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Hölder norm estimate for the Hilbert transform in Clifford analysis

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a Jordan domain with d-summable boundary Γ. The main gol of this paper is to estimate the Hölder norm of a fractal version of the Hilbert transform in the Clifford analysis context acting from Hölder spaces of Clifford algebra valued functions defined on Γ. The explicit expression for the upper bound of the norm provided here is given in terms of the Hölder exponents, the diameter of Γ and certain $\mathbf{d}$-sum ($\mathbf{d} > d$) of the Whitney decomposition of Ω. The result obtained is applied to standard Hilbert transform for domains with left Ahlfors-David regular surface.

Keywords: Clifford analysis, Hilbert transform, fractals boundaries.

Mathematical subject classification: 30G35.

1 Introduction

Standard Clifford analysis is a higher dimensional function theory offering both a generalization of holomorphic functions theory in the complex plane and at the same time a refinement of classical harmonic analysis. Its roots go back as early as the 1930's. For a classical account of this function theory we refer e.g. to the monograph [7].

Recently, making heavy use of the interaction between harmonic analysis and geometric measure theory, Clifford analysis has emerged as yet a particularly suitable framework for the treatment of higher-dimensional boundary values phenomena for domains with highly irregular boundaries, see [2, 3, 4, 5, 6].

Hence, this can be regarded of as a good motivation for finding conditions on the boundary, which give boundedness of certain singular integral operators, such as the Hilbert transform when the boundary is permitted to be fractal. This is the question we shall be concerned in this work.

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2 Preliminaries

2.1 Clifford algebras and monogenic functions

The real Clifford algebra associated with $\mathbb{R}^n$ endowed with the Euclidean metric is the minimal enlargement of $\mathbb{R}^n$ to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^n$.

It thus follows that if $\{e_j\}_{j=1}^n$, is the standard basis of $\mathbb{R}^n$, then we must have that

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

Every element $a \in \mathbb{R}_{0,n}$ is of the form $a = \sum_{A\subseteq N} a_A e_A$, $N = \{1, \dots, n\}$, $a_A \in \mathbb{R}$, where $e_\emptyset = e_0 = 1$, $e_{\{j\}} = e_j$, and $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$ for $A = \{\alpha_1, \dots, \alpha_k\}$ where $\alpha_j \in \{1, \dots, n\}$ and $\alpha_1 < \cdots < \alpha_k$.

The conjugation is defined by $\overline{a} := \sum_A a_A \overline{e_A}$, where

$$\overline{e_A} = (-1)^k e_{i_k} \cdots e_{i_2} e_{i_1}, \ \text{ if } e_A = e_{i_1} e_{i_2} \cdots e_{i_k}.$$

Notice that for $x \in \mathbb{R}^n$, we thus have that

$$x\,\overline{x} = \overline{x}\,x = |x|^2.$$

By means of the conjugation, a norm $|a|$ may be defined for each $a \in \mathbb{R}_{0,n}$ by putting

$$|a|^2 = \sum_A |a_A|^2.$$

A function defined in some subset $\mathbf{E}$ of $\mathbb{R}^n$ with values in the Clifford algebra $\mathbb{R}_{0,n}$ is a map $u : \mathbf{E} \to \mathbb{R}_{0,n}$ of the form

$$u(x) = \sum_A u_A(x) e_A, \quad x \in \mathbf{E},$$

where u_A are real components of u, then notions of continuity and differentiability of u are introduced by means of those corresponding for its real components.

If $\mathbf{E} \subset \mathbb{R}^n$ is a compact set, then $C^{0,\alpha}(\mathbf{E})$, $0 < \alpha < 1$ stands for the class of all Hölder continuous $\mathbb{R}_{0,n}$-valued functions u of exponent α (see [10]), for which

$$|u|_{\alpha,\mathbf{E}} := \sup_{x,y\in\mathbf{E};\, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is finite and define

$$\|u\|_{\alpha,\mathbf{E}} = \max_{x\in\mathbf{E}} |u(x)| + |u|_{\alpha,\mathbf{E}}.$$

For a thorough treatment we refer the reader to [16].

Clifford analysis focuses on the null solutions of various special partial differential operators arising naturally with the Clifford algebras language, the most important of them being the so-called Dirac operator in $\mathbb{R}^n$ given by

$$\mathcal{D} := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

It is a first order elliptic operator whose fundamental solution is given by

$$e(x) = \frac{1}{\sigma_n} \frac{\overline{x}}{|x|^n}, \quad x \neq 0,$$

where σ_n is the area of the unit sphere in $\mathbb{R}^n$.

If Ω is open in $\mathbb{R}^n$ and $u \in C^1(\Omega)$, then u is said to be monogenic (left) in Ω if $\mathcal{D} u = 0$ in Ω.

In closing this introductory subsection let us remember that a Whitney extension (see [15]) of $u \in C^{0,\alpha}(\mathbf{E})$, $\mathbf{E}$ being compact, is a compactly supported function $\tilde{u} \in C^\infty(\mathbb{R}^n \setminus \mathbf{E}) \cap C^{0,\alpha}(\mathbb{R}^n)$ such that $\tilde{u}|_{\mathbf{E}} = u$ and

$$|\mathcal{D}\tilde{u}(x)| \leq c\,|u|_{\alpha,\mathbf{E}}(\operatorname{dist}(x, \mathbf{E}))^{\alpha-1} \text{ for } x \in \mathbb{R}^n \setminus \mathbf{E}.$$

It is worth remarking that $c > 0$ depends only on α and n.

2.2 Some geometry

By definition, presented in [11], a set $\mathbf{E} \subset \mathbb{R}^n$ is said to be d-summable if the improper integral

$$\int_0^1 N_{\mathbf{E}}(\tau)\tau^{d-1} d\tau$$

converges.

Here and subsequently, $N_{\mathbf{E}}(\tau)$ stands for the least number of τ-balls needed to cover $\mathbf{E}$.

The diameter of $\mathbf{E} \subset \mathbb{R}^n$ will be denoted by $|\mathbf{E}|$.

It should be noted that if $\mathbf{E}$ is d-summable, then it is also $(d + \epsilon)$-summable for every $\epsilon > 0$.

In all that follows, $\Omega \subset \mathbb{R}^n$ denotes a Jordan domain, what means a bounded oriented connected open subset of $\mathbb{R}^n$ whose boundary Γ is a compact topological surface. When looked at the case $n = 2$ this leads to that usual Jordan domain in the plane.

The following lemma reveals the specific importance of the notion of d-summability applied to the boundary Γ of a Jordan domain Ω and relating to the Whitney decomposition $\mathcal{W}$ of Ω by binary cubes, see [15] for more details.

Lemma 2.1 [11]. *If Ω is a Jordan domain of $\mathbb{R}^n$ and Γ is d-summable, then the d-sum $\sum_{Q\in\mathcal{W}}|Q|^d$ of the Whitney decomposition $\mathcal{W}$ of Ω is finite.*

For simplicity of notation, we let $\mathbf{s}(d)$ stand for the d-sum of a Jordan domain Ω with d-summable boundary.

An important special case is when Γ is assumed to be left Ahlfors-David regular ((l) AD-regular for short), which is understood to mean that it has Hausdorrf $(n-1)$-measure finite ($\mathcal{H}(\Gamma)<\infty$) and there exists a constant c_Γ such that

$$c_\Gamma\, r^{n-1}\leq \mathcal{H}(\Gamma\cap B(x,r))\ \text{ for } x\in\Gamma,\ 0<r\leq|\Gamma|, \tag{1}$$

where $B(x,r)$ denotes the closed ball with center x and radius r (see [3, 4, 5]).

A nice link between this geometric notion, introduced in [3], and that of d-summability is given by the following lemma. Precisely, under (l) AD-regular assumption the $(n-1+\epsilon)$-sum is bounded above by a constant times $\mathcal{H}(\Gamma)/\epsilon$.

Lemma 2.2. *If Γ is (l) AD-regular, then it is $(n-1+\epsilon)$-summable for any $\epsilon>0$ and*

$$\sum_{Q\in\mathcal{W}}|Q|^{n-1+\epsilon}\leq c\frac{\mathcal{H}(\Gamma)}{\epsilon}, \tag{2}$$

where c depend only on n and c_Γ.

Proof. The proof of the $(n-1+\epsilon)$-summability easily follows by noticing that

$$N_\Gamma(\tau)\leq P_\Gamma\left(\frac{\tau}{2}\right),$$

where $P_\Gamma(\frac{\tau}{2})$ is the greatest number of disjoint $\frac{\tau}{2}$-balls with centers in Γ, precisely what is the well-known packing number of Γ (see for instance [13]).

Then, in accordance with (1) we have

$$c_\Gamma\frac{\tau^{n-1}}{2^{n-1}}P_\Gamma\left(\frac{\tau}{2}\right)\leq\mathcal{H}(\Gamma)$$

and hence that

$$N_\Gamma(\tau)\leq c\mathcal{H}(\Gamma)\tau^{1-n},$$

where the constant c depends on n and c_Γ.

Consequently,

$$\int_0^1 N_\Gamma(\tau)\tau^{n-1+\epsilon-1}d\tau \le c\mathcal{H}(\Gamma)\int_0^1 \tau^{\epsilon-1}d\tau = c\frac{\mathcal{H}(\Gamma)}{\epsilon}, \tag{3}$$

which establishes the $(n-1+\epsilon)$-summability of Γ.

We can now proceed analogously to the proof of Lemma 2 in [11], which, together with (3), gives (2). □

More generally, the above proof reveals that if Γ is a d-set $(0 < d \le n)$ i.e., a regular set with dimension d in the sense of [8, 13], then it is $(d+\epsilon)$-summable for any $\epsilon > 0$.

3 Hilbert transform on d-summable surfaces

The inspiration for the following definition is the Cauchy type formula established in [1], Theorem 3.1.

Definition 3.1. *Let Γ be d-summable with $d \in (n-1, n]$. If $\alpha > d+1-n$, then we define the Cauchy transform of a function $u \in C^{0,\alpha}(\Gamma)$ by*

$$(C_\Gamma^* u)(x) := \tilde{u}(x)\chi_\Omega(x) + \int_\Omega e(y-x)\mathcal{D}\tilde{u}(y)dy, \quad x \in \mathbb{R}^n \setminus \Gamma, \tag{4}$$

where $\tilde{u}$ stands for a Whitney extension of u and $\chi_\Omega(x)$ denotes the characteristic function of Ω.

A trivial verification shows that $C_\Gamma^* u$, being monogenic in $\mathbb{R}^n \setminus \Gamma$, vanishes at infinity. Meanwhile, when Γ is sufficiently regular (e.g. (l) AD-regular), the Cauchy transform (4) becomes the more standard one defined by

$$(C_\Gamma u)(x) := \int_\Gamma e(y-x)\nu_\Gamma(y)u(y)d\mathcal{H}(y), \ x \notin \Gamma,$$

where $\nu_\Gamma(y)$ is the outward pointing unit normal to Γ introduced in [9].

For a deeper discussion of higher dimensional analogue of the Plemelj Sokhotzki formula in Clifford analysis setting, where the existence of the boundary limits of the last-mentioned Cauchy transform is satisfactory can be found in [3, 5, 6].

Definition 3 is legitimate, because the right member of (4) exists for any $x \in \mathbb{R}^n \setminus \Gamma$ and its value does not depend on the particular choice of $\tilde{u}$. The proof of this last assertion can be found in [1], Proposition 3.2.

A natural question to ask is whether $C^*_\Gamma u$ has a continuous extension to $\overline{\Omega} := \Omega \cup \Gamma$. It is generally a highly nontrivial question. However, on the positive side, the next theorem sheds some light on the answer and one can therefore also introduce the following "fractal" multidimensional Hilbert transform

$$(\mathrm{H}^*_\Gamma u)(x) = 2(C^*_\Gamma u)^+(x) - u(x), \quad x \in \Gamma.$$

Here $(C^*_\Gamma u)^+$ denotes the trace on Γ of the continuous extension of $C^*_\Gamma u$ to $\overline{\Omega}$.

This approach is an alternative to the more conventional Hilbert transform in Clifford analysis, which is defined to be the Cauchy principal value singular integral

$$(\mathrm{H}_\Gamma u)(x) := \int\limits_\Gamma e(y-x)\nu_\Gamma(u(y)-u(x))d\mathcal{H}(y) + u(x), \; x \in \Gamma. \tag{5}$$

For a recent account of the description of Hölder-boundedness of (5) we refer the reader to [4].

In case of Lipschitz domains the Hilbert transform $\mathrm{H}_\Gamma u$ can be rewriten as

$$(\mathrm{H}_\Gamma u)(x) := \int\limits_\Gamma e(y-x)\nu_\Gamma u(y)d\mathcal{H}(y) + \left[1 - \int\limits_\Gamma e(y-x)\nu_\Gamma d\mathcal{H}(y)\right]u(x),$$

where the second term depends on the interior angle of each corner point of Γ, see [14] for more details.

Theorem 3.1. *Let Γ be d-summable and $\alpha > \frac{d}{n}$. Then $C^*_\Gamma u$ has continuous extension to $\overline{\Omega}$. Furthermore, $H^*_\Gamma u \in C^{0,\beta}(\Gamma)$ whenever $\beta < \frac{n\alpha - d}{n-d}$.*

Proof. Our proof starts with the observation that from $\alpha > \frac{d}{n}$ it follows that $n < \frac{n-d}{1-\alpha}$. Then, we are at liberty to choose p such that $n < p < \frac{n-d}{1-\alpha}$.

The next claim is that $\mathcal{D}\tilde{u} \in L^p(\Omega)$. Indeed,

$$\begin{aligned}\int\limits_\Omega |\mathcal{D}\tilde{u}(y)|^p dy &= \sum_{Q\in\mathcal{W}} \int\limits_Q |\mathcal{D}\tilde{u}(y)|^p dy \le c\,|u|_{\alpha,\Gamma} \sum_{Q\in\mathcal{W}} \int\limits_Q (\mathrm{dist}(y,\Gamma))^{p(\alpha-1)} dy \\ &\le c|u|_{\alpha,\Gamma} \sum_{Q\in\mathcal{W}} |Q|^{p(\alpha-1)}|Q|^n = c|u|_{\alpha,\Gamma} \sum_{Q\in\mathcal{W}} |Q|^{n-p(1-\alpha)}.\end{aligned}$$

The finiteness of the last sum follows from the d-summability of Γ together with the fact that $n - p(1-\alpha) > d$.

From what has already been proved it follows that the integral term of (4), which will be denoted by

$$U(x) := \int_{\Omega} e(y-x)\mathcal{D}\tilde{u}(y)dy$$

represents a continuous function in $\mathbb{R}^n$ (see [10], for instance).

This clearly forces $(C^*_\Gamma u)(x)$ to has continuous extension to $\overline{\Omega}$. Moreover, $U \in C^{0,\frac{p-n}{p}}(\mathbb{R}^n)$, combined with $\beta < \frac{n\alpha-d}{n-d}$, implies that $\mathrm{H}^*_\Gamma u \in C^{0,\beta}(\Gamma)$. □

3.1 Hölder norm estimate for $\mathbf{H}^*_\Gamma$

Theorem 3 gains in interest if we realize that the Hilbert transform H^*_Γ acts from $C^{0,\alpha}(\Gamma)$ into $C^{0,\beta}(\Gamma)$ whenever

$$0 < \beta < \frac{n\alpha - d}{n-d} < 1. \tag{6}$$

We next show that H^*_Γ is bounded between these spaces and present upper bound for its norm.

Theorem 3.2. *Let Γ be d-summable and suppose* (6) *occurs. Then H^*_Γ is bounded from $C^{0,\alpha}(\Gamma)$ into $C^{0,\beta}(\Gamma)$ and*

$$\|H^*_\Gamma\| \leq 1 + |\Gamma|^{\alpha-\beta} + c_1(\mathbf{s}(\mathbf{d}))^{\frac{1-\beta}{n}} + c_2(\mathbf{s}(\mathbf{d}))^{\frac{1-\beta}{n}}|\Gamma|^{\beta}, \tag{7}$$

where $\mathbf{d} := n\frac{\alpha-\beta}{1-\beta}$ and c_1, c_2 depend only on α, β and n.

Proof. We begin by choosing $p = \frac{n}{1-\beta}$. A brief inspection of the proof of Theorem 3 reveals that

$$\begin{aligned}
\int_{\Omega} |\mathcal{D}\tilde{u}(y)|^p dy &\leq c\,|u|^p_{\alpha,\Gamma} \sum_{Q\in\mathcal{W}} |Q|^{n-p(1-\alpha)} \\
&= c\,|u|^p_{\alpha,\Gamma} \sum_{Q\in\mathcal{W}} |Q|^{p(\alpha-\beta)} \\
&= c\,|u|^p_{\alpha,\Gamma}\,\mathbf{s}(p(\alpha-\beta)) = c\,|u|^p_{\alpha,\Gamma}\,\mathbf{s}(\mathbf{d}).
\end{aligned}$$

Since $\mathbf{d} > d$, we have

$$\|\mathcal{D}\tilde{u}\|_{L_p} \leq c^{1/p}\, |u|_{\alpha,\Gamma}(\mathbf{s}(\mathbf{d}))^{1/p}.$$

At this stage we appeal to Proposition 8.1 in [10] to deduce that

$$|U(x)| \leq c\|\mathcal{D}\tilde{u}\|_{L_p}\, |\Gamma|^{\frac{p-n}{p}} = c\|\mathcal{D}\tilde{u}\|_{L_p}\, |\Gamma|^{\beta} \leq c_1\, |u|_{\alpha,\Gamma}(\mathbf{s}(\mathbf{d}))^{1/p}|\Gamma|^{\beta},$$

and

$$|U|_{\beta,\mathbb{R}^n} \leq c\|\mathcal{D}\tilde{u}\|_{L_p} \leq c_2\, |u|_{\alpha,\Gamma}(\mathbf{s}(\mathbf{d}))^{1/p}.$$

Consequently, for any $x \in \Gamma$

$$\begin{aligned} |\mathrm{H}^*_\Gamma u(x)| &\leq |u(x)| + 2|U(x)| \\ &\leq |u(x)| + c_1\, |u|_{\alpha,\Gamma}(\mathbf{s}(\mathbf{d}))^{1/p}|\Gamma|^{\beta} \\ &\leq (1 + 2c_1(\mathbf{s}(\mathbf{d}))^{1/p}|\Gamma|^{\beta})\|u\|_{\alpha,\Gamma}, \end{aligned}$$

$$\begin{aligned} |\mathrm{H}^*_\Gamma u|_{\beta,\Gamma} &\leq |u|_{\beta,\Gamma} + 2|U|_{\beta,\mathbb{R}^n} \\ &\leq |\Gamma|^{\alpha-\beta}|u|_{\alpha,\Gamma} + 2c_2\, |u|_{\alpha,\Gamma}(\mathbf{s}(\mathbf{d}))^{1/p} \\ &= (|\Gamma|^{\alpha-\beta} + 2c_2(\mathbf{s}(\mathbf{d}))^{1/p})|u|_{\alpha,\Gamma}. \end{aligned}$$

This finally yields

$$\|\mathrm{H}^*_\Gamma u\|_{\beta,\Gamma} \leq (1+|\Gamma|^{\alpha-\beta} + c_1(\mathbf{s}(\mathbf{d}))^{\frac{1-\beta}{n}} + c_2(\mathbf{s}(\mathbf{d}))^{\frac{1-\beta}{n}}|\Gamma|^{\beta})\|u\|_{\alpha,\Gamma},$$

which completes the proof. □

Remark 3.1. Note that in the proof of both Theorems 3 and 3.1 we have used the d-summability of Γ just to ensure the finiteness of some $(d+\epsilon)$-sum. In the following section we exploit this argument when the surface is assumed to be (l) AD-regular.

3.2 The case of (l) AD-regular boundary

Theorem 3.3. *Let Γ be (l) AD-regular. Then the Hilbert transform H_Γ is bounded from $C^{0,\alpha}(\Gamma)$ into $C^{0,\beta}(\Gamma)$, whenever $0 < \beta < n\alpha + 1 - n < 1$. Moreover,*

$$\|H_\Gamma\| \leq 1 + c_1[\mathcal{H}(\Gamma)]^{\frac{\alpha-\beta}{n-1}} + c_2[\mathcal{H}(\Gamma)]^{\frac{1-\beta}{n}} + c_3[\mathcal{H}(\Gamma)]^{\frac{n-1+\beta}{n(n-1)}}, \qquad (8)$$

where c_1, c_2 and c_3 depend on α, β, n and c_Γ.

Proof. Let us take $p = \frac{n}{1-\beta}$. By assumption we have $1 - p(1-\alpha) > 0$. Lemma 2.2 now shows that the $(n - p(1-\alpha))$-sum is finite.

Then, according to Remark 3.1, the estimate (7) follows by the same method as in the proof of Theorem 3.1.

The proof is now completed by the following observations:

i) $|\Gamma| \le (c_\Gamma)^{-\frac{1}{n-1}}[\mathcal{H}(\Gamma)]^{\frac{1}{n-1}}$, which is clear from (1).

ii) Lemma 2.2, taking $\epsilon = 1 - p(1-\alpha)$ gives

$$\mathbf{s}(p(\alpha-\beta)) \le c\frac{\mathcal{H}(\Gamma)}{1-p(1-\alpha)}. \qquad \square$$

Remark 3.2. Employing a more intrinsic strategy, the Hölder boundedness of H_Γ was carried out in [4]. Much to our surprise, this permitted to show that H_Γ is precisely bounded from $C^{0,\alpha}(\Gamma)$ into $C^{0,n\alpha+1-n}(\Gamma)$. However, the technique employed there does not seem to be available anymore for fractal domains.

3.3 Rectifiable Jordan curves in $\mathbb{R}^2$

An especially interesting case occurs when Γ is a rectifiable closed Jordan curve in $\mathbb{R}^2$. Since the (l) AD-regularity condition is automatic for such a boundaries, it follows that the constants in (8) depend only on α and β. This particular situation was early considered in [12] for piecewise smooth curves. Our approach offers certain improvement to the main result obtained there.

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