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Torsion-free sheaves on nodal curves and triples

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Abstract. Let X be a reduced irreducible curve with at most nodes as singularities with normalization $\pi : \widetilde{X} \to X$. We study the description of torsion free sheaves on X in terms of vector bundles with an additional structure on $\widetilde{X}$ which was introduced by Seshadri.

Keywords: torsion-free sheaves, moduli vector bundles.

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1 Introduction

Let X be an irreducible reduced curve over an algebraically closed field with at most ordinary double points as singularities and let $\pi : \widetilde{X} \to X$ denote its normalization. In [10, Ch. 8] Seshadri described torsion free sheaves on X in terms of vector bundles on $\widetilde{X}$ with some additional structure. Let for simplicity X just have one node x with $\pi^{-1}(x) = \{p_1, p_2\}$. Then Seshadri showed that there is a canonical bijection between torsion free sheaves $\mathcal{F}$ on X and triples $(E, (\Delta_1, \Delta_2), \sigma)$ consisting of a vector bundle E on $\widetilde{X}$, a pair of vector subspaces (Δ_1, Δ_2) with $\Delta_i \subset E(p_i)$ and an isomorphism $\sigma : \Delta_1 \to \Delta_2$. To the best of our knowledge, this beautiful and clear result has not been applied very much in the literature (apart from [11]and [1]). The main papers on vector bundles on nodal curves apply different methods (see [2], [3], [4]).

The starting point of this paper was to understand Seshadri's result. We give a new proof and discuss the functoriality of this construction. We introduce the notion of morphisms of such triples and show that the categories of torsion free sheaves on X and the category of these triples on $\widetilde{X}$ are equivalent. Moreover, we define stability of triples over $\widetilde{X}$ using the corresponding notion for

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torsion free sheaves on X. Using this, we study the relation between the stability of a torsion free sheaf $\mathcal{F}$ on X and the stability of the vector bundle E occurring in the corresponding triple.

To be more precise, assume that $x_1, \dots, x_n$ are exactly the nodes of X and $\pi^{-1}(x_i) = \{p_i, q_i\}$ for $i = 1, \dots, n$. For any torsion free sheaf $\mathcal{F}$ of rank r on X there are unique integers a_i, $0 \le a_i \le r$ such that $\mathcal{F}_{x_i} \simeq \mathcal{O}_{x_i}^{a_i} \oplus m_{x_i}^{r-a_i}$. Moreover there is a unique subsheaf $\mathcal{E} \subset \mathcal{F}$ of the form $\mathcal{E} = \pi_* E$ with E a vector bundle of rank r on $\widetilde{\mathrm{X}}$ fitting into an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \oplus_{i=1}^{n} \mathcal{W}_{x_i} \to 0$$

where $\mathcal{W}_{x_i}$ is the skyscraper sheaf on X with fibre a vector space W_{x_i} of dimension a_i at the node x_i. Starting with this exact sequence, the vector bundle E is given by

$$E = \pi^* \mathcal{E} / \mathcal{T}$$

where $\mathcal{T}$ denotes the torsion subsheaf of $\pi^*(\mathcal{E})$. We show then that there is a canonical isomorphism

$$\Phi = \oplus \Phi_i : \mathrm{Ext}^1(\oplus_{i=1}^{n} \mathcal{W}_{x_i}, \mathcal{E}) \overset{\simeq}{\to}$$
$$\overset{\simeq}{\to} \oplus_{i=1}^{n} (\mathrm{Hom}_k(W_{x_i}, E(p_i)) \oplus \mathrm{Hom}_k(W_{x_i}, E(q_i)))$$

(Note that this description of Ext^1 is different from the description in [10, Ch. 8, Lemme 15].) Given an exact sequence as above, the vector spaces Δ_1 and Δ_2 as well as the isomorphism σ are constructed out of the corresponding homomorphisms $W_{x_i} \to E(p_i)$ and $W_{x_i} \to E(q_i)$ (see Remark 2.5).

In Section 3, we introduce morphisms of triples and prove the above mentioned equivalence of categories (see Theorem 3.2). Since the category of torsion free sheaves on X admits kernels and images, so does the category of triples on $\widetilde{\mathrm{X}}$. We describe them and give a criterion for a cokernel to exist.

In Section 4, we translate the notion of stability of torsion free sheaves to the corresponding triples. This is used to relate the stability properties of $\mathcal{F}$ to the stability of the vector bundle E. We show in particular that if E is stable, so are all triples $(E, (\Delta_1, \Delta_2), \sigma)$ which correspond to a vector bundle on X. We also give an example of an unstable vector bundle on $\widetilde{\mathrm{X}}$ admitting stable triples. Finally in Section 5, we give some applications.

Notation: Let X be a curve over k. By an $\mathcal{O}_X$-module we always understand a coherent $\mathcal{O}_X$-module. Similarly a torsion free $\mathcal{O}_X$-module means a coherent torsion free $\mathcal{O}_X$-module. A point of X always means a closed point. If $x \in X$

is a point, m_x denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and $\underline{m}_x$ denotes the corresponding ideal sheaf in $\mathcal{O}_X$. For any $\mathcal{O}_X$-module $\mathcal{F}$ and any $x \in X$ we denote by $\mathcal{F}_x$ its stalk and by $\mathcal{F}(x) = \mathcal{F}_x / m_x \mathcal{F}_x$ its fibre at the point x and we abbreviate $\mathcal{O}_x := \mathcal{O}_{X,x}$.

2 Relation to vector bundles on the normalization

2.1 The set up

Let X be an irreducible reduced curve over an algebraically closed field k with at most ordinary double points as singularities. We assume that X admits exactly n ordinary double points $x_1, \ldots, x_n$.

According to [10, Ch. 8, Prop. 2], for any torsion-free $\mathcal{O}_{x_i}$-module M of rank r there is a uniquely determined non-negative integer a_i such that $M \simeq \mathcal{O}_{x_i}^{a_i} \oplus m_{x_i}^{r-a_i}$. In particular, for any torsion-free sheaf $\mathcal{F}$ of rank r and degree d on X and any $i = 1, \ldots, n$ there is an integer a_i, uniquely determined with $0 \leq a_i \leq r$ such that

$$\mathcal{F}_{x_i} \cong \mathcal{O}_{x_i}^{a_i} \oplus m_{x_i}^{r-a_i}. \tag{2.1}$$

This gives surjective homomorphisms

$$\mathcal{F}_{x_i} \to W_{x_i} := k_{x_i}^{a_i},$$

where $k_{x_i} \simeq k$ denotes the residue field at the point x_i. If we denote by $\mathcal{W}_{x_i}$ the skyscraper sheaf concentrated at x_i with fibre W_{x_i}, we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \oplus_{i=1}^{n} \mathcal{W}_{x_i} \to 0, \tag{2.2}$$

where the kernel $\mathcal{E}$ is uniquely determined by $\mathcal{F}$, although the homomorphism $\mathcal{F} \to \oplus_{i=1}^{n} \mathcal{W}_{x_i}$ itself is not. This implies that $\mathcal{F}$ is an extension of $\oplus_{i=1}^{n} \mathcal{W}_{x_i}$ by a torsion-free sheaf $\mathcal{E}$ with

$$\mathcal{E}_{x_i} \cong m_{x_i}^{r} \quad \text{and} \quad \deg(\mathcal{E}) = \deg(\mathcal{F}) - \sum_{i=1}^{n} a_i. \tag{2.3}$$

Now consider the normalization map

$$\pi : \widetilde{X} \to X$$

and denote the two points of $\pi^{-1}(x_i)$ by p_i and q_i. According to [10, Ch. 8, Prop. 10], for a torsion free sheaf $\mathcal{E}$ of rank r and degree d on X there is a vector bundle E on $\widetilde{X}$ such that

$$\mathcal{E} = \pi_*(E) \tag{2.4}$$

if and only if $\mathcal{E}_{x_i} \cong m_{x_i}^r$ for all i. In this case E is uniquely determined by $\mathcal{E}$ and

$$\deg(E) = \deg(\mathcal{E}) - nr. \tag{2.5}$$

(Note that [10, Ch. 8, Prop. 10] states that $\deg(E) = \deg(\mathcal{E}) + nr$. However there is an error in the proof: the degree of $\pi_* \mathcal{O}_{\widetilde{X}}$ is considered to be -1).

We use the following two statements of [11].

Let k_{x_i} (respectively k_{p_i} and k_{q_i}) be the skyscraper sheaf on X (respectively $\widetilde{X}$) with fibre k at the point x_i (respectively p_i and q_i). Then we have

$$\underline{\mathrm{Tor}}^1_{\mathcal{O}_X}\left(k_{x_i}, \mathcal{O}_{\widetilde{X}}\right) = k_{p_i} \oplus k_{q_i}. \tag{2.6}$$

considered as sheaves on $\widetilde{X}$. This a consequence of [11, Ch. II, Lemma 2.1].

There exists a locally free sheaf $\mathcal{G}$ of rank r and degree $\deg \mathcal{F} + nr - \sum_{i=1}^n a_i$ on X such that

$$\mathcal{F} \subset \mathcal{G}. \tag{2.7}$$

and the quotient $\mathcal{G}/\mathcal{F}$ is supported at the nodes x_i. This is [11, Ch. II, Lemma 2.3].

Proposition 2.1. *Let $\mathcal{F}$ be a torsion free sheaf of rank r on X satisfying* (2.1) *and let $\mathcal{T}$ denote the torsion subsheaf of $\pi^*\mathcal{F}$. Then*

$$\deg(\pi^*\mathcal{F}) = \deg \mathcal{F} + nr - \sum_{i=1}^n a_i \quad \textit{and} \quad \deg \mathcal{T} = 2\left(nr - \sum_{i=1}^n a_i\right).$$

Proof. Let $\mathcal{G}$ be the locally free sheaf of (2.7). Pulling back the exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$ by π, we get the exact sequence

$$0 \to \mathcal{T} \to \pi^*\mathcal{F} \to \pi^*\mathcal{G} \to \pi^*(\mathcal{G}/\mathcal{F}) \to 0.$$

with $\mathcal{T} = \underline{\mathrm{Tor}}^1_{\mathcal{O}_X}\left(\mathcal{G}/\mathcal{F}, \mathcal{O}_{\widetilde{X}}\right)$. Since

$$\deg \mathcal{G}/\mathcal{F} = nr - \sum_{i=1}^n a_i$$

we get $\deg \pi^*(\mathcal{G}/\mathcal{F}) = 2\left(nr - \sum_{i=1}^n a_i\right)$. This implies

$$\begin{aligned} \deg(\pi^*\mathcal{F}/\mathcal{T}) &= \deg \pi^*\mathcal{G} - \deg \pi^*(\mathcal{G}/\mathcal{F}) \\ &= \deg \mathcal{F} + nr - \sum_{i=1}^n a_i - 2\left(nr - \sum_{i=1}^n a_i\right) \\ &= \deg \mathcal{F} - \left(nr - \sum_{i=1}^n a_i\right) \end{aligned}$$

On the other hand, from (2.6) we deduce

$$\deg \mathcal{T} = 2 \deg \mathcal{G}/\mathcal{F} = 2\left(nr - \sum_{i=1}^{n} a_i\right).$$

This implies also the first assertion. □

For the following corollary see [10, p.175, l.3].

Corollary 2.2. *Let $\mathcal{F}$ be a torsion free sheaf of rank r satisfying* (2.1) *and let $\mathcal{E} = \pi_* E$ its subsheaf as in* (2.4). *If $\mathcal{T}$ denotes the torsion subsheaf of $\pi^*\mathcal{E}$, then we have*

$$\pi^*\mathcal{E}/\mathcal{T} \simeq E.$$

Proof. As an immediate consequence of Proposition 2.1 and equation (2.5) we get

$$\deg\left(\pi^*\mathcal{E}/\mathcal{T}\right) = \deg E. \tag{2.8}$$

Now consider the canonical map $\pi^*\mathcal{E} \to \pi^*\mathcal{E}/\mathcal{T}$. Adjunction gives a map

$$\pi_* E = \mathcal{E} \to \pi_*\left(\pi^*\mathcal{E}/\mathcal{T}\right)$$

which is of maximal rank outside the nodes x_i. Since E is a vector bundle on $\widetilde{\mathrm{X}}$, we have

$$\begin{aligned} \operatorname{Hom}\left(\pi_* E, \pi_*(\pi^*\mathcal{E}/\mathcal{T})\right) &= H^0\left(\pi_* E^* \otimes \pi_*(\pi^*\mathcal{E}/\mathcal{T})\right) \\ &\subset H^0\left(\pi_*(E^* \otimes \pi^*\mathcal{E}/\mathcal{T})\right) \\ &\simeq H^0\left(E^* \otimes \pi^*\mathcal{E}/\mathcal{T}\right) \\ &= \operatorname{Hom}\left(E, \pi^*\mathcal{E}/\mathcal{T}\right). \end{aligned}$$

Hence we get a homomorphism $E \to \pi^*\mathcal{E}/\mathcal{T}$ which clearly is of maximal rank. Hence it is injective and thus an isomorphism by (2.8). □

2.2 Extensions

Recall that the extensions $0 \to \mathcal{E} \to \mathcal{G} \to \oplus_{i=1}^{n} \mathcal{W}_{x_i} \to 0$ are classified by the group $\operatorname{Ext}^1(\oplus_{i=1}^{n} \mathcal{W}_{x_i}, \mathcal{E})$. The next proposition gives a characterization of $\operatorname{Ext}^1(\oplus_{i=1}^{n} \mathcal{W}_{x_i}, \mathcal{E})$. Since Ext^1 is additive, it suffices to do this for each node x_i separately. The following proposition appeared in [10, Ch. 8, Lemme 12] in a slightly different form.

Proposition 2.3. *Let $\mathcal{E} = \pi_* E$ with E a vector bundle of rank r on $\widetilde{X}$. Then there is a canonical isomorphism*

$$\Phi = \oplus \Phi_i : \operatorname{Ext}^1\left(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E}\right) \stackrel{\simeq}{\to}$$
$$\stackrel{\simeq}{\to} \oplus_{i=1}^n \left(\operatorname{Hom}_k(W_{x_i}, E(p_i)) \oplus \operatorname{Hom}_k(W_{x_i}, E(q_i))\right)$$

where $\mathcal{W}_{x_i}$ is the skyscraper sheaf concentrated at x_i with fibre $W_{x_i} := k_{x_i}^{a_i}$ for integers a_i, $0 \le a_i \le r$.

For the proof we need the following lemma.

Lemma 2.4. *There is a canonical isomorphism*

$$\operatorname{Hom}_{\mathcal{O}_X}\left(\underline{m}_{x_i}, \mathcal{E}\right) \simeq \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\mathcal{O}_{\widetilde{X}}, E(D_i)\right).$$

where $D_i := p_i + q_i$ and as usual $E(D_i) := E \otimes \mathcal{O}_{\widetilde{X}}(D_i)$.

Proof. Let $\mathcal{T}_i$ denote the torsion subsheaf of $\pi^*(\underline{m}_{x_i})$. Certainly the divisor D_i induces a canonical isomorphism

$$\pi^*\left(\underline{m}_{x_i}\right)/\mathcal{T}_i \simeq \mathcal{O}_{\widetilde{X}}(-D_i).$$

Now the exact sequence $0 \to \mathcal{T}_i \to \pi^*(\underline{m}_{x_i}) \to \pi^*(\underline{m}_{x_i})/\mathcal{T}_i \to 0$ induces an isomorphism

$$\operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\pi^*(\underline{m}_{x_i})/\mathcal{T}_i, E\right) \to \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\pi^*(\underline{m}_{x_i}), E\right)$$

since $\operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{T}_i, E) = 0$. Using moreover adjunction, we finally get the following canonical isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{O}_X}(\underline{m}_{x_i}, \mathcal{E}) &\cong \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\pi^*(\underline{m}_{x_i}), E\right) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\pi^*(\underline{m}_{x_i})/\mathcal{T}_i, E\right) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\mathcal{O}_{\widetilde{X}}(-D_i), E\right) \\
&\cong \operatorname{Hom}_{\mathcal{O}_{\widetilde{X}}}\left(\mathcal{O}_{\widetilde{X}}, E(D_i)\right).
\end{aligned}$$

□

Proof of Proposition 2.3. We may assume $h^1(\mathcal{E}) = 0$, since tensorizing $\mathcal{E}$ and $\mathcal{W}_{x_i}$ by a line bundle changes both sides of the equation only by a canonical isomorphism. Using this, the exact sequence

$$0 \to W_{x_i} \otimes_k \underline{m}_{x_i} \to W_{x_i} \otimes_k \mathcal{O}_X \to \mathcal{W}_{x_i} \to 0$$

induces an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \mathcal{O}_X, \mathcal{E}) \to \mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \underline{m}_{x_i}, \mathcal{E}) \to \\ \to \mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E}) \to 0, \end{aligned} \tag{2.9}$$

where the last 0 uses $h^1(\mathcal{E}) = 0$.

The canonical sequence $0 \to E \to E(D_i) \to E(p_i) \oplus E(q_i) \to 0$ (here we identify the fibres $E(p_i)$ and $E(q_i)$ with the corresponding skyscraper sheaf concentrated at p_i and q_i respectively) induces an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E) \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(D_i)) \to \\ \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(p_i) \oplus E(q_i)) \to 0 \end{aligned} \tag{2.10}$$

where we used $\mathrm{Ext}^1(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E) = W_{x_i} \otimes H^1(\widetilde{X}, E) = W_{x_i} \otimes H^1(X, \mathcal{E}) = 0$.

Moreover, by adjunction we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \mathcal{O}_X, \mathcal{E}) \simeq \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(\pi^*(W_{x_i} \otimes \mathcal{O}_X), E) = \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E).$$

Using this, Lemma 2.4 and the exact sequences (2.9) and (2.10), we obtain the following diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \mathcal{O}_X, \mathcal{E}) & \to & \mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \underline{m}_{x_i}, \mathcal{E}) & \to & \mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E}) \to 0 \\ \| \wr & & \| \wr & & \\ 0 \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E) & \to & \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(D_i)) & \stackrel{\theta_i}{\to} & \\ & & \stackrel{\theta_i}{\to} \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(p_i) \oplus E(q_i)) \to 0, & & \end{array} \tag{2.11}$$

Since this diagram is certainly commutative, it induces a canonical isomorphism

$$\mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(p_i) \oplus E(q_i)).$$

Finally, observe that $\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{O}_{\widetilde{X}}, E(p_i)) = \mathrm{Hom}_k(k, E(p_i))$ and similarly for q_i. Hence we can identify

$$\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\widetilde{X}}, E(p_i) \oplus E(q_i)) = \mathrm{Hom}_k(W_{x_i}, E(p_i)) \oplus \mathrm{Hom}_k(W_{x_i}, E(q_i)).$$

Combining both isomorphisms completes the proof of the proposition. □

Remark 2.5. According to the proof and in particular diagram (2.11), the image of an extension $(e_i)\colon 0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{W}_{x_i} \to 0$ under the map Φ_i is given as follows: choose a preimage $\psi_{e_i} \in \mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \underline{m}_{x_i}, \mathcal{E})$ of $(e_i) \in \mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E})$ and consider it as an element in $\mathrm{Hom}_{\mathcal{O}_{\tilde{X}}}(W_{x_i} \otimes \mathcal{O}_{\tilde{X}}, E(D_i))$. Then

$$\Phi_i(e_i) = \theta_i(\psi_{e_i}).$$

Conversely, given a pair $(\alpha_i, \beta_i) \in \mathrm{Hom}_k(W_{x_i}, E(p_i)) \oplus \mathrm{Hom}_k(W_{x_i}, E(q_i))$, the corresponding extension $\Phi_i^{-1}(\alpha_i, \beta_i) \in \mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E})$ is constructed as follows: choose a preimage ψ_i of (α_i, β_i) under the map θ_i, considered as an element of $\mathrm{Hom}_{\mathcal{O}_X}(W_{x_i} \otimes \underline{m}_{x_i}, \mathcal{E})$. Then $\Phi_i^{-1}(\alpha_i, \beta_i)$ is the push-out of the canonical sequence defining $\mathcal{W}_{x_i}$ by ψ_i:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_{x_i} \otimes \underline{m}_{x_i} & \longrightarrow & W_{x_i} \otimes \mathcal{O}_X & \longrightarrow & \mathcal{W}_{x_i} & \longrightarrow & 0 \\
 & & \downarrow \psi_i & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_i & \longrightarrow & \mathcal{W}_{x_i} & \longrightarrow & 0
\end{array}$$

It follows from Proposition 2.3 that $\Phi_i(e_i)$ and $\Phi_i^{-1}(\alpha_i, \beta_i)$ do not depend on the choices of the preimages.

The extension $\Phi^{-1}((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)) \in \mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E})$ is then the sum of the extensions $0 \to \mathcal{E} \to \mathcal{F}_i \to \mathcal{W}_{x_i} \to 0$ in the group $\mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E})$, where $\mathrm{Ext}^1(\mathcal{W}_{x_i}, \mathcal{E})$ is considered as a subgroup of $\mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E})$.

2.3 Torsion free extensions

Let x denote any of the nodes x_i of X and let p and q be the points of $\tilde{X}$ above x. Recall that we denote by W_x a k_x-vector space of dimension a, $1 \leq a \leq r$ and by $\mathcal{W}_x$ the skyscraper sheaf on X with fibre W_x at x. The $\mathcal{O}_x$-module $\mathrm{Ext}^1_{\mathcal{O}_x}(W_x, \mathcal{E}_x)$ classifies the extensions $0 \to \mathcal{E}_x \to \mathcal{F}_x \to W_x \to 0$ of modules over the local ring $\mathcal{O}_x$. The following lemma shows that every such module is the restriction of a unique exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{W}_x \to 0$.

Lemma 2.6. *There is a canonical isomorphism* $\mathrm{Ext}^1(\mathcal{W}_x, \mathcal{E}) \simeq \mathrm{Ext}^1_{\mathcal{O}_x}(W_x, \mathcal{E}_x)$.

Proof. The edge homomorphism of the local-global spectral sequence (see [5, (4.2.7)]) is an isomorphism

$$\mathrm{Ext}^1(\mathcal{W}_x, \mathcal{E}) \simeq H^0(X, \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\mathcal{W}_x, \mathcal{E})),$$

since $H^i(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{W}_x, \mathcal{E})) = 0$ for $i = 1$ and 2. This implies the assertion, since the sheaf $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(\mathcal{W}_x, \mathcal{E})$ is a skyscraper sheaf with fibre $\mathrm{Ext}^1_{\mathcal{O}_x}(W_x, \mathcal{E}_x)$ concentrated at the point x. □

Combining the isomorphisms of Proposition 2.3 and Lemma 2.6, we get the following description of the extensions of $\mathcal{O}_x$-modules of W_x by $\mathcal{E}_x$:

Let

$$\widetilde{\mathcal{O}}_x := \mathcal{O}_p \cap \mathcal{O}_q$$

(intersection in the function field of X) denote the normalization of $\mathcal{O}_x$. It is a semilocal ring with two maximal ideals m_p and m_q and we have

$$m_x = m_p \cap m_q.$$

Moreover we denote by

$$E_x := E \otimes_{\mathcal{O}_{\widetilde{X}}} \widetilde{\mathcal{O}}_x \quad \text{and} \quad E(D)_x := E(D) \otimes_{\mathcal{O}_{\widetilde{X}}} \widetilde{\mathcal{O}}_x$$

the $\widetilde{\mathcal{O}}_x$-modules defined by E and $E(D)$, where D denotes the divisor $p+q$ on $\widetilde{X}$. As in the proof of Proposition 2.3, the exact sequences $0 \to W_x \otimes_k m_x \to W_x \otimes_k \mathcal{O}_x \to W_x \to 0$ and $0 \to E_x \to E(D)_x \to E(p) \oplus E(q) \to 0$ induce the following local version of diagram (2.11).

$$\begin{array}{l} 0 \to \mathrm{Hom}_{\mathcal{O}_x}(W_x \otimes \mathcal{O}_x, \mathcal{E}_x) \to \mathrm{Hom}_{\mathcal{O}_x}(W_x \otimes m_x, \mathcal{E}_x) \to \mathrm{Ext}^1_{\mathcal{O}_x}(W_x, \mathcal{E}_x) \to 0 \\ \qquad\qquad \wr\| \qquad\qquad\qquad\qquad\qquad \wr\| \\ 0 \to \mathrm{Hom}_{\widetilde{\mathcal{O}}_x}(W_x \otimes \widetilde{\mathcal{O}}_x, E_x) \to \mathrm{Hom}_{\widetilde{\mathcal{O}}_x}(W_x \otimes \widetilde{\mathcal{O}}_x, E(D)_x) \xrightarrow{\theta} \\ \qquad \xrightarrow{\theta} \mathrm{Hom}_{\widetilde{\mathcal{O}}_x}(W_x \otimes \widetilde{\mathcal{O}}_x, E(p) \oplus E(q)) \to 0 \\ \qquad\qquad\qquad\qquad \wr\| \\ \qquad \mathrm{Hom}_k(W_x, E(p)) \oplus \mathrm{Hom}_k(W_x, E(q)). \end{array} \tag{2.12}$$

Now consider $(\alpha, \beta) \in \mathrm{Hom}_k(W_x, E(p)) \oplus \mathrm{Hom}_k(W_x, E(q))$. Let $(\widetilde{\alpha}, \widetilde{\beta})$ denote the corresponding element in $\mathrm{Hom}_{\widetilde{\mathcal{O}}_x}(W_x \otimes \widetilde{\mathcal{O}}_x, E(p) \oplus E(q))$. Choose a preimage ψ of $(\widetilde{\alpha}, \widetilde{\beta})$ under the map θ and consider it as an element of $\mathrm{Hom}_{\mathcal{O}_x}(W_x \otimes m_x, \mathcal{E}_x)$. Then the extension in $\mathrm{Ext}^1_{\mathcal{O}_x}(\mathcal{W}_x, \mathcal{E}_x)$ corresponding to (α, β) is the pushout of the canonical sequence defining W_x by ψ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_x \otimes m_x & \longrightarrow & W_x \otimes \mathcal{O}_x & \longrightarrow & W_x & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{F}_x & \longrightarrow & W_x & \longrightarrow & 0. \end{array} \tag{2.13}$$

Recall that $\mathcal{E}_x \simeq m_x^r$. Hence, fixing isomorphisms $\mathcal{E}_x \simeq m_x^r$ and $W_x \otimes m_x \simeq m_x^a$, any homomorphism $\psi : W_x \otimes m_x \to \mathcal{E}_x$ is given by a matrix

$$A = (\alpha_{ij}) \in M(r \times a, \widetilde{\mathcal{O}}_x).$$

We say that ψ has birank (b_1, b_2), if $\operatorname{rk} A \mod m_p = b_1$ and $\operatorname{rk} A \mod m_q = b_2$. This definition does not depend on the choice of the isomorphisms. Certainly ψ has birank (b_1, b_2) if and only if $\operatorname{rk}\alpha = b_1$ and $\operatorname{rk}\beta = b_2$.

Proposition 2.7. *Let $0 \to \mathcal{E}_x \to \mathcal{F}_x \to W_x \to 0$ be the extension corresponding to the pair $(\alpha, \beta) \in \operatorname{Hom}_k(W_x, E(p)) \oplus \operatorname{Hom}_k(W_x, E(q))$. The following statements are equivalent:*

(1) *$\mathcal{F}_x$ is torsion-free,*

(2) *$\mathcal{F}_x \simeq \mathcal{O}_x^a \oplus m_x^{r-a}$,*

(3) *α and β are injective.*

Proof. The equivalence of (1) and (2) is clear, we have already used it. We have to show that (2) is equivalent to (3). Now α and β are injective if and only if they are both of rank a. As we saw just before the proposition, this is the case if an only if ψ is of birank (a, a). Since W_x is of dimension a, this is the case if and only if ψ is injective. From diagram (2.13) we deduce that this is the case if and only if the push-out $W_x \otimes \mathcal{O}_x \to \mathcal{F}_x$ is injective. But this is injective if and only if $\mathcal{F}_x$ is torsion-free. □

As an immediate consequence we get,

Corollary 2.8. *Let $(e)\colon 0 \to \mathcal{E} \to \mathcal{F} \to \oplus_{i=1}^n \mathcal{W}_{x_i} \to 0$ be an extension as in subsection 2.2 and let $\Phi((e)) = ((\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)) \in \oplus_{i=1}^n(\operatorname{Hom}_k(W_{x_i}, E(p_i)) \oplus \operatorname{Hom}_k(W_{x_i}, E(q_i)))$ (see Proposition 2.3). Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is torsion free;*

(2) *α_i and β_i are injective for $i = 1, \dots, n$.*

2.4 Triples on $\widetilde{X}$

In this subsection we outline the relation between torsion free sheaves on X and vector bundles on $\widetilde{X}$ with an additional structure.

Given a torsion free sheaf $\mathcal{F}$ on X, let $\mathcal{E} = \pi_*(E)$ be its subsheaf such that $0 \to \mathcal{E} \to \mathcal{F} \to \oplus_{i=1}^n \mathcal{W}_{x_i} \to 0$ is exact, where $\oplus_{i=1}^n \mathcal{W}_{x_i}$ is the torsion sheaf as above. Let

$$((\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)) \in \oplus_{i=1}^n(\operatorname{Hom}_k(W_{x_i}, E(p_i)) \oplus \operatorname{Hom}_k(W_{x_i}, E(q_i)))$$

denote the corresponding element according to Proposition 2.3. Then by Corollary 2.8 the homomorphisms α_i and β_i are all injective and we can consider the following vector spaces:

$$\Delta_1^i := \operatorname{Im}\alpha_i \subset E(p_i), \quad \Delta_2^i := \operatorname{Im}\beta_i \subset E(q_i),$$
$$\Delta_1 := \oplus_{i=1}^n \Delta_1^i, \qquad \Delta_2 := \oplus_{i=1}^n \Delta_2^i.$$

Combining the inverse of the isomorphism α_i onto its image with β_i we obtain an isomorphism

$$\sigma_i = \beta_i \circ \alpha_i^{-1} \colon \Delta_1^i \to \Delta_2^i.$$

Finally we denote the direct sum by

$$\sigma := \oplus_{i=1}^n \sigma_i \colon \Delta_1 \to \Delta_2.$$

Summing up, we associated to the sheaf $\mathcal{F}$ on X the object $(E, (\Delta_1, \Delta_2), \sigma)$ in a canonical way. We call these objects *triples on* $\widetilde{X}$ in the sequel.

Conversely, given a triple $(E, (\Delta_1, \Delta_2), \sigma)$ on $\widetilde{X}$, we denote for $i = 1, \cdots, n$

$$W_{x_i} := \Delta_1^i$$

and if $\alpha_i \colon W_{x_i} \hookrightarrow E(p_i)$ is the natural inclusion, we define $\beta_i \colon \Delta_2^i \to E(q_i)$ to be the composition

$$W_{x_i} = \Delta_1^i \xrightarrow{\sigma_i} \Delta_2^i \hookrightarrow E(q_i).$$

According to Proposition 2.3 there is a unique extension $0 \to \pi_* E \to \mathcal{F} \to \oplus_{i=1}^n \mathcal{W}_{x_i} \to 0$ associated to the n pairs $((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, where $\mathcal{F}$ is torsion free of rank r according to Corollary 2.8.

Summing up, we associated to the triple $(E, (\Delta_1, \Delta_2), \sigma)$ the torsion free sheaf $\mathcal{F}$ in a unique way. The following theorem is due to Seshadri (see [10, p. 178, Theorem 17]).

Theorem 2.9. *Given intergers $a_1, \ldots, a_n$ and r such that $0 \le a_i \le r$, there is a canonical bijection between the sets:*

(1) *of isomorphism classes of torsion free sheaves $\mathcal{F}$ of rank r and degree d on X such that for $i = 1, \cdots, n$,*

$$\mathcal{F}_{x_i} \cong \oplus_{i=1}^{a_i} \mathcal{O}_{x_i} \oplus \oplus_{i=1}^{(r-a_i)} m_{x_i} \tag{2.14}$$

and

(2) *of isomorphism classes of triples $(E, (\Delta_1, \Delta_2), \sigma)$, where*

- *E is a vector bundle of degree $d - nr - \sum_{i=1}^{n} a_i$ and rank r on $\widetilde{X}$,*
- *for $j = 1$ and 2, $\Delta_j = \oplus_{i=1}^{n} \Delta_j^i$ with Δ_1^i and Δ_2^i vector subspaces of dimension a_i of $E(p_i)$ and $E(q_i)$ respectively,*
- *$\sigma = \oplus_{i=1}^{n} \sigma_i$ with isomorphisms $\sigma_i \colon \Delta_1^i \to \Delta_2^i$.*

Proof. It is easy to see that the maps of one set to the other set given above are inverse to each other. The statement relating the degrees of E and $\mathcal{F}$ is a consequence of equations (2.3) and (2.5). □

As a direct consequence we get the following corollary.

Corollary 2.10. *Let $\mathcal{F}$ correspond to the triple $(E, (\Delta_1, \Delta_2), \sigma)$ as in Theorem 2.9. Then $\mathcal{F}$ is a vector bundle if and only if $a_1 = \cdots = a_n = r$.*

3 The functor Ψ

Let $(E, (\Delta_1, \Delta_2), \sigma)$ and $(E', (\Delta_1', \Delta_2'), \sigma')$ be triples on $\widetilde{X}$. A *homomorphism*

$$\tilde{g} \colon (E, (\Delta_1, \Delta_2), \sigma) \to (E', (\Delta_1', \Delta_2'), \sigma')$$

of triples on $\widetilde{X}$ is by definition a homomorphism $g \colon E \to E'$ of the underlying vector bundles satisfying $g(p_i)(\Delta_1^i) \subset \Delta_1'^i$ and $g(q_i)(\Delta_2^i) \subset \Delta_2'^i$ for $i = 1, \ldots, n$ such that the following diagram commutes

$$\begin{array}{ccc} \Delta_1 = \oplus_i \Delta_1^i & \xrightarrow{\sigma} & \Delta_2 = \oplus_i \Delta_2^i \\ \Big\downarrow \oplus_i g(p_i) & & \Big\downarrow \oplus_i g(q_i) \\ \Delta_1' = \oplus_i \Delta_1'^i & \xrightarrow{\sigma'} & \Delta_2' = \oplus_i \Delta_2'^i. \end{array} \tag{3.1}$$

With this notion of morphisms the set of triples on $\widetilde{X}$ forms a category which we denote by $\mathcal{TR}$.

Let $f \colon \mathcal{F} \to \mathcal{F}'$ denote a homomorphism of torsion free sheaves on X and denote by

$$\Psi(\mathcal{F}) := (E, (\Delta_1, \Delta_2), \sigma) \quad \text{and} \quad \Psi(\mathcal{F}') := (E', (\Delta_1', \Delta_2'), \sigma')$$

the corresponding triples according to Theorem 2.9. Let $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{E}' \subset \mathcal{F}'$ be the subsheaves defined in (2.2). It is easy to see that f maps $\mathcal{E}$ into $\mathcal{E}'$. We then see from Corollary 2.2 that

$$\pi^*(f|\mathcal{E})/\text{torsion} \colon E = \pi^*\mathcal{E}/\mathcal{T} \to \pi^*\mathcal{E}'/\mathcal{T}' = E'$$

is a homomorphism of vector bundles. Moreover we have,

Lemma 3.1. *The homomorphism f induces a homomorphism of triples*

$$\Psi(f)\colon (E, (\Delta_1, \Delta_2), \sigma) \to (E', (\Delta'_1, \Delta'_2), \sigma')$$

on $\widetilde{X}$.

Proof. Consider the extension (2.2) defined by $\mathcal{F}$ and let $0 \to \mathcal{E}' \to \mathcal{F}' \to \oplus_i \mathcal{W}'_{x_i} \to 0$ be the corresponding extension for $\mathcal{F}'$. Since f maps $\mathcal{E}$ into $\mathcal{E}'$ we get the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \oplus_i \mathcal{W}_{x_i} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f|\mathcal{E}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \oplus_i f_{x_i}} & & \\ 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & \oplus_i \mathcal{W}'_{x_i} & \longrightarrow & 0. \end{array} \tag{3.2}$$

Let $((\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)) \in \oplus_i (\mathrm{Hom}(W_{x_i}, E(p_i)) \oplus \mathrm{Hom}(W_{x_i}, E(q_i)))$ and similarly

$$((\alpha'_1, \beta'_1), \dots, (\alpha'_n, \beta'_n)) \in \oplus_i (\mathrm{Hom}(W'_{x_i}, E'(p_i)) \oplus \mathrm{Hom}(W'_{x_i}, E'(q_i)))$$

denote the n-tuples of pairs of homomorphisms associated to the extensions of diagram (3.2) according to Proposition 2.3. By definition the following diagrams commute for $i = 1, \dots, n$:

$$\begin{array}{ccc} W_{x_i} & \xrightarrow{\alpha_i} & E(p_i) \\ {\scriptstyle f_{x_i}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi(f)(p_i)} \\ W'_{x_i} & \xrightarrow{\alpha'_i} & E'(p_i) \end{array} \qquad \begin{array}{ccc} W_{x_i} & \xrightarrow{\beta_i} & E(q_i) \\ {\scriptstyle f_{x_i}}\big\downarrow & & \big\downarrow{\scriptstyle \Psi(f)(q_i)} \\ W'_{x_i} & \xrightarrow{\beta'_i} & E'(q_i). \end{array}$$

which implies $\Psi(f)(p_i)(\Delta_1^i) \subset \Delta_1^{'i}$ and $\Psi(f)(q_i)(\Delta_2^i) \subset \Delta_2^{'i}$ for $i = 1, \dots, n$. It remains to show that the diagram (3.1) commutes, but this is straightforward using Proposition 2.7 and $\sigma = \oplus_i \sigma_i = \oplus_i \big(\beta_i \circ \alpha_i^{-1}\big)$ and similarly for σ'. $\square$

If $\mathcal{TS}$ denotes the category of torsion free sheaves on X, then clearly Ψ is a functor from the category $\mathcal{TS}$ to the category $\mathcal{TR}$ of triples on $\widetilde{X}$.

Theorem 3.2. *The functor $\Psi\colon \mathcal{TS} \to \mathcal{TR}$ is an equivalence of categories.*

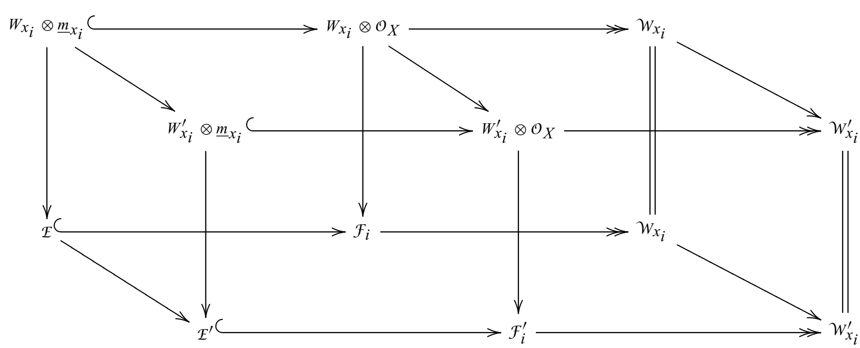
Proof. According to Theorem 2.9 the functor Ψ is a bijection on the objects. We have to show that it is a bijection on the sets of morphisms. So let $\underset{\sim}{\tilde{g}}\colon (E, (\Delta_1, \Delta_2), \sigma) \to (E', (\Delta'_1, \Delta'_2), \sigma')$ be a homomorphism of triples on $\widetilde{X}$. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the torsion free sheaves on X with $\Psi(\mathcal{F}) = (E, (\Delta_1, \Delta_2), \sigma)$ and $\Psi(\mathcal{F}') = (E', (\Delta'_1, \Delta'_2), \sigma')$.

Define $W_{x_i} := \Delta_1^i$, $W'_{x_i} := \Delta_1^{'i}$. So α_i and α'_i are its canonical inclusions into $E(p_i)$ and $E'(p_i)$ respectively. Similarly $\beta_i = \iota_i \circ \sigma_i$ and $\beta'_i = \iota'_i \circ \sigma'_i$, where ι_i and ι'_i are the canonical inclusions $\Delta_2^i \hookrightarrow E(q_i)$ and $\Delta_2^{'i} \hookrightarrow E'(q_i)$ respectively. Then diagram (3.1) implies that the following diagrams are commutative

$$\begin{array}{ccc} W_{x_i} & \xrightarrow{\alpha_i} & E(p_i) \\ {\scriptstyle g(p_i)}\downarrow & & \downarrow{\scriptstyle g(p_i)} \\ W'_{x_i} & \xrightarrow{\alpha'_i} & E'(p_i) \end{array} \qquad \begin{array}{ccc} W_{x_i} & \xrightarrow{\beta_i} & E(q_i) \\ {\scriptstyle g(p_i)}\downarrow & & \downarrow{\scriptstyle g(q_i)} \\ W'_{x_i} & \xrightarrow{\beta'_i} & E'(q_i) \end{array} \tag{3.3}$$

for $i = 1, \dots, n$.

Now choose for $i = 1, \dots, n$ preimages ψ_i and ψ'_i of (α_i, β_i) and (α'_i, β'_i) under the map θ_i of diagram (2.11) respectively and consider them as homomorphisms $\psi_i : W_{x_i} \otimes \underline{m}_{x_i} \to \mathcal{E} = \pi_* E$ and $\psi'_i\colon W'_{x_i} \otimes \underline{m}_{x_i} \to \mathcal{E}' = \pi_* E'$. According to remark 2.5, $\mathcal{F}$ is given by the sum in $\mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E})$ of the push-outs $0 \to \mathcal{E} \to \mathcal{F}_i \to \mathcal{W}_{x_i} \to 0$ by ψ_i of the canonical exact sequences $0 \to W_{x_i} \otimes \underline{m}_{x_i} \to W_{x_i} \otimes \mathcal{O}_X \to \mathcal{W}_{x_i} \to 0$ defining the sheaf $\mathcal{W}_{x_i}$. Similarly $\mathcal{F}'$ is defined. For every $i = 1, \dots, n$ we obtain the following commutative diagram:



where the left hand vertical maps are ψ_i and ψ'_i and the commutativity of the left hand vertical square is a consequence of the commutativity of diagram (3.3). The universal property of the push-out gives us a homomorphism

$$f_i\colon \mathcal{F}_i \to \mathcal{F}'_i$$

such that the whole diagram commutes. Finally, taking the sum of the lower bottom diagrams we get a map $\mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}_{x_i}, \mathcal{E}) \to \mathrm{Ext}^1(\oplus_{i=1}^n \mathcal{W}'_{x_i}, \mathcal{E}')$ given by a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \oplus_{i=1}^n \mathcal{W}_{x_i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & \oplus_{i=1}^n \mathcal{W}'_{x_i} & \longrightarrow & 0 \end{array}$$

which defines $f : \mathcal{F} \to \mathcal{F}'$.

It is easy to see that $\tilde{g} = \Psi(f)$ which completes the proof of the theorem. □

Since any morphism in the category $\mathcal{TS}$ admits a kernel and an image, so does any morphism in the category $\mathcal{TR}$. They are given in the following lemma the proof of which we omit.

Lemma 3.3. *Let $\tilde{g} : (E, (\Delta_1, \Delta_2), \sigma) \to (E', (\Delta'_1, \Delta'_2), \sigma')$ be a morphism of triples on $\widetilde{\mathrm{X}}$ with underlying map $g : E \to E'$. Then*

(a) $$\mathrm{Ker}\,\tilde{g} = (\mathrm{Ker}(g), (\Delta_{1,g}, \Delta_{2,g}), \sigma|\Delta_{1,g})$$

with $\Delta_{1,g} = \oplus_{i=1}^n \Delta_{1,g}^i$ and $\Delta_{1,g}^i = \Delta_1^i \cap \mathrm{Ker}\, g(p_i) \cap \sigma_i^{-1}(\mathrm{Ker}\, g(q_i))$ and $\Delta_{2,g} = \oplus_{i=1}^n \Delta_{2,g}^i$ with $\Delta_{2,g}^i = \sigma_i(\Delta_1^i \cap \mathrm{Ker}\, g(p_i)) \cap \mathrm{Ker}\, g(q_i)$.

(b) $$\mathrm{Im}\,\tilde{g} = (\mathrm{Im}(g), (\Delta'_{1,g}, \Delta'_{2,g}), \sigma|\Delta'_{1,g})$$

where $\Delta'_{1,g} = \oplus_{i=1}^n \Delta'^i_{1,g}$ with $\Delta'^i_{1,g} = \Delta'^i_1 \cap g(p_i)(\Delta_1^i) \cap \sigma'^{-1}_i(g(q_i)(\Delta_2^i))$ and $\Delta'_{2,g} = \oplus_{i=1}^n \Delta'^i_{2,g}$ with $\Delta'^i_{2,g} = \sigma'_i(\Delta'^i_1 \cap g(p_i)(\Delta_1^i)) \cap g(q_i)(\Delta_2^i)$.

Proposition 3.4. *Let $f : \mathcal{F} \to \mathcal{F}'$ be a homomorphism of torsion free sheaves and $\tilde{g} = \Psi(f)$ with underlying map of vector bundles $g : E \to E'$. Then*

(a) *$f : \mathcal{F} \to \mathcal{F}'$ is injective if and only if $g : E \to E'$ is injective.*

(b) *$f : \mathcal{F} \to \mathcal{F}'$ is surjective if and only if $g : E \to E'$ is surjective.*

Proof.

(a) It is clear from Corollary 2.2 that if f is injective, then so is g. So assume $g : E \to E'$ is injective. Since, for every i, the map $f_{x_i} : \mathcal{W}_{x_i} \to \mathcal{W}'_{x_i}$ can be identified with a restriction of the injective map $g(p_i) : E(p_i) \to E'(p_i)$ and $g : E \to E'$ is injective if and only if $f|\mathcal{E} : \mathcal{E} \to \mathcal{E}'$ is injective, the map $f : \mathcal{F} \to \mathcal{F}'$ is injective by diagram (3.2).

(b) Suppose f is surjective. We claim first that then also $f|\mathcal{E}\colon \mathcal{E} \to \mathcal{E}'$ is surjective. It suffices to show this over each node x_i separately. We have the following diagram of $\mathcal{O}_{x_i}$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & m_{x_i}^r & \longrightarrow & \mathcal{O}_{x_i}^{a_i} \oplus m_{x_i}^{r-a_i} & \longrightarrow & k_{x_i}^{a_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (f|\mathcal{E})_{x_i}} & & \downarrow{\scriptstyle f_{x_i}} & & \downarrow{\scriptstyle f(x_i)} & & \\ 0 & \longrightarrow & m_{x_i}^s & \longrightarrow & \mathcal{O}_{x_i}^{b_i} \oplus m_{x_i}^{s-b_i} & \longrightarrow & k_{x_i}^{b_i} & \longrightarrow & 0\,. \end{array}$$

Since f_{x_i} is surjective, so is $f(x_i)$. Hence the diagram implies that $m_{x_i}^{r-s} \subset \ker(f|\mathcal{E})_{x_i}$ with equality if and only if θ_i is surjective. The surjectivity of θ_i follows from its definition, i.e. from the definition of the of surjection in the top sequence in the diagram. This implies that θ_i is surjective, which in turn implies that $(f|\mathcal{E})_{x_i}$ and thus $f|\mathcal{E}$ is surjective. Now it follows from Corollary 2.2 and the right exactness of π^* that $g\colon E \to E'$ is surjective. The proof of the converse implication is similar to the corresponding proof in (a). □

The cokernel of a morphism of torsion free sheaves on X is not necessarily torsion free. The following proposition works out what it means for the corresponding triples that this is the case.

Proposition 3.5. *Let $f: \mathcal{F}' \to \mathcal{F}$ be a morphism of torsion free sheaves on X and $\tilde{g} = \Psi(f)\colon (E', (\Delta_1', \Delta_2'), \sigma') \to (E, (\Delta_1, \Delta_2), \sigma)$ be the corresponding map of triples on $\widetilde{\mathrm{X}}$. Then the following conditions are equivalent*

(1) *$\mathcal{F}/f(\mathcal{F}')$ is torsion free;*

(2) *E' is a vector subbundle of E, i.e. $E/g(E')$ is a vector bundle on $\widetilde{\mathrm{X}}$ and for $i = 1, \dots, n$ we have $g(p_i)(\Delta'^{i}_1) = \Delta_1^i \cap g(p_i)(E'(p_i))$ and $g(q_i)(\Delta'^{i}_2) = \Delta_2^i \cap g(q_i)(E'(q_i))$.*

Proof. Suppose that $\mathcal{F}/f(\mathcal{F}')$ is torsion free with corresponding triple $(E'', (\Delta_1'', \Delta_2''), \sigma'')$ on $\widetilde{\mathrm{X}}$. Pulling back the exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/f(\mathcal{F}') \to 0$ by π^* and moding out the torsion parts, we get, applying Corollary 2.2, the exact sequence

$$0 \to E' \xrightarrow{g} E \to E'' \to 0. \tag{3.4}$$

which implies that $E/g(E') \simeq E''$ is a vector bundle on $\widetilde{\mathrm{X}}$. As in the proof of Proposition 3.4 we see that the map $\Delta_j \to \Delta_j'$ is surjective for $j = 1$ and 2. This implies that $g(p_i)(\Delta'^i_1) = \Delta^i_1 \cap g(p_i)(E'(p_i))$ for $i = 1, \ldots, n$ and similarly for the points q_i.

Conversely, suppose that $E/g(E')$ is a vector bundle and that $g(p_i)(\Delta'^i_1) = \Delta^i_1 \cap g(p_i)(E'(p_i))$ and $g(q_i)(\Delta'^i_2) = \Delta^i_2 \cap g(q_i)(E'(q_i))$ for all i. Then we have the exact sequence (3.4) with $E'' = E'/g(E')$. Defining $\Delta''_1 := \oplus_i \Delta''^i_1$ with $\Delta''^i_1 = \Delta^i_1/g(p_i)(\Delta'^i_1)$ and $\Delta''_2 := \oplus_i \Delta''^i_2$ with $\Delta''^i_2 = \Delta^i_2/g(q_i)(\Delta'^i_2)$ for $i = 1, \ldots, n$, the assumptions imply that Δ''^i_1 is a subspace of $E''(p_i)$ and Δ''^i_2 is a subspace of $E''(q_i)$. Denote by $\sigma'' \colon \Delta''_1 \to \Delta''_2$ the isomorphism induced by σ. Then, if $\mathcal{F}''$ denotes the torsion free sheaf on X corresponding to the triple $(E'', (\Delta''_1, \Delta''_2), \sigma'')$, we have $\mathcal{E}'' = \pi_* E''$. So applying π_* to the exact sequence (3.4), we get the exact sequence $0 \to \mathcal{E}' \xrightarrow{f} \mathcal{E} \to \mathcal{E}'' \to 0$. Now consider the diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F}' & \longrightarrow & \oplus_{i=1}^n \mathcal{W}'_{x_i} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \oplus_{i=1}^n \mathcal{W}_{x_i} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}'' & \longrightarrow & \mathcal{F}'' & \longrightarrow & \oplus_{i=1}^n \mathcal{W}''_{x_i} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

By the assumption on the Δ'^i_j the right hand vertical sequence is exact. The obvious morphisms of triples give us maps $f \colon \mathcal{F}' \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{F}''$. Now the exactness of the diagram gives us an exact sequnce $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. In particular $\mathcal{F}/f(\mathcal{F}') \simeq \mathcal{F}''$ is torsion free. □

4 Stability

4.1 Definitions

For any torsion free sheaf $\mathcal{F}$ on X we define the slope $\mu(\mathcal{F})$ by

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}}.$$

Recall $\mathcal{F}$ is called *semistable*, respectively *stable*, if

$$\mu(\mathcal{G}) \leq \mu(\mathcal{F}), \quad \text{respectively} \quad \mu(\mathcal{G}) < \mu(\mathcal{F}),$$

for all proper subsheaves $\mathcal{G}$ of $\mathcal{F}$. Theorem 3.2 suggests to translate these notions to the category of triples on $\widetilde{X}$.

Let $(E, (\Delta_1, \Delta_2), \sigma)$ be a triple of rank r on $\widetilde{X}$ with $\Delta_1 = \oplus_{i=1}^n \Delta_1^i$ and $\dim \Delta_1^i = a_i$. If $\mathcal{F}$ denotes the corresponding torsion free sheaf on X, then according to (2.3) and (2.5) the degrees of E and $\mathcal{F}$ are related by

$$\deg E = \deg \mathcal{F} - nr - \sum_{i=1}^{n} a_i. \tag{4.1}$$

Hence it makes sense to define the *degree* of the triple $(E, (\Delta_1, \Delta_2), \sigma)$ of rank r by

$$\deg(E, (\Delta_1, \Delta_2), \sigma) := \deg E + \sum_{i=1}^{n} a_i + nr. \tag{4.2}$$

If $(E, (\Delta_1, \Delta_2), \sigma)$ is a triple of rank r and $d := \deg E$, we call it a triple *of type* $(r, d, a_1, \dots, a_n)$. Note that d is the degree of the vector bundle E, not the degree of the triple. We then define the *slope* of this triple by

$$\mu(E, (\Delta_1, \Delta_2), \sigma) := \frac{\deg(E, (\Delta_1, \Delta_2), \sigma)}{\operatorname{rk} E}.$$

A *subtriple* of $(E, (\Delta_1, \Delta_2), \sigma)$ is by definition a triple $(E', (\Delta_1', \Delta_2'), \sigma')$ on $\widetilde{X}$ such that $E' \subset E$ is a subbundle, $\Delta_i' \subset \Delta_i$ are vector subspaces respecting the direct sums for $i = 1$ and 2 such that $\sigma' = \sigma|\Delta_1'$. With these definitions the above notions translate as follows: A triple $(E, (\Delta_1, \Delta_2), \sigma)$ on $\widetilde{X}$ is called *(semi-) stable* if

$$\mu(E', (\Delta_1', \Delta_2'), \sigma') \overset{(\leq)}{<} \mu(E, (\Delta_1, \Delta_2), \sigma)$$

for all subtriples $(E', (\Delta_1', \Delta_2'), \sigma')$ of $(E, (\Delta_1, \Delta_2), \sigma)$.

It is convenient to define for any integer k, $1 \leq k \leq r$, the invariant s_k of a triple $(E, (\Delta_1, \Delta_2), \sigma)$ of type $(r, d, a_1, \dots, a_n)$ by

$$\begin{aligned} s_k(E, (\Delta_1, \Delta_2), \sigma) := k \deg(E, (\Delta_1, \Delta_2), \sigma) \\ -r \max \deg(E', (\Delta_1', \Delta_2'), \sigma') \end{aligned} \tag{4.3}$$

where the maximum is taken over all subtriples $(E', (\Delta_1', \Delta_2'), \sigma')$ of rank k of the given triple. It is clear that this maximum exists and that $(E, (\Delta_1, \Delta_2), \sigma)$ is stable, respectively semistable, if and only if $s_k(E, (\Delta_1, \Delta_2), \sigma) > 0$, respectively ≥ 0, for $1 \leq k \leq \operatorname{rk} E - 1$.

4.2 Relation to stability of E

In this subsection we want to compare the invariants $s_k(E, (\Delta_1, \Delta_2), \sigma)$ with the corresponding invariants of the vector bundle E. Recall (see e.g. [7]) that for any integer k, $1 \le k \le r-1$ the invariant s_k of the vector bundle E of rank r on $\widetilde{X}$ is defined by

$$s_k(E) = k \deg E - r \max \deg E' \tag{4.4}$$

where the maximum is taken over all subbundles E' of rank k of E. Certainly this maximum exists and E is stable, respectively semistable, if and only if $s_k(E) > 0$, respectively ≥ 0.

Proposition 4.1. *Let $(E, (\Delta_1, \Delta_2), \sigma)$ be a triple of type $(r, d, a_1, \ldots, a_n)$ on $\widetilde{X}$. Then we have for $1 \le k \le r$*

$$\begin{aligned} s_k(E, (\Delta_1, \Delta_2), \sigma) + r \sum_i a_i'' &\le s_k(E) + k \sum_i a_i \\ &\le s_k(E, (\Delta_1, \Delta_2), \sigma) + r \sum_{i=1}^n a_i' \end{aligned}$$

where for all $i = 1, \ldots, n$,

$$\begin{aligned} &\max(k + a_i - r, 0) \le a_i' \le \min(a_i, k) \quad \textit{and} \\ &\max(2k + a_i - 2r, 0) \le a_i'' \le \min(a_i, k). \end{aligned}$$

Proof. Suppose the subtriple $(E', (\Delta_1', \Delta_2'), \sigma')$ of type $(k, d', a_1', \ldots, a_n')$ takes the maximum in (4.3), i.e. $s_k(E, (\Delta_1, \Delta_2), \sigma) = k \deg(E, (\Delta_1, \Delta_2), \sigma) - r \deg(E', (\Delta_1', \Delta_2'), \sigma')$. According to Proposition 3.5, E' is a subbundle of E. This implies, applying (4.2),

$$\begin{aligned} s_k(E) &\le kd - rd' \\ &= k\left(\deg(E, (\Delta_1, \Delta_2), \sigma) - nr - \sum_i a_i\right) \\ &\quad -r\left(\deg(E', (\Delta_1', \Delta_2'), \sigma') - nk - \sum_i a_i'\right) \\ &= s_k(E, (\Delta_1, \Delta_2), \sigma) + r \sum_i a_i' - k \sum_i a_i \end{aligned}$$

Since $(E', (\Delta_1', \Delta_2'), \sigma')$ is a subtriple of maximal degree, we must have

$$\Delta_1^{'i} = \Delta_1^i \cap E'(p_i)$$

for all i. Considering $\Delta_1^{'i}$ as an intersection in $E(p_i)$, the dimension formula implies the left hand inequalities for the a_i'. The right hand inequalities are trivial.

Now suppose the subbundle $E'' \subset E$ of rank k takes the maximum in (4.4). Then $(E'', (\Delta_1'', \Delta_2''), \sigma'')$ with $\Delta_1'' = \oplus_i \Delta_1^{''i}$ and $\Delta_2'' = \oplus_i \Delta_2^{''i}$ such that

$$\Delta_1^{''i} = E''(p_i) \cap \Delta_1^i \cap \sigma^{-1}(E''(q_i)) \quad \text{and} \quad \Delta_2^{''i} = \sigma(\Delta_1^{''i})$$

and

$$\sigma'' = \sigma | \Delta_1''$$

is a subtriple of rank k, not necessarily of maximal degree. This implies, denoting $a_i'' = \dim \Delta_1''$ and applying (4.2) again,

$$\begin{aligned} s_k(E, (\Delta_1, \Delta_2), \sigma) &\leq k \deg(E, (\Delta_1, \Delta_2), \sigma) - r \deg(E'', (\Delta_1'', \Delta_2''), \sigma'') \\ &= k\Big(\deg E + nr + \sum_i a_i\Big) - r\Big(\deg E'' + nk + \sum_i a_i''\Big) \\ &= s_k(E) + k\sum_i a_i - r\sum_i a_i''. \end{aligned}$$

The right hand inequalities for a_i'' is trivial. The left hand inequalities follow from the definition of $\Delta_1^{''i}$ considering it as an intersection in the r-dimensional vector space $E(p_i)$. □

As a direct consequence we obtain the following corollary.

Corollary 4.2. *Let $(E, (\Delta_1, \Delta_2), \sigma)$ be a triple of type $(r, d, 0, \dots, 0)$ on $\widetilde{X}$. Then*

$$s_k(E, (\Delta_1, \Delta_2), \sigma) = s_k(E)$$

for all k. In particular $s_k(E, (\Delta_1, \Delta_2), \sigma)$ is (semi-) stable if and only if E is (semi-) stable.

Recall from Corollary 2.10 that $\mathcal{F}$ is a vector bundle if and only if the corresponding triple $(E, (\Delta_1, \Delta_2), \sigma)$ is of type $(r, d, r, \dots, r)$. In this case the inequality of Proposition 4.1 implies

$$s_k(E) \leq s_k(E, (\Delta_1, \Delta_2), \sigma).$$

This gives the following corollary.

Corollary 4.3. *If E is (semi-) stable, so is any triple $(E, (\Delta_1, \Delta_2), \sigma)$ of type $(r, d, r, \dots, r)$.*

The following proposition shows that the converse statement is not valid in general.

Proposition 4.4. *Let X be a rational curve with one node. So X is of arithmetic genus 1 with $\widetilde{X} \simeq \mathbb{P}^1$. If*

$$E = \oplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1},$$

then a general triple $(E, (\Delta_1, \Delta_2), \sigma)$ of type $(r, r-1, r)$ is stable.

Proof. We have to show that for a general $(E, (\Delta_1, \Delta_2), \sigma)$ of type $(r, r-1, r)$ on $\widetilde{X}$ and all subtriples $(E', (\Delta_1', \Delta_2'), \sigma')$ of type (k, d', a'), $1 \leq k < r$ the number

$$\begin{aligned} s_k((E, (\Delta_1, \Delta_2), \sigma), (E', (\Delta_1', \Delta_2'), \sigma')) &:= k \deg(E, (\Delta_1, \Delta_2), \sigma) \\ &\quad -r \deg(E', (\Delta_1', \Delta_2'), \sigma') \end{aligned}$$

is positive.

By (4.2) we have

$$\begin{aligned} & s_k\big((E, (\Delta_1, \Delta_2), \sigma), (E', (\Delta_1', \Delta_2'), \sigma')\big) \\ &\quad = k(r - 1 + 2r) - r(\deg E' + a' + k) \\ &\quad = r(k - \deg E') - k + r(k - a'). \end{aligned}$$

If $\deg E' < k$, then we have, since $a' \leq k$,

$$s_k((E, (\Delta_1, \Delta_2), \sigma), (E', (\Delta_1', \Delta_2'), \sigma')) > r - k > 0.$$

Hence we may assume that $\deg E' = k$, i.e. $E' \simeq \oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(1)$. Then we have

$$s_k((E, (\Delta_1, \Delta_2), \sigma), (E', (\Delta_1', \Delta_2'), \sigma')) > 0 \quad \Leftrightarrow \quad a' < k - \frac{k}{r}$$

Suppose this is not the case. Then $a' \geq k - \frac{k}{r} > k - 1$, which implies $a' = k$ and thus

$$\Delta_1' = E'(p_1) \subset \Delta_1, \ \ \Delta_2' = E'(q_1) \subset \Delta_2 \ \text{ and } \ \sigma(E'(p_1)) = E'(q_1). \qquad (4.5)$$

Hence it suffices to show that there exists $\sigma : E(p_1) \to E(q_1)$ such that for all subbundles E' of rank k of E,

$$\sigma(E'(p_1)) \neq E'(q_1). \tag{4.6}$$

Now the dimension of the space of subbundles of E of rank and degree k is less or equal to

$$\dim \operatorname{Hom}\left(\oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(1), \oplus_{j=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(1)\right) - \dim \operatorname{Aut}\left(\oplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^1}(1)\right) = k(r-1) - k^2.$$

Moreover, given a subbundle of rank and degree k, the space of isomorphisms $\sigma : E(p_1) \to E(q_1)$ such that $\sigma(E'(p_1)) = E'(q_1)$ is of dimension $r(r-k) + k^2$. But

$$k(r-1) - k^2 + r(r-k) + k^2 = r^2 - k < r^2.$$

Hence a general $\sigma : E(p_1) \to E(q_1)$ satisfies (4.6) which completes the proof. □

Proposition 4.5. *Let X be of geometric genus $g \geq 3$ with one node x_1 and E be a general stable vector bundle of rank r and degree d on $\widetilde{X}$. Then any triple $(E, (\Delta_1, \Delta_2), \sigma)$ on $\widetilde{X}$ is stable.*

Proof. Let (r, d, a_1) be the type of the triple $(E, (\Delta_1, \Delta_2), \sigma)$ and k an integer with $1 \leq k \leq r$. Since E is general, we have by [8]:

$$s_k(E) \geq k(r-k)(g-1).$$

Using this, we get from Proposition 4.1

$$\begin{aligned} s_k(E, (\Delta_1, \Delta_2), \sigma) &\geq s_k(E) + ka_1 - r\min(a_1, k) \\ &\geq k(r-k)(g-1) + ka_1 - r\min(a_1, k) \end{aligned}$$

If $a_1 \leq k$, this gives

$$s_k(E, (\Delta_1, \Delta_2), \sigma) \geq (r-k)(k(g-1) - a_1) > 0.$$

If $a_1 > k$, it gives

$$s_k(E, (\Delta_1, \Delta_2), \sigma) \geq k(r-k)(g-2) > 0$$

which completes the proof of the proposition. □

For X of geometric genus 2 with one node the same proof gives that for general stable E on $\widetilde{X}$ any triple $(E, (\Delta_1, \Delta_2), \sigma)$ is semistable. This is best possible as the following example shows.

Example 4.6. Let X be of geometric genus $g = 2$ with one node and let E be a general stable vector bundle of rank 2 and degree 1 on $\widetilde{X}$. So $s_1(E) = 1$. If E' is a line subbundle of maximal degree of E, we deduce from $1 = s_1(E) = \deg E - 2\deg E'$ that $\deg E' = 0$. Choose $\Delta_1 = \Delta_1' = E'(p_1)$ and $\Delta_2 = \Delta_2' = E'(q_1)$ and let $\sigma = \sigma' \colon E(p_1) \to E'(q_1)$. Then we have

$$\begin{aligned} s_1(E, (\Delta_1, \Delta_2), \sigma) &\leq \deg(E, (\Delta_1, \Delta_2), \sigma) - 2\deg(E', (\Delta_1', \Delta_2'), \sigma') \\ &= (\deg E + a_1 - 2) - 2(\deg E' + a_1' - 1) \\ &= (1 + 1 - 2) - 2(0 + 1 - 1) = 0. \end{aligned}$$

Remark 4.7. It is not difficult to work out analogous results of Propositions 4.4 and 4.5 in the case of n nodes.

5 Applications

5.1 Sheaves of type $(r, d, a_1, \ldots, a_n)$

Let the notations be as above. In particular X is an irreducible curve, singular only at exactly n nodes x_i with normalization $\widetilde{X}$ of genus g. So X is of arithmetic genus $p_a(X) = g + n$. We call a torsion free sheaf $\mathcal{F}$ on X *of type* $(r, d, a_1, \ldots, a_n)$ if it is of rank r, degree d and

$$\mathcal{F}_{x_i} \simeq \mathcal{O}_{x_i}^{a_i} \oplus m_{x_i}^{r-a_i} \tag{5.1}$$

for $i = 1, \ldots, n$. Since the degree of the corresponding triple

$$(E, (\Delta_1, \Delta_2), \sigma) := \Psi(\mathcal{F})$$

equals by definition the degree of $\mathcal{F}$, we conclude from (4.2),

Lemma 5.1. *A torsion free sheaf $\mathcal{F}$ is of type $(r, d, a_1, \ldots, a_n)$ if and only if the corresponding triple $(E, (\Delta_1, \Delta_2), \sigma)$ is of type*

$$\left(r, d - \sum_i a_i - nr, a_1, \ldots, a_n \right).$$

According to [10, Ch. 8, Prop. 9] the stable torsion free sheaves of rank r and degree d form an irreducible moduli space $\mathcal{M}_X(r,d)$. For any a_i, $0 \le a_i \le r$ for $i = 1, \dots, n$ consider the subset

$$\mathcal{M}_X(r,d,a_1,\dots,a_n) := \big\{\mathcal{F} \in \mathcal{M}(r,d) \mid \mathcal{F} \text{ of type } (r,d,a_1,\dots,a_n)\big\}.$$

Lemma 5.2. *$\mathcal{M}_X(r,d,a_1,\dots,a_n)$ is a locally closed subset of $\mathcal{M}_X(r,d)$.*

Proof. For the proof only note that the function $d_i \colon \mathcal{M}(r,d) \to \mathbb{Z}$, $\mathcal{F} \mapsto \dim \mathcal{F}(x_i)$ is upper semi-continuous. □

Proposition 5.3.

(a) *For any type $(r,d,a_1,\dots,a_n)$ we have*

$$\dim \mathcal{M}_X(r,d,a_1,\dots,a_n) \le r^2(g-1) + 1 + 2r\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^2. \quad (5.2)$$

(b) *If for a general $E \in \mathcal{M}_{\tilde{X}}(r, d - \sum_i a_i - nr)$ all triples $(E,(\Delta_1,\Delta_2),\sigma)$ of type $(r, d - \sum_i a_i - nr, a_1,\dots,a_n)$ are stable, then we have equality in* (5.2).

Proof.

(a) Suppose $\mathcal{F} \in \mathcal{M}_X(r,d,a_1,\dots,a_n)$ with corresponding triple $(E,(\Delta_1,\Delta_2),\sigma)$. According to Lemma 5.1 we have

$$\tilde{d} := \deg E = d - \sum_i a_i - nr.$$

Recall that an algebraic family $\mathcal{V}$ of vector bundles of rank r degree $\tilde{d}$ on $\widetilde{X} \times S$ is called effective, if for any point $s \in S$ there are at most finitely many points $s' \in S$ such that $\mathcal{V}|X \times \{s\} \simeq \mathcal{V}|X \times \{s'\}$. It is well known (see [9, Prop. 2.6]) that any effective family of such vector bundles is of dimension $\le \dim \mathcal{M}_{\tilde{X}}(r,d')$.

Recall moreover that an isomorphism of triples $\tilde{g} \colon (E,(\Delta_1,\Delta_2),\sigma) \to (E',(\Delta_1',\Delta_2'),\sigma')$ is by definition an isomorphism $g \colon E \to E'$ satisfying $g(p_i)(\Delta_1) = \Delta_1'$ and $g(q_i)(\Delta_2) = \Delta_2'$ for all i such that (3.1) commutes.

For a fixed E, the pairs (Δ_1,Δ_2) vary over $(\times_i Gr(a_i,r)) \times (\times_i Gr(a_i,r))$. For each fixed pair (Δ_1,Δ_2), the isomorphism σ varies over $Iso(\Delta_1,\Delta_2)$. Moreover, $\mathbb{P}(\operatorname{Aut} E)$ acts on the set of triples with the underlying bundle E by

$$g(\Delta_1,\Delta_2,\sigma) := ((g(p_i)(\Delta_1^i))_i, (g(q_i)(\Delta_2^i))_i, g(\sigma_i)_i),$$

where $g(\sigma_i)\colon g(p_i)(\Delta_1^i) \to g(q_i)(\Delta_2^i)$ is the isomorphism induced by σ_i. The quotient by this action is the variety of isomorphism classes of triples with the underlying bundle E. Hence the isomorphism classes of triples with the underlying bundle E are determined by

$$(\times_i Gr(a_i, r)) \times (\times_i Gr(a_i, r)) \times (\times_i k^{a_i^2})/\mathbb{P}(\mathrm{Aut}\, E).$$

This quotient has maximum dimension for stable vector bundles E.

Using Theorem 3.2, this implies

$$\begin{aligned}\dim \mathcal{M}_X(r, d, a_1, \dots, a_n) &\le \dim \mathcal{M}_{\tilde{X}}(r, \tilde{d}) + 2\sum_{i=1}^{n} \dim Gr(a_i, r) + \sum_{i=1}^{n} a_i^2\\ &= r^2(g-1) + 1 + 2\sum_{i=1}^{n} a_i(r - a_i) + \sum_{i=1}^{n} a_i^2\\ &= r^2(g-1) + 1 + 2r\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^2.\end{aligned}$$

(b) Assume now that for a general $E \in \mathcal{M}_X(r, \tilde{d})$ all triples $(E, (\Delta_1, \Delta_2), \sigma)$ of type $(r, \tilde{d}, a_1, \dots, a_n)$ are stable. Then there is a non-empty open set U of $\mathcal{M}_X(r, \tilde{d})$ with this property.

The commutativity of (3.1) is automatically fullfilled for any σ if E is stable, since an automorphism of a stable bundle is a constant. This implies that

$$\dim \mathcal{M}_X(r, d, a_1, \dots, a_n) \ge \dim \mathcal{M}_{\tilde{X}}(r, \tilde{d}) + 2\sum_{i=1}^{n} \dim Gr(a_i, r) + \sum_{i=1}^{n} a_i^2,$$

which by the same computation as above completes the proof of the proposition. □

Remark 5.4. Note that $r^2(g-1) + 1 + 2\sum_i a_i - \sum_i a_i^2 = r^2(g_X - 1) + 1 - \sum_i (r - a_i)^2$, where g_X denotes the genus of X. Thus Proposition 5.3 agrees with the results of [2, Proposition 2.7].

The following corollary is well known (see e.g. [2]).

Corollary 5.5. *The dimension of the moduli space of stable torsion free sheaves of rank r and degree d on a curve X of genus $g \ge 2$ with n nodes is*

$$\dim \mathcal{M}_X(r, d) = r^2(g + n - 1) + 1.$$

Proof. Since the functions $d_i \colon \mathcal{M}(r, d) \to \mathbb{Z}$, $\mathcal{F} \mapsto \dim \mathcal{F}(x_i)$ are upper semi-continuous for all i, there is an open dense set U in $\mathcal{M}_X(n, d)$ parametrizing vector bundles. Since the corresponding triples on $\widetilde{\mathrm{X}}$ are of type $(r, d, r, \dots, r)$, Corollary 4.3 and Proposition 5.2 (b) imply $\dim \mathcal{M}_X(r, d) = r^2(g-1)+r^2n+1$ which completes the proof. □

5.2 Upper bounds for s_k

Proposition 4.1 allows a cheap proof of an upper bound for the invariant $s_k(E, (\Delta_1, \Delta_2), \sigma)$ which however is not best possible (see [2]).

Proposition 5.6. *For any torsion free sheaf $\mathcal{F}$ of rank r and degree d on X and any k, $1 \le k \le r-1$ we have*

$$s_k(\mathcal{F}) \le kr(g+n-1) - k^2(g-1) + r - 1.$$

Proof. Let $(E, (\Delta_1, \Delta_2), \sigma)$ be the triple corresponding to $\mathcal{F}$. According to the left hand inequality of Proposition 4.1 we have

$$s_k(\mathcal{F}) = s_k(E, (\Delta_1, \Delta_2), \sigma) \le s_k(E) + k \sum_i a_i.$$

Now according to [6], $s_k(E) \le k(r-k)(g-1) + r - 1$. Moreover $a_i \le r$ for all i. This gives the assertion. □

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References

[1] U.N. Bhosle. *Generalized parabolic bundles and applications to torsionfree sheaves on nodal curves.* Ark. Mat., **30** (1992), 187–215.

[2] U.N. Bhosle. *Maximal subsheaves of torsion-free sheaves on nodal curves.* Journ. London Math. Soc., **74** (2006), 59–74.

[3] I. Burban and Y. Drozd. *Coherent sheaves on rational curves with simple double points and transversal intersections.* Duke Math. J., **121** (2004), 189–229.

[4] I. Burban, Y. Drozd and G.-M. Greuel. *Vector bundles on singular projective curves.* Kluwer Acad. Publ., NATO Sci. Ser. II, Math. Phys. Chem., **36** (2001), 1–15.

[5] A. Grothendieck. *Sur quelques points d'algèbre homologique.* Tohoku Math. J., **9** (1957), 119–221.

[6] A. Hirschowitz. *Problème de Brill-Noether en rang supérieur.* Preprint 91, Université de Nice (1986).

[7] H. Lange. *Zur Klassifikation von Regelmannigfaltigkeiten.* Math. Ann., **262** (1083), 447–459

[8] S. Mukai and F. Sakai. *Maximal subbundles of vector bundles on a curve*. Manuscr. Math., **52** (1985), 251–256.

[9] M.S. Narasimhan and S. Ramanan. *Deformations of the moduli space of vector bundles over an algebraic curve.* Ann. of Math., **101** (1975), 391–417.

[10] C.S. Seshadri. *Fibrés vectoriels sur les courbes algébriques.* Astérisque, **96** (1982).

[11] S. Sundaram. *Special divisors and vector bundles.* Tohoku Math. J., **39** (1987), 175–213.

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