$ be a double cover of $\mathcal{R}_g$, ramified along Δ. Then from the Leray spectral sequence, one has

$$H^0(\mathcal{M}_g, \mathcal{K}_{\mathcal{M}_g}) \simeq H^0(\mathcal{R}_g, \mathcal{K}_{\mathcal{R}_g}) \oplus H^0(\mathcal{R}_g, \mathcal{K}_{\mathcal{R}_g} + \delta) \tag{6.7.1}$$

where $\mathcal{K}_{\mathcal{M}_g}$ is the canonical bundle of $\mathcal{M}_g$. Since $\mathcal{R}_g$ is a ruled surface, the first summand in (6.7.1) is zero. Furthermore (1) and (2) will guarantee that $\mathcal{M}_g$ is a surface of general type.

Step 2: Let $\Theta := \pi^*(\Gamma)$. It follows readilly from (6.5.1) that

$$\Theta^2 = 0 \tag{6.7.2}$$

Furthermore, $\mathfrak{U}_g := \mathcal{M}_g \setminus \Theta$, being a finite cover of the Stein surface $\mathcal{R}_g$, is itself Stein. Since $\mathcal{M}_g$ is of general type, it follows readily that $\mathsf{k}_a(\mathfrak{U}_g) = 2$. In view of (6.7.2), $\mathfrak{U}_g$ is not affine. Since $\mathsf{k}_a(\mathfrak{U}_g) = 2$, Theorem 5.11 tells us that analytic compactifications of $\mathfrak{U}_g$ are biholomorphic. We infer readily that $\mathfrak{U}_g$ do not admit any affine structure.

Remark 6.8. Notice that the above construction applies *mutatis mutandis* in order to provide a negative answer to Problem 4 in [29].

7 Some further prospects

7.0.1. The above constructions provide us concrete examples of compact algebraic surfaces M which are ruled (resp. of general type) namely $\mathcal{R}_g$ (resp. $\mathcal{M}_g$) for any $g > 1$ (resp. any $g > 0$) such that M is the compactification of a Stein surface X, namely $\mathfrak{X}_g$ (resp. $\mathfrak{U}_g$) which do not admit any affine structure. Such a construction was motivated by a question raised by Hartshorne, namely

Problem 7.1. [7] (Problem 3.4 p. 235) Let M be an *arbitrary* compact surface, let $\Gamma \subset M$ be an irreducible compact curve and let $X := M \setminus \Gamma$.

Assume that X is free of compact curves.
Is X always Stein, provided $\Gamma^2 \geq 0$.

7.1.1. Digression. From the analysis by Ogus and a construction by Arnold, it is known that Problem 7.1 admits a negative answer if M is *non minimal*, see [34] for precise references (see also [20] p. 37); indeed, Arnold constructed *non minimal* rational surfaces M with an embedded non singular elliptic curve $\Gamma \subset M$ admitting a tubular neighborhood $U \subset M$ such that:

(1) $\Gamma^2 = 0$;

(2) $N_{\Gamma/M}$, the normal bundle of Γ in M is non torsion;

(3) $X := M \setminus \Gamma$ is free of compact curves.

Then Ogus' argument will imply that X is not Stein. Notice that since M is rational, the technique in 6.7 will apply in order to produce new counterexamples for Problem 7.1 where the compact surfaces M can be selected to be of *general type*. So let us take this opportunity to discuss further new challenges which we'll encounter, when *non minimal* compactifications are taking into account.

In fact, in [19] a family of algebraic surfaces Å was exhibited such that:

(1) the *algebraic* cohomology groups $\mathcal{H}^1$Å, Ω^p) vanish for any $p \geq 0$;

(2) Å are not affine;

(3) k_l(Å) $= -\infty$;

(4) Å admits non minimal rational compactifications Ñ;

(5) Γ :=Ñ\Å consisted of 9 components.

Consequently, one would like to raise the following:

7.2. Problem. Is Å always Stein?

- If the answer to 7.2 is negative, Å will provide us a second counterexample to Problem 7.1, if the irreducibility assumption for Γ is dropped; then in view of Remark 3.23 one has the following:

  7.3. Question. Does Å admit a non algebraic compactification?

  This will direct us to the follow up:

  7.4. Question. Do the *analytic* cohomology groups H^1(Å, Ω^p) also vanish for for $p \geq 0$?

  A positive answer to 7.4, will reinforce our belief that the assumption of minimality for compact surfaces in [32] is crucial. On the other hand, a negative answer to 7.4 will tell us, once again, the limitations of GAGA principle even in dimension 2, in view of (1) in 7.1.1.

- If the answer to 7.2 is positive, then it will direct us to the follow up:

7.5. Question.

(1) Does Å admit some non rational compactification?

(2) Does Å always admit some affine structure?

Remark 7.6. Similarly, since Ñ is rational, one can construct a series of algebraic surfaces, say Ã with similar properties as Å, but with a single exception, namely $\mathsf{k}_l(\tilde{A})= 2$.

Now let us get back to our initial framework, i.e. minimal compactifications. Then in [33] an affirmative answer to Problem 7.1 was given, provided $\mathsf{k}(M) = -\infty$; concisely, one has

Theorem 7.7. *Let M be a compactification of some surface X.*

Assume that $\mathsf{k}(M) = -\infty$.

(1) *Assume that M is non algebraic.*

Then X is Stein iff either $X \simeq \mathbb{A}_\alpha^\nu$ for some $\nu \leq 0$ and $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$.

(2) *Assume that M is algebraic.*

Then X is Stein iff X is affine or $X \simeq \mathfrak{X}_g$ for any $g > 0$.

Complementing this result, it is not very hard to prove the following

Theorem 7.8. *Let M be a compact algebraic surface, let $\Gamma \subset M$ be a compact analytic subvariety and let $X := M \setminus \Gamma$. Assume that* $\mathsf{k}(M) = 0$ *or 1.*

Then X is Stein iff X is affine.

Thus, in order to complete this picture, one naturally would like to raise the following:

Problem 7.9. Are $\mathcal{M}_g$, for any $g > 0$, up to biholomorphism, the only compact surfaces of general type which compactify Stein surfaces which are not affine, namely $\mathfrak{U}_g$?

For further generalizations, let us introduce the following:

7.10. Proposition-Definition. Let $\Gamma := \cup_i \Gamma_i$ be a connected compact curve in a given compact surface M, where Γ_i are the irreducible components of Γ. Let $a_{i,j} := \Gamma_i \cdot \Gamma_j$ and let $\mathsf{A} := (a_{ij})$ be the intersection matrix. Then one of the following three alternatives will occur: either

- A has at least one positive eigenvalue. Then one can select $r_i \in \mathbb{Z}$ such that $\mathsf{D}^2 > 0$ where $\mathsf{D} := \sum_i r_i \Gamma_i$. In this case, Γ is said to be *topologically of positive* type, or
- A is negative definite. Then Γ is said to be *topologically of negative* type, or
- A is negative semi-definite. Then one can find $s_j \in \mathbf{Z}$ such that $\mathsf{D}'^2 = 0$ where $\mathsf{D}' := \sum_j s_j \Gamma_j$; in which case, Γ is said to be *topologically of zero* type.

Now, in order to round off this discussion, we would like to provide some applications of Theorem 4.3 to another aspect of Problem 1.7 above, namely:

Problem 7.11. Let M_i be given compact surfaces and let $\Gamma_i \subset M_i$ be compact connected curves, with $i = 1$ or 2. Assume that $X_i := M_i \setminus \Gamma_i$ are biholomorphic.

Are M_i bimeromorphically equivalent if Γ_i are topologically of the same type?

In this direction, we have the following:

Theorem 7.12. *Problem* 7.11 *admits an affirmative answer, provided*

(♠) both Γ_i *are topologically of negative (resp. positive) type.*

Proof.

(1) Assume that both Γ_i are topologically of negative type. Grauert's criterion tells us that there exist 2-dimensional normal compact C-analytic spaces, say Y_i with one isolated singular point $\{\gamma_i\}$ and morphisms $\pi_i : M_i \to Y_i$ inducing biholomorphisms

$$X_i = M_i \setminus \Gamma_i \simeq Y_i \setminus \{\gamma_i\} \tag{7.12.1}$$

Since Y_i are normal, Hartogs extension Theorem implies that the isomorphism $X_1 \simeq X_2$ will extend to a biholomorphism $Y_1 \simeq Y_2$; consequently, $M_1 \simeq M_2$ since they are non singular resolutions of the same space.

(2) Assume that Γ_i are both topologically of positive type. Then one deduces from Chow-Kodaira Theorem that M_i are projective algebraic. Also we infer from [29] that X_i are 1-convex with exceptional set S_i.

   (a) Assume that $\dim_C S_i > 0$. Then Theorem 4 and 5 in [29] tell us that M_i are biholomorphic.

(b) Assume that $\dim_C S_i = 0$, i.e. X_i are Stein. Then we have 2 alternatives:

- Assume that $X_i \simeq \check{\mathbb{T}}$. Since Γ_i are topologically of positive type, M_i are necessarily rational surfaces. Thus we are done.
- Assume that $X_i \not\simeq \check{\mathbb{T}}$. Then Theorem 4.3 tells us that M_i are bimeromorphically equivalent. □

Corollary 7.13. *Theorem* 7.12 *will hold if one replaces the hypothesis (♠) by*

(♣) none of the Γ_i *are topologically of zero type.*

Proof.

(1) Assume that Γ_1 is topologically of negative type. It follows readily from (7.7.1) and Hartogs extension theorem that

$$\Gamma(X_1, \mathcal{O}_{X_1}) = \mathbb{C} \tag{7.13.1}$$

Now if Γ_2 is topologically of positive type, then X_2 is 1-convex, i.e.

$$\dim_C \Gamma(X_2, \mathcal{O}_{X_2}) = \infty \tag{7.13.2}$$

in view of Definition 3.8. Therefore (7.13.1) and (7.13.2) will contradict the hypothesis that $X_1 \simeq X_2$. Thus Γ_2 must also topologically of negative type. Hence our conclusion will follow from Theorem 7.12.

(2) Assume that Γ_1 is topologically of positive type. Then the same argument as above will tell us that Γ_2 also must be of positive type. Again our conclusion will follow from Theorem 7.12. □

7.14. Question. Does Problem 7.11 admit a positive answer if both Γ_i are topologically of zero type?

Obviously, the answer is "No". However, our current study shows that, drastically, an answer to Question 7.14 is still negative, even if one assumes, furthermore that

(1) M_i are both algebraic (resp. both non algebraic), and

(2) X_i are Stein.

However on the positive side, we have

Proposition 7.15. *Question* 7.14 *admits a positive answer, provided, either*

(1) $X_1 \simeq X_2 =: X$ *is Stein and* $X \not\simeq \check{\mathbb{T}}$*; or*

(2) $X_1 \simeq X_2 =: X$ *is properly 1-convex.*

Proof.

(1) **Step 1**: Assume that M_1 is non algebraic. Since $\Gamma_1^2 = 0$, it follows from Theorem 3.20 that $X \simeq \mathcal{A}_{k,\alpha,\mathbf{t}}$ with $\mathbf{t} \neq 0$. Assume that M_2 is algebraic. Since $X_2 \simeq \mathcal{A}_{k,\alpha,\mathbf{t}}$ and $\Gamma_2^2 = 0$, Lemma 3.10 tells us that this is not possible. Hence M_2 got to be non algebraic. We infer from Theorem 1.4 that $M_1 \simeq M_2$.

Step 2: Assume that M_1 is algebraic. Since $\Gamma_1^2 = 0$, it follows from Theorem 3.20 that $X \not\simeq \mathcal{A}_{k,\alpha,\mathbf{t}}$ and, by hypothesis , $X \not\simeq \check{\mathbb{T}}$. Assume that M_2 is non algebraic. Then Corollary 3.22 tells us that this is not possible. Consequently M_2 is also algebraic. Since $X \not\simeq \check{\mathbb{T}}$, the Main Theorem will apply and our conclusion will follow.

(2) Assume that M_1 is non algebraic. We infer from Theorem B in [29] that

($\diamondsuit$) $\quad X \simeq \mathcal{A}_{k,\alpha,0}$

Now if M_2 is algebraic, then in view of ($\diamondsuit$) and $\Gamma_2^2 = 0$, Lemma 3.10 tells us that this is not possible. Hence M_2 is also non algebraic and our conclusion will follow from Theorem 4 in [29].

Now assume that M_1 is algebraic. Then following Theorem B in [29], for any $k \geq 1$ and $\alpha \in \mathbb{C}$, one has

($\heartsuit$) $\quad X \not\simeq \mathcal{A}_{k,\alpha,0}$

On the other hand, if M_2 is non algebraic, it follows from Theorem B in [29] and $\Gamma_2^2 = 0$ that, for some $k \geq 1$ and $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$, $X \simeq \mathcal{A}_{k,\alpha,0}$. This certainly contradicts ($\heartsuit$). Hence M_2 is also algebraic and our conclusion again will follow from Theorem B in [29]. $\square$

Remark 7.16. Notice that Proposition 7.15 is false if X is not assumed to be 1-convex.

Indeed, let us use the same notations as in Definition 3.13. Let $M_1 := \mathbb{H}_{\alpha,\omega_1}$ and let $M_2 := \mathbb{H}_{\alpha,\omega_2}$ be non elliptic Hopf surfaces of type (II). Now let $X_1 := M_1 \setminus \gamma_{\omega_1}$ and let $X_2 := M_2 \setminus \gamma_{\omega_2}$. Then one can check that

(1) $X_1 \simeq X_2 =: X$;

(2) γ_{ω_1} and γ_{ω_2} are of zero type.

However M_1 and M_2 are bimeromorphically equivalent iff $\omega_1 = \omega_2$. In fact, one can check that [34] X is topologically (but not analytically) a trivial line bundle, say $\mathcal{L}$ on the elliptic curve γ_α and $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

Consequently, X also admits a third compactification which is projective algebraic, say M_3 where

$$M_3 := S_e^\alpha \simeq \mathbb{P}(\mathfrak{E})$$

where $e = 0$ and $\mathfrak{E}$ is the trivial extension of

$$0 \longrightarrow \mathbf{O}_{\gamma_\alpha} \longrightarrow \mathfrak{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

Acknowledgements. The major part of this work was completed when the author was on sabbatical leave at the Institute of Mathematics of the University of Copenhagen, Denmark (resp., HCM International University in Sàigòn, Viêt Nam) during the fall semester 2006 (resp., the spring semester 2007). He would like to thank Prof. H. Jakobsen (resp., Prof. Phan quôc Khánh) for their hospitality. Our sincere thanks go to the referee for helpful comments and thoughtful suggestions. Last, but not least, the author would like to thank his colleague, Dr. A. Glesser, a LaTeXguru, for his invaluable technical assistance.

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