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On the deformation theory of Calabi-Yau structures in strongly pseudo-convex manifolds

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Abstract. We study the deformation theory of Calabi-Yau structures in strongly pseudo-convex manifolds with trivial canonical bundles. Our approach could be considered as an alternative proof for a theorem of H. Laufer on the deformation of strongly pseudo-convex surfaces.

Keywords: strongly pseudo-convex manifolds, deformation theory, Hodge theory.

Mathematical subject classification: 53C55, 53C26.

1 Introduction

There are traditionally two main approaches for the study of the deformation theory of complex closed manifolds: The algebraic method based on Çech cohomology and the analytic method which uses Hodge theory [5]. The theory which is known in the literature as Kodaira-Spencer theory shows that the first order infinitesimal deformations of a complex manifold M is characterized by $H^1(M, \mathcal{T}')$ and the obstruction to formally extending the deformation to higher orders is determined by $H^2(M, \mathcal{T}')$ where $\mathcal{T}'$ denotes the holomorphic tangent sheaf to M.

One of the most fundamental families of complex manifolds for which local and global deformation theory are extensively studied and very well understood is the special case of K3 surfaces. By definition K3 surfaces are simply connected compact complex surfaces with trivial canonical bundles. It can be seen that for these types of surfaces the above mentioned obstruction vanishes and so the moduli of complex structures form a smooth manifold of dimension dim $H^{1,1}(M)$. This is a well-known theorem in the literature, named local Torelli theorem for K3 surfaces. Local Torelli theorem can also be proved by

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a third method using an observation of Andereotti. We recall that K3 surfaces are topologically unique and their variation is associated to different complex structures on this unique background. Suppose that a K3 surface is described by a complex structure I on the differentiable manifold M. We write $X = (M, I)$. The holomorphic 2-form σ which is unique upto scaling can be viewed as a complex two form $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$. The complex form σ obviously satisfies the following three conditions:

$$\text{i) } \sigma \text{ is closed i.e. } d\sigma = 0, \quad \text{ii) } \sigma \wedge \sigma = 0, \quad \text{and} \quad \text{iii) } \sigma \wedge \bar{\sigma} > 0.$$

The two form σ is also called the holomorphic volume form or the *Calabi-Yau structure* of $X = (M, I)$. The observation of Andereotti is that the converse also holds. Indeed, any complex two form $\sigma \in A^2_{\mathbb{C}}(M)$ satisfying $i)$-$iii)$ is induced by a complex structure in the above sense. More precisely, one defines $T^{0,1}M$ as the kernel of $\sigma \colon T_{\mathbb{C}}M \to T^*_{\mathbb{C}}M$ and $T^{1,0}M$ as its complex conjugate. Conditions $ii)$ and $iii)$ ensure that this results in a decomposition of $T_{\mathbb{C}}M$ which defines an almost complex structure. This almost complex structure is integrable due to $i)$. Thus the space of complex structures on M can be identified with the space of complex two forms σ on M satisfying the conditions $i)$-$iii)$ up to natural equivalences. This description of complex structures is the basis of a third method for studying the deformation theory of complex structures on K3 surfaces [4].

In this note we would like to extend this method into strongly pseudo-convex surfaces with trivial canonical bundles. The deformation theory of strongly pseudo-convex surfaces has already been studied by H. Laufer [8] using algebraic methods. The theorem we prove can also be deduced from the following theorem of Laufer, but as far as we know no analytic method has yet been developed for the deformation theory in this case:

Theorem 1.1 ([8]). *Let M be a strongly pseudo-convex surface with trivial canonical bundle. Then there exists a versal deformation $\omega \colon \mathcal{M} \to Q$ of $M = \omega^{-1}(0)$, where Q is a complex manifold and the Kodaira-Spencer map $\rho_0 \colon T_{0,Q} \to H^1(M, \mathcal{T}')$ is an isomorphism.*

K3 surfaces constitute the two dimensional examples of Calabi-Yau manifolds (the definition is quite similar). CY manifolds have been the subject of extensive studies and several conjectures since more than 3 decades ago. The generalization of local Torelli theorem showing the non-obstruction for deformation theory of higher dimensional CY manifolds was proved by Tian-Todorv-Bogomolov in [11].

As far as we know the existence of Ricci flat metrics on strongly pseudo-convex manifolds with trivial canonical bundles (which we also call strongly pseudo-convex CY manifolds) has been confirmed in some special cases [1, 2]. The theorem of Laufer gives another evidence for the existence of similarities between strongly-pseudoconvex CY surfaces and K3 surfaces. The moduli of complex structures play an important role in establishing the so-called dualities in superstring theories and one might hope that strongly pseudo-convex CY manifolds can also be studied in the context of string theory. Strongly pseudo-convex surfaces with trivial canonical bundles benefit a wider range of topologies than K3 surfaces and thus their study is much more complicated. An important class of examples for these manifolds that have been extensively studied can be obtained by considering neighborhoods of zero section in negative line bundles on complex varieties [9]. By using the adjunction formula one can also see that neighborhoods of zero section in the canonical bundle of a complex curve with genus $g \geq 2$ are complex surfaces with trivial canonical bundles and so provide a CY pseudo-convex manifold. We would like to give in this note a new proof for the following theorem:

Theorem 1.2. *If (M, M') is a strongly pseudo-convex CY manifold then the space of $\widetilde{CY}$ structures on M upto natural equivalences is locally isomorphic to $H^{(1,1)}(M)$.*

Here the word "natural" refers to two types of equivalence groups acting on CY structures: 1) the group of diffeomorphisms of M and 2) the group of now-here zero holomorphic functions in M acting by multiplication on σ.

In sections 2 and 3 we provide some preliminaries and review some known results about strongly pseudo convex manifolds. Section 4 develops formal deformation theory and section 5 treats the convergence of the associated formal series.

2 Preliminaries

In this section we briefly review some standard definitions regarding strongly pseudo-convex manifolds.

Let M' be a complex manifold and M be an open submanifold of M' with the following properties:

(a) $\overline{M}$, the closure of M, is compact.

(b) ∂M, the boundary of M, is a C^∞ submanifold of M'.

(c) If $p \in \partial M$, there exists local coordinates $t^1, \dots, t^{2n-1}, r$ on an open neighborhood U of the point p in M' s.t. $r(Q) < 0$ if $Q \in U \cap M$ and $r(Q) > 0$ for $Q \in U \cap (M' - \overline{M})$.

We call such a pair (M, M') a finite manifold. Let $A^{p,q}$ denote the space of C^∞ (p,q)-forms on M and $\dot{A}^{p,q}$ be the space of (p,q)-forms which are restrictions of (p,q)-forms on M'. We also define:

$$\dot{A}_0^{p,q} = \left\{\alpha \in \dot{A}^{p,q} \ s.t. \ \alpha = 0 \ \text{ on } \ \partial M\right\}$$

A finite manifold $\{M, M'\}$ is called strongly pseudo-convex if for each holomorphic coordinate system on a domain $U \subset M'$ there exists a C^∞ function $f \in C^\infty(U)$ s.t.

(a) $f(p) < 0$ if $p \in M$ and $f(p) > 0$ if $p \in U \cap (M' - \overline{M})$.

(b) $(df)_p \neq 0$ if $p \in \partial M$.

(c) If $(a_1, \dots, a_n) \in \mathbb{C}^n$ and $\sum f_{z_i}(p)a_i = 0$ for a $p \in \partial M$ then

$$\sum f_{z_i\overline{z_j}}(p)a_i\overline{a_j} > 0.$$

We also have the following cohomology groups:

$$H^{p,q}(M) = \frac{Z^{p,q}}{B^{p,q}} \qquad \dot{H}^{p,q}(M) = \frac{\dot{Z}^{p,q}}{\dot{B}^{p,q}}$$

where Z and $\dot{Z}$ (resp. B and $\dot{B}$) denote the space of $\overline{\partial}$-closed (resp. $\overline{\partial}$-exact) forms in A and $\dot{A}$. By introducing a Hermitian metric $\mathcal{G}$ on M' we can also define the space $\mathcal{H}^{(p,q)} \subset \dot{A}^{p,q}$ consisting of harmonic (p,q) forms with respect to $\mathcal{G}$ (see theorem 3.2 for the relation between different cohomology groups defined above).

Definition 2.1. *By a Calabi-Yau strongly pseudoconvex surface we mean a strongly pseudo-convex finite surface (M, M') with a trivial canonical bundle $K_{M'}$.*

Definition 2.2. *Let σ be a smooth complex 2-form defined in a neighborhood of M. We say that σ defines a Calabi-Yau structure if and only if the following conditions are satisfied:*

$$1)\ \sigma \wedge \sigma = 0, \qquad 2)\ d\sigma = 0, \quad \textit{and} \quad 3)\ \sigma \wedge \overline{\sigma} > 0.$$

We use the notation $\widetilde{CY}$ for the space of Calabi-Yau structures on M.

3 A review on the theory of Kohn and Rossi

In this section we briefly review the Hodge theory developed by Kohn and Rossi for strictly pseudo-convex complex manifolds [KR]. Let $\varphi, \psi \in \dot{A}^{p,q}$, then the usual inner product (φ, ψ) and the norm $|\varphi|_2$ are defined as follows:

$$(\varphi, \psi) = \int_M \varphi \wedge *\overline{\psi}\ , \quad |\varphi|_2^2 = (\varphi, \varphi).$$

The operator $\delta\colon A^{p,q} \longrightarrow A^{p,q-1}$ is defined by

$$\delta\varphi = - * \partial * \varphi\ .$$

Let $\mathcal{L}^{p,q}$ denote the Hilbert space obtained by completing $\dot{A}^{p,q}$ under the above inner product and denote by $T\colon \mathcal{D}_T^{p,q} \longrightarrow \mathcal{L}^{p,q+1}$ the closure of $\overline{\partial}$, i.e.,

$$\mathcal{D}_T^{p,q} = \big\{\varphi \in \mathcal{L}^{p,q} |\ \exists(\varphi_k),\ \varphi_k \in \dot{A}^{p,q}\ \ s.t.\ \ \varphi = \lim \varphi_k\ \text{ and } (\overline{\partial}\varphi_k) \text{ is Cauchy}\big\},$$

and T is defined by $T\varphi = \lim \overline{\partial}\varphi_k$. Also the operator $T^*\colon \mathcal{D}_{T^*}^{p,q} \longrightarrow \mathcal{L}^{p,q-1}$ denotes the Hilbert space adjoint of T, where

$$\mathcal{D}_{T^*}^{p,q} = \Big\{\varphi \in \mathcal{L}^{p,q} |\ \exists\theta \in \mathcal{L}^{p,q-1} \text{ such that } (\varphi, T\alpha) = (\theta, \alpha) \text{ for all } \alpha \in \mathcal{D}_T^{p,q-1}\Big\},$$

and T^* is defined by $T^*\varphi = \theta$. Further we define $L\colon \mathcal{D}_L^{p,q} \longrightarrow \mathcal{L}^{p,q}$ by

$$L = TT^* + T^*T$$

and

$$\mathcal{D}_L^{p,q} = \Big\{\varphi \in \mathcal{D}_T^{p,q} \cap \mathcal{D}_{T^*}^{p,q} |\ T\varphi \in \mathcal{D}_{T^*}^{p,q-1} \text{ and } T^*\varphi \in \mathcal{D}_T^{p,q}\Big\}.$$

Finally the space $\mathcal{H}^{p,q}$ is defined as

$$\mathcal{H}^{p,q} = \Big\{\varphi \in \mathcal{D}_L^{p,q} | L\varphi = 0\Big\},$$

and it can be seen that

$$\mathcal{H}^{p,q} = \Big\{\varphi \in \mathcal{D}_T^{p,q} \cap \mathcal{D}_{T^*}^{p,q} |\ T\varphi = T^*\varphi = 0\Big\}.$$

In [9] it is proved that L is self-adjoint and that we have the weak decomposition:

$$\mathcal{L}^{p,q} = \big[L\mathcal{D}_L^{p,q}\big] \oplus \mathcal{H}^{p,q}\ ,$$

where $[S]$ denotes the closure of S in $\mathcal{L}^{p,q}$.

The following theorem is proved in [9]:

Theorem 3.1. *If $M \subset M'$ is strongly pseudo convex then there exists a bounded operator $N\colon \mathcal{L}^{p,q} \longrightarrow \mathcal{L}^{p,q}$ such that:*

(a) $N\mathcal{L}^{p,q} \subset \mathcal{D}_L^{p,q}$, *and we have the strong orthogonal decomposition:*

$$\mathcal{L}^{p,q} = LN\mathcal{L}^{p,q} \oplus \mathcal{H}^{p,q}.$$

(b) If $H\colon \mathcal{L}^{p,q} \longrightarrow \mathcal{H}^{p,q}$ is the orthogonal projection on $\mathcal{H}$, then $HN = NH = 0$. If $\varphi \in \mathcal{D}_L^{p,q}$, then $TN\varphi = NT\varphi$, if $\varphi \in \mathcal{D}_{T^*}^{p,q}$, then $T^*N\varphi = NT^*\varphi$ and, if $\varphi \in \mathcal{D}_L^{p,q}$, then $LN\varphi = NL\varphi$.

Moreover we have

$$|TN\phi|_2 + |T^*N\phi|_2 + |N\phi|_2 \leq c|\phi|_2$$

(c) N and H preserve differentiability up to the boundary, i.e., $N(\dot{A}^{p,q}) \subset \dot{A}^{p,q}$ and $H(\dot{A}^{p,q}) \subset \dot{A}^{p,q}$.

The following theorem establishes the relation between different cohomology groups defined in the introduction:

Theorem 3.2 ([K]). *If $M \subset M'$ is strongly pseudo convex then $H^{p,q}(M) \cong \mathcal{H}^{p,q}$ and if $q \neq 0$ then $\dot{H}^{p,q}(M) \cong H^{p,q}(M) \cong \mathcal{H}^{p,q}$.*

4 Formal deformation of Calabi-Yau structures

In this section we will show that the space of formal deformations of a Calabi-Yau structure upto natural equivalences is isomorphic to $H^{1,1}(M)$.

4.1 A primary characterization for $H^{1,1}(M)$

In this section we give a characterization of the cohomology group $H^{1,1}(M)$ in terms of special subspaces of differential forms which will be used in the proof of theorem 1. We will frequently use the following lemmas through the next two sections:

Proposition 4.1 ([KR]). *If M is a finite manifold then $H^{(p,n)}(M) = 0$.*

Using the extension theorem of Kohn and Rossi in [KR] one can easily show that for each strongly pseudo convex finite manifold we have $H_0^{0,1}(M) = 0$. Applying the duality between $H_0^{n-p,n-q}(M)$ and $H_0^{p,q}(M)$ established in [KR] for strongly pseudo convex finite manifolds one can deduce the following lemma:

Lemma 4.1. *Let M be a strongly pseudo convex finite manifold of complex dimension n. If M satisfies $H^{0,1}(M) = 0$ then we have $H^{n,n-1}(M) = 0$.*

Assume now that W and W' be defined as follows:

$$W = \left\{\alpha \in \dot{A}^{2,0} \oplus \dot{A}^{1,1} \mid d\alpha = 0\right\}, \quad W' = W \cap \left\{d\beta + \gamma^{2,0} | \gamma^{2,0} \in \dot{Z}^{2,0}\right\}.$$

Lemma 4.2. *If (M', M) is a Calabi-Yau strongly pseudo-convex finite surface then $H^{1,1}(\dot{M}, \mathbb{C}) \cong \frac{W}{W'}$.*

Proof. Let α be an element in W with the decomposition $\alpha = \alpha^{2,0} + \alpha^{1,1}$ where $\alpha^{1,1} \in \dot{A}^{1,1}$ and $\alpha^{2,0} \in \dot{A}^{2,0}$. We define the application ϕ as follows:

$$\phi\colon W \longrightarrow H^{1,1}, \qquad \phi(\alpha) = [\alpha^{1,1}].$$

We first show that ϕ is surjective. Let $[\beta^{1,1}] \in H^{1,1}(M)$ where $\beta^{1,1} \in A^{1,1}(M)$. For surjectivity we should find a $(2, 0)$-form $\beta^{2,0}$ such that $d(\beta^{2,0} + \beta^{1,1}) = 0$ or equivalently by writing $d = \partial + \overline{\partial}$ we should have $\overline{\partial}\beta^{2,0} + \partial\beta^{1,1} = 0$. To show the existence of $\beta^{2,0}$ we first note that $[\partial\beta^{1,1}] \in H^{2,1}(M)$ and according to lemma 4.1 we have $H^{2,1}(M) = 0$ so there exist a (2,0)-form $\gamma^{2,0}$ which satisfies $\partial\beta^{1,1} = \overline{\partial}\gamma^{2,0}$ thus $\beta^{2,0} = -\gamma^{2,0}$ is exactly what we need to derive the surjectivity of ϕ.

We now prove that $Ker(\phi) = W'$. Let $\alpha = \alpha^{2,0} + \alpha^{1,1} \in \mathrm{Ker}(\phi)$. This means that $[\alpha^{1,1}] = 0$ and so there exists a $(1, 0)$-form $\beta^{1,0} \in A^{1,0}(M)$ for which we have $\alpha^{1,1} = \overline{\partial}\beta^{1,0}$. Now if we define $\gamma^{2,0} := \alpha^{2,0} - \partial\beta^{1,0}$ then by using the relation $d\alpha = 0$ one can easily see that $\overline{\partial}\gamma^{2,0} = 0$ and therefore we have $\alpha^{2,0} + \alpha^{1,1} = \gamma^{2,0} + d\beta \in W'$. This shows that $\mathrm{Ker}(\phi) \subset W'$. Conversely let $d\beta + \gamma^{2,0} \in W'$ and let $\beta = \beta^{1,0} + \beta^{0,1}$ be the decomposition of β into $(1, 0)$ and $(0, 1)$ parts. According to the definition of W' we should have $\overline{\partial}\beta^{0,1} = 0$. On the other hand we know that $H^{0,1}(M) = 0$ so there exists a C^∞ function f s.t. $\beta^{0,1} = \overline{\partial} f$ and therefore

$$\phi\big(d\beta + \gamma^{2,0}\big) = \big[\overline{\partial}\beta^{1,0} + \partial\beta^{0,1}\big] = \big[\partial\beta^{0,1}\big] = \big[\partial\overline{\partial} f\big] = \big[-\overline{\partial}\partial f\big] = 0$$

this shows that $W' \subset \mathrm{Ker}(\phi)$ and the proof of the lemma is completed. □

4.2 Infinitesimal deformation

Let (M', M) be a Calabi-Yau strongly pseudo-convex finite manifold and let σ be a Calabi-Yau structure on M' (c.f. definition 2.2 in §2). Consider the *formal deformation* $\sigma(t) = \sum_{i=0}^{\infty} \sigma_i t^i$ of σ_0 with $\sigma_i \in \dot{A}^2$, $i \geq 1$. We say that $\sigma(t) \in \widetilde{CY}$ defines a *formal* Calabi-Yau structure if and only if the following two conditions are satisfied:

1) $\sigma_i \in \dot{Z}^2$,

2) $\sum_{k=0}^{i} \sigma_k \wedge \sigma_{i-k} = 0$, $\forall i \geq 0$.

Note that if the formal series $\sigma(t)$ is convergent then the above two conditions lead to a Calabi-Yau structure in M. For $i = 1$ condition (2) implies that $d\sigma_1 = 0$ and $\sigma_1 \wedge \sigma_0 = 0$ thus $\sigma_1 \in A^{1,1}(M) \oplus A^{2,0}(M)$ and we obtain $\sigma_1 \in W$. In fact we have the following proposition:

Proposition 4.2. *If $T_{\sigma_0}\widetilde{CY}$ denotes the tangent space at σ_0 of the space of Calabi-Yau structures then we have $T_{\sigma_0}\widetilde{CY} = W$.*

Proof. Let $\sigma(t) = \sum_{i=0}^{\infty} \sigma_i t^i$ be a first order deformation and $\sigma_i \in \dot{A}^2$ for $i \geq 0$ satisfy the two conditions mentioned above. For $i = 1$ the conditions $d\sigma_1 = 0$ and $\sigma_1 \wedge \sigma_0 = 0$ imply that $\sigma_1 \in W$.

Conversely we prove that given $\sigma_1 \in W$ one can construct a formal Calabi-Yau deformation series for σ_0 with first order term σ_1. To prove this we use the induction on i. Assume that sequence of differential complex 2-forms like $\{\sigma_k\}_{k=1}^{i-1}$ satisfy conditions (1) and (2) above. Let the complex 2-form σ_i be decomposed as follows:

$$\sigma_i = f_i \overline{\sigma}_0 + \alpha_i^{1,1} + \beta_i^{2,0}$$

Clearly the condition (2) for the sequence $\{\sigma_k\}_{k=1}^{i}$ is equivalent to

$$\sigma_i \wedge \sigma_0 = -\sum_{k=1}^{i-1} \sigma_k \wedge \sigma_{i-k}$$

thus by using the decomposition of σ_i we get:

$$f_i\, \overline{\sigma_0} \wedge \sigma_0 = -\sum_{k=1}^{i-1} \sigma_k \wedge \sigma_{i-k}$$

the unique inductive solution of this equation is given by

$$f_i = \frac{-\sum_{k=1}^{i-1} \sigma_k \wedge \sigma_{i-k}}{\overline{\sigma}_0 \wedge \sigma_0}.$$

Now we should find $\alpha_i^{1,1}$ and $\beta_i^{2,0}$ in such a way that condition (1) is satisfied i.e. $d\sigma_i = 0$. Using the decomposition $d = \partial + \overline{\partial}$ we obtain: $(\partial\alpha_i^{1,1} + \overline{\partial}\beta_i^{2,0}) + (\partial f \wedge \overline{\sigma}_0 + \overline{\partial}\alpha_i^{1,1}) = 0$ so the condition (1) is equivalent to the following two equations:

$$\partial f_i \wedge \overline{\sigma}_0 + \overline{\partial}\alpha_i^{1,1} = 0 \tag{4.1}$$

$$\partial\alpha_i^{1,1} + \overline{\partial}\beta_i^{2,0} = 0 \tag{4.2}$$

Now we can use the following $\bar{\partial}$-Neumann lemma to obtain $\alpha_i^{1,1}$ and $\beta_i^{2,0}$ from the above system of equations.

Lemma 4.3 ([7]). *The equation $\overline{\partial}\alpha = \gamma$ is solvable for α if and only if $\overline{\partial}\gamma = 0$.*

Remark 4.1. Note that the above equations are not well defined unless other supplementary conditions are added. We will impose the local condition $\bar{\partial}^*\alpha_i^{1,1} = 0$ and a boundary condition for the equation 4.2 in the last section.

4.3 Action of the group of isomorphisms

There exist two groups of isomorphisms acting on the space of Calabi-Yau structures $\widetilde{CY}$:

1) *The group of diffeomorphisms of M*: Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of M and let $G_1 = \mathrm{Diff}_0(M)$ be the connected component of the identity. The action of the group G_1 on the space $\widetilde{CY}(M)$ can be described as follows:

$$\phi\colon G_1 \times \widetilde{CY}(M) \longrightarrow \widetilde{CY}(M)$$
$$\phi(f, \sigma) = f^*\sigma\ .$$

It can be easily verified that the tangent space to the orbit passing through σ_0 of the action of the group G_1 on $\widetilde{CY}$ is equal to

$$S_1 = W \cap \left\{d i_v \sigma_0 \mid v \in \Gamma(TM)\right\} = W \cap \left\{d\alpha \mid \alpha \in \dot{A}^{1,0}\right\} \tag{4.3}$$

2) *The group of the isomorphisms of the canonical bundle*: Let G_2 be the group of nonzero holomorphic functions on $\overline{M}$, i.e. $G_2 = \{f\colon M \longrightarrow \mathbb{C}^\times \mid f \in \Omega^0(M)\}$. This can be considered as a multiplicative group acting by multiplication on the space of Calabi-Yau structures $\widetilde{CY}$. The tangent space to the orbit of this action can be identified to

$$S_2 = W \cap \left\{\gamma \in \dot{A}^{2,0} \mid \overline{\partial}\gamma = 0\right\} \tag{4.4}$$

Now we can easily prove the following theorem:

Theorem 4.1. $T_{\sigma_0}\widetilde{CY}/S_1 + S_2 \simeq H^{1,1}(M)$

Proof. According to proposition 4.2 we know that $T_{\sigma_0}\widetilde{CY} = W$. Now using the relations (4.3) and (4.4) and lemma 4.2 the proof of the theorem will follow. □

5 Convergence

Now we would like to show that the formal deformation series of $\sigma(t)$ defined in the previous section is convergent for small values of t and defines a smooth complex structure on M. To this end we will prove that $\sigma(t) \in H_2^s(M)$ for $s \in \mathbb{R}$ great enough and for sufficiently small values of t , where $H_2^s(M)$ denotes the Sobolev space of 2-forms in M. We begin by the following lemma:

Lemma 5.1. *Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers and let $c \in \mathbb{R}^+$ be a given constant s.t.*

$$a_i = c \sum_{k=1}^{k=i-1} a_k a_{i-k}$$

then the sequence $f(t) = \sum_{i=0}^{\infty} a_i t^i$ is convergent for small values of t.

Proof. To prove this lemma consider the following formal calculation:

$$\begin{aligned}\big(f(t) - a_0\big)^2 = \left(\sum_{i=1} a_i t^i\right)^2 &= \sum_{i=2}^{\infty}\left(\sum_{k=1}^{k=i-1} a_k a_{i-k}\right) t^i \\ &= \frac{1}{c}\sum_{i=2}^{\infty} a_i t^i = \frac{1}{c}\big(f(t) - a_0 - a_1 t\big)\end{aligned}$$

The solution of this functional equation leads to

$$f(t) = A \pm \sqrt{Bt + C}$$

for appropriate values of A, B and C and this shows that the formal series defining f is in fact convergent for small values of t. □

Now according to the special solutions of the equations 4.1 and 4.2 (see the remark 4.1) and using the theorem 3.1 one can easily deduce that

$$|\alpha_i^{1,1}|_2 = |\bar{\partial}^* N(\partial f_i \wedge \sigma_0)|_2 \leq c|\partial f_i \wedge \sigma_0|_2 \tag{5.1}$$

Writing $\beta_i^{2,0} = g_i\sigma_0$ the equation 4.2 reduces to an equation for g_i:

$$\bar\partial g_i = \frac{\partial\alpha_i^{1,1}}{\sigma_0} \tag{5.2}$$

As is standard in the literature from the equation 4.1 along with $(\bar\partial^*)^2 = 0$ it follows that:

$$|\nabla\alpha_i^{1,1}|_2 \le c\big(|\bar\partial\alpha_i^{1,1}|_2 + |\bar\partial^*\alpha_i^{1,1}|_2 + |\alpha_i^{1,1}|_2\big) = c\big(|\bar\partial\alpha_i^{1,1}|_2 + |\alpha_i^{1,1}|_2\big)$$

where the constant c depends only on the properties of the domain M. From this inequality and the equations (4.1), (4.2), (5.1) and (5.2) we can deduce that:

$$|\sigma_i|_{1,2} \le c|f_i|_{1,2}$$

using the inductive definition of f_i one can also see that:

$$|f_i|_{1,2} \le c\sum_{k=1}^{i-1}|\sigma_k|_{1,2}|\sigma_{i-k}|_{1,2} \le c\sum_{k=1}^{i-1}|f_k|_{1,2}|f_{i-k}|_{1,2} \tag{5.3}$$

Now from lemma (5.1) and the inequalities (5.1) and (5.3) it follows that for small values of t we have:

$$\sum_{i=0}^{\infty}|\sigma_i|_{1,2}t^i < \infty$$

The same argument can be applied to show that $\sigma(t) \in H_2^s(M)$ for $s \in \mathbb{Z}$ arbitrarily large and for t small enough. It is not difficult to see that the constant c in the inequality (5.3) can be chosen to work uniformly for all values of s and thus an identical radius of convergence is found for all the series $\sum_{i=0}^{\infty}|\sigma_i|_{1,s}t^i$ with different values of s. On the other hand by the well-known Sobolev lemma, we have:

$$H_2^{s+2n}(M) \subset C_2^s(M)$$

Here $C_2^s(M)$ denotes the space of s-times continuously differentiable 2-forms on M. This shows that the induced complex structure is in fact C^∞ and thus analytic. □

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