On Henselizations of Valued Fields*

OTTO ENDLER[1]

This paper is intended to be a complement to Chapter IV of our book on Valuation Theory [3], concerning the case of infinite field extensions.

The main part of that chapter deals with the existence of finite separable field extensions with prescribed valuations. More precisely, Krull's results about the possibility of prescribing value groups and residue fields for finitely many non-archimedean valuations were derived from similar but stronger results about the prescribing of completions, where also archimedean valuations are admitted. On the other hand, a theorem on the existence of an infinite separable field extension with prescribed value groups and residue fields (cf [3], (28.1)) was proved directly since we found it difficult to obtain a similar result for completions.

Actually completions are not appropriate for the study of infinite extensions, because infinite separable extensions of complete valued fields are never complete. Therefore a suitable substitute for completions has to be found. For the study of extensions of Krull valuations, the notion of "henselization" is a good substitute for "completion" as was shown in [3], Chapter III. This fact suggests that completions should be replaced by henselizations also in the context of Chapter IV. However for this purpose, we first have to extend the notion of "henselization" to archimedean valuations.[2]

In the present paper we define henselizations for arbitrary real valuations by means of a universal property and show that many results which are well-known in the non-archimedean case hold in general. We give a survey of the set of all henselizations of a field K (Theorem 1) and show

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[2]Henselizations of archimedean valuations were already considered in Geyer's paper [5], with which I became acquainted after the present paper was completed.

that the normal closure of any henselization is separably closed (Theorem 2), as is the product of any two different henselizations (Proposition 5). Finally Theorem 3, which generalizes a theorem due to Neukirch [7], yields the desired result on the existence of an infinite separable field extension with prescribed henselizations, from which the above mentioned theorem [3], (28.1) can be obtained as a corollary. It shows also that the intersection of finitely many non-isomorphic henselizations is large in the sense that almost all of its henselizations are antihenselian.

We recall that a (real) valuation of K is a mapping φ from K into the set $\mathbb{R}_{\geq}$ of all non-negative real numbers such that $\varphi x = 0 \Leftrightarrow x = 0$, $\varphi(x \cdot y) = \varphi x \cdot \varphi y$, and $\varphi(x+y) \leq \varphi x + \varphi y$, for all $x, y \in K$. Non-archimedean valuations are those for which the stronger inequality $\varphi(x+y) \leq \max\{\varphi x, \varphi y\}$ holds; they are in 1-1 correspondence with the Krull valuations of rank 1. The usual absolute value $|\cdot|$ is archimedean. Valuations which induce the same topology of K are called equivalent; they are real powers of each other. The trivial valuation of K (which maps 0 to 0 and any $x \neq 0$ to 1) will always be excluded from consideration. Every field has infinitely many non-equivalent valuations, except absolutely algebraic fields of prime characteristic, which have none.

Let Ω be a separable closure of the field K. We recall that any valuation φ of K has at least one extension to Ω, and φ is called *henselian* if there is only one extension. In particular, all valuations of any separably closed field are henselian. A non-archimedean valuation is henselian if and only if it satisfies "Hensel's condition" (cf [3], §16). By a *henselization* of a valued field (K, φ) we understand any extension (K', φ') of (K, φ) such that φ' is henselian and the following universal property holds: Any imbedding λ of (K, φ) in any valued field (L, ψ) such that ψ is henselian extends uniquely to an imbedding λ' of (K', φ') in (L, ψ). It is obvious that any two henselizations of (K, φ) are K-isomorphic and that any extension (L, ψ) of (K, φ) such that ψ is henselian contains at most one henselization of (K, φ). Actually, "at most one" can be replaced by "exactly one", as soon as the existence of a henselization of (K, φ) is proved.

In the case of a non-archimedean valuation φ, it is well-known that $(K^Z_\omega, \omega|K^Z_\omega)$ is a henselization of (K, φ), where ω is any valuation of Ω extending φ, K^Z_ω is the decomposition field of ω over K, and $\omega|K^Z_\omega$ is the restriction of ω to K^Z_ω. (This holds even for Krull valuations, cf [3], §17). A henselization can also be obtained by means of a completion $(\hat{K}, \hat{\varphi})$ of (K, φ); in fact, the relative separable closure $\hat{K}_s$ of K in $\hat{K}$, endowed with the restriction $\hat{\varphi}_s = \hat{\varphi}|\hat{K}_s$, is the unique henselization of (K, φ) which is contained in $(\hat{K}, \hat{\varphi})$ (cf [3], (17.18)).

We are going to prove that the last statement holds also for archimedean valuations.[3] We first recall that any archimedean valuation φ of K is either *real-archimedean* or *complex-archimedean*, i.e., any completion $(\hat{K}, \hat{\varphi})$ of (K, φ) is isomorphic either to $(\mathbb{R}, |\cdot|^\alpha)$ or to $(\mathbb{C}, |\cdot|^\alpha)$ for some $\alpha > 0$. Assuming, without loss of generality, that Ω is contained in a separable closure of $\hat{K}$, we have $\hat{K}_s = \Omega$ whenever φ is complex-archimedean, whereas $\hat{K}_s$ is a real archimedean closure (i.e., a real closed subfield whose unique ordering is archimedean) whenever φ is real-archimedean.

PROPOSITION 1. *For any archimedean valuation φ of K, $(\hat{K}_s, \hat{\varphi}_s)$ is a henselization of (K, φ).*

PROOF. Let λ be an imbedding of (K, φ) in a valued field (L, ψ) such that ψ is henselian, and let $\hat{\lambda}$ be the unique imbedding of $(\hat{K}, \hat{\varphi})$ in some completion $(\hat{L}, \hat{\psi})$ of (L, ψ) which extends λ. We claim that $\hat{\lambda}\hat{K}_s \subseteq L$. In fact, if ψ is complex- (resp. real-) archimedean then every finite extension of L has degree 1 (resp. ≤ 2, cf [3], (2.13)), hence L is algebraically closed (resp. real-closed); therefore $L = L.\hat{\lambda}\hat{K}_s$ (resp. $L \subseteq L.\hat{\lambda}\hat{K}_s \subseteq L(\sqrt{-1})$). Actually, the equality $L = L.\hat{\lambda}\hat{K}_s$ holds also in the real-archimedean case, since otherwise we would have $\sqrt{-1} \in \hat{L}$, contradicting $\hat{L} \cong \mathbb{R}$. Therefore in both cases we have $\hat{\lambda}\hat{K}_s \subseteq L$, i.e., $\hat{\lambda}$ is actually an imbedding of $(\hat{K}_s, \hat{\varphi}_s)$ in (L, ψ). Moreover, any imbedding μ of $(\hat{K}_s, \hat{\varphi}_s)$ in (L, ψ) which extends λ is extended by an imbedding $\hat{\mu}$ of $(\hat{K}, \hat{\varphi})$ in $(\hat{L}, \hat{\psi})$, which necessarily coincides with the unique extension $\hat{\lambda}$ of λ to $\hat{K}$; therefore $\mu = \hat{\lambda}$. □

From the existence of henselizations we conclude easily that all valuations of Ω which extend a given valuation φ of K are K-conjugate. In fact, denoting by $\mathcal{G}$ the Galois group Aut $(\Omega|K)$ of Ω over K, we prove the following fact, which is well-known in the case of Krull valuations (cf [3], (14.3)):

[3] Note that if K admits an archimedean valuation then it has characteristic zero; therefore "separably closed" means "algebraically closed".

COROLLARY 1. *Let ω be a valuation of Ω which extends φ. Then $\{\omega \circ \sigma \mid \sigma \in \mathcal{G}\}$ is the set of all valuations of Ω which extend φ.*

PROOF. Obviously $\omega \circ \sigma$ is a valuation of Ω which extends φ. On the other hand, let ω_i be a valuation of Ω which extends φ and (K_i', φ_i') be the unique henselization of (K, φ) contained in (Ω, ω_i) $(i = 1, 2)$. Then there is a K-isomorphism μ of (K_1', φ_1') onto (K_2', φ_2'), which extends to a K-automorphism σ of Ω. Since ω_1 and $\omega_2 \circ \sigma$ are extensions of the same henselian valuation φ_1', they must coincide. □

The valuations ω, $\omega \circ \sigma$ of Ω need not be distinct. In fact, similarly as in [3], §15, it is shown that $\mathcal{G}_\omega^Z = \{\sigma \in \mathcal{G} \mid \omega \circ \sigma = \omega\}$ is a closed subgroup of the topological Galois group $\mathcal{G}$, which is called the *decomposition group* of ω over K. The valuations of Ω which extend φ are in a 1-1 correspondence with the right cosets of $\mathcal{G}_\omega^Z$ in $\mathcal{G}$, by $\omega \circ \sigma \leftrightarrow \mathcal{G}_\omega^Z \circ \sigma$. The fixed field K_ω^Z of $\mathcal{G}_\omega^Z$ is called the *decomposition field* of ω over K. Obviously, φ is henselian if and only if $\mathcal{G}_\omega^Z = \mathcal{G}$, if and only if $K_\omega^Z = K$, for some (and actually any) valuation ω of Ω which extends φ. Moreover we have $\mathcal{G}_{\omega \circ \sigma}^Z = \sigma^{-1} \circ \mathcal{G}_\omega^Z \circ \sigma$ and $K_{\omega \circ \sigma}^Z = \sigma^{-1} K_\omega^Z$, and, similarly as in [3], (15.7), we get the following characterization of K_ω^Z by a minimal property:

COROLLARY 2. *For any field L between K and Ω we have $K_\omega^Z \subseteq L$ if and only if $\omega \mid L$ is henselian.*

We obtain as an immediate consequence:

COROLLARY 3. *For any valuation ω of Ω which extends φ, $(K_\omega^Z, \omega \mid K_\omega^Z)$ is the unique henselization of (K, φ) contained in (Ω, ω).*

In particular, if φ is complex-archimedean then $K_\omega^Z = \Omega$ and $\mathcal{G}_\omega^Z$ consists only of the identical automorphism ι of Ω. If φ is real-archimedean then K_ω^Z is real-closed, $[\Omega\colon K_\omega^Z] = 2$, and $\mathcal{G}_\omega^Z = \{\iota, \sigma_\omega\}$, where σ_ω is the only non-identical K-automorphism of Ω such that $\omega \circ \sigma_\omega = \omega$.

Obviously, φ is henselian if and only if (K, φ) is a henselization of itself. Considering the opposite case, we say that φ is *antihenselian* if (Ω, ω) is a henselization of (K, φ) for some (and actually any) valuation ω of Ω which extends φ. It is clear that φ is antihenselian if and only if $\mathcal{G}_\omega^Z = \{\iota\}$, if and only if $K_\omega^Z = \Omega$, for some (and actually any) valuation ω of Ω which extends φ. In particular, any complex-archimedean, but no real-archimedean, valuation is antihenselian. Moreover, it is obvious that any valuation of a separably closed field is henselian and antihenselian. Conversely, if K has a valuation which is henselian and antihenselian then $K = \Omega$.

Antihenselian valuations can also be characterized by means of their finite separable extensions or by their completions; in fact:

PROPOSITION 2. *For any valuation φ of K the following conditions are equivalent:*

(i) *φ is antihenselian.*
(ii) *Any finite separable extension L of K has exactly $[L:K]$ valuations which extend φ.*
(iii) *Any finite separable proper extension of K has at least two valuations which extend φ.*
(iv) *$\hat{K}$ is separably closed (where $(\hat{K}, \hat{\varphi})$ is a completion of (K, φ)).*

A proof of the equivalences (ii) ⇔ (iii) ⇔ (iv) can be found in [3], (26.7). The equivalence (i) ⇔ (ii) is proved in [3], (17.15) for Krull valuations. It holds also for archimedean valuations, as follows easily from [3], (2.12). Note that in [3], §26 "antihenselian" was defined by condition (iii) of this proposition, whereas in [2], §5, in the case of arbitrary Krull valuations, it was defined by condition (ii).

For non-archimedean valuations φ, we are giving another characterization of "antihenselian", by means of the value group and the residue field. We say that φ is *saturated* if its value group is divisible and its residue field is algebraically closed. It is obvious that, in this case, any algebraic extension (L, ψ) of (K, φ) is immediate (i.e., ψ and φ have the same value group and the same residue field). Conversely, if any finite separable extension of (K, φ) is immediate then φ is saturated, as follows from Krull's existence theorem (cf [3], (27.1)). Moreover, we recall that for any finite extension L of K we have the fundamental inequality $\sum e_\psi \cdot f_\psi \leq [L:K]$ (summation over all valuations ψ of L which extend φ), where e_ψ (resp. f_ψ) is the ramification index (resp. residue degree) of ψ over φ. We say that φ is *defectless* if the equality sign holds for any separable finite extension L

of K. For example, φ is defectless whenever it is discrete or its residue field has characteristic zero (cf [3], (18.7) and (20.23)).

PROPOSITION 3. *For any non-archimedean valuation φ the following conditions are equivalent:*

(i) *φ is antihenselian.*
(ii) *φ is saturated and defectless.*

PROOF. (i) $\Rightarrow$ (ii) follows from the fundamental inequality. (ii) $\Rightarrow$ (i) follows from Proposition 2. $\square$

Note that there exist non-archimedean valuations which are saturated but not antihenselian. In fact, let $(\hat{\mathbb{Q}}_p, \hat{\varphi}_p)$ be the field of p-adic numbers; then the field L obtained by adjoining to $\hat{\mathbb{Q}}_p$ all roots of unity and all roots of p is not separably closed, and the unique extension of $\hat{\varphi}_p$ to L is saturated and henselian.

Obviously, discrete valuations are not saturated and therefore are not antihenselian. Therefore many fields, for example, all finite extensions of $\mathbb{Q}$ and of $\mathbb{F}_p(X)$, have no antihenselian valuations. On the other hand, there are fields which are not separably closed and have very many antihenselian valuations, as is shown in the following proposition.

PROPOSITION 4. *If K has a henselian valuation φ then any valuation of K which is non-equivalent to φ is antihenselian.*

This proposition is proved in [3], (26.5) by means of the existence theorem (25.6), and it yields the following well-known theorem, which is due to F. K. Schmidt.

COROLLARY 1. *Any field which is not separably closed has at most one henselian valuation (up to equivalence).*

Let $\mathfrak{H}$ be the set of all those subfields H of Ω which occur as henselizations of K, i.e., such that (H, ψ) is a henselization of (K, φ) for appropriate valuations φ, ψ.

PROPOSITION 5. *If $H_1, H_2 \in \mathfrak{H}$ and $H_1 \neq H_2$ then $H_1 \cdot H_2 = \Omega$.*

PROOF. By Corollary 3 of Proposition 1 there exist non-equivalent valuations ω_1, ω_2 of Ω such that $H_1 = K^Z_{\omega_1}$, $H_2 = K^Z_{\omega_2}$, and $\omega_1 | H_1, \omega_2 | H_2$ are henselian and non-equivalent valuations. So are the restrictions of ω_1 and ω_2 to $H_1 \cdot H_2$, and therefore $H_1 \cdot H_2 = \Omega$ by the preceding corollary. $\square$

We know already that Ω is in $\mathfrak{H}$ if and only if K admits an antihenselian valuation, and Ω is the henselization of K with respect to any valuation of that type. Moreover, it is clear that $\mathfrak{H}$ is closed under K-isomorphisms, so we can consider the classes $[H] = \{\sigma H \mid \sigma \in \mathscr{G}\}$ of K-conjugate henselizations. The following theorem gives a survey of the set of these classes.

THEOREM 1. *The classes $[H]$ such that $H \in \mathfrak{H}$, $H \neq \Omega$, are in 1-1 correspondence with the equivalence classes $[\varphi]$ of those valuations φ of K which are not antihenselian.*
Under this 1-1 correspondence, the classes $[R]$ defined by archimedean real closures R of K correspond to the equivalence classes of real-archimedean valuations of K.

PROOF. If φ is not antihenselian and (H, ψ) is a henselization of (K, φ) then $H \in \mathfrak{H}$, $H \neq \Omega$, and the class $[H]$ depends only on the equivalence class $[\varphi]$. Obviously, $[\varphi] \mapsto [H]$ is a mapping *onto* the set of those classes. It is injective, as follows immediately from Proposition 5.

If φ is real-archimedean and $[H]$ corresponds to $[\varphi]$ then H is an archimedean real closure. On the other hand, any archimedean real closure R of K can be imbedded in $\mathbb{R}$ and therefore has a real-archimedean valuation ψ. Since ψ has only one extension to $R(\sqrt{-1}) = \Omega$, by [3], (2.12), ψ is henselian. Let $(H, \psi | H)$ be the unique henselization of $(K, \psi | K)$ which is contained in (R, ψ). It follows from [3], (2.13) that H is real-closed, hence $R = H \in \mathfrak{H}$. Therefore R corresponds to the class of the real-archimedean valuation $\psi | K$ of K. $\square$

It is well known that the archimedean orderings of K are in 1-1 correspondence with the classes of K-isomorphic real closures of K. Using the second statement of Theorem 1, we conclude that they are also in 1-1 correspondence with the equivalence classes of real-archimedean valuations of K.

Obviously, K has a henselian valuation if and only if $K \in \mathfrak{H}$, and in this case we have $\mathfrak{H} = \{K, \Omega\}$, by Proposition 4. Moreover, we conclude from Theorem 1:

COROLLARY. *For any field K the following conditions are equivalent:*
(i) *K has an archimedean ordering and a henselian valuation.*
(ii) *K is real closed and has a real-archimedean valuation.*
In this case, these valuations are unique and coincide, up to equivalence.

PROOF. (i) $\Rightarrow$ (ii): K admits an archimedean real closure $R \subset \Omega$, and by Theorem 1, (R, ψ) is a henselization of (K, φ) for appropriate real-archimedean valuations φ, ψ. Since K has a henselian valuation, we have $R \in \mathfrak{H} = \{K, \Omega\}$, hence $(K, \varphi) = (R, \psi)$.

(ii) $\Rightarrow$ (i): Let φ be a real-archimedean valuation of K and (R, ψ) be a henselization of (K, φ). Since R is an archimedean real closure of K, by Theorem 1, we have $(R, \psi) = (K, \varphi)$. Hence (i) holds and φ is henselian. The uniqueness statement follows from Corollary 1 of Proposition 4. □

As to real-closed fields whose unique ordering is non-archimedean, we mention without proof that they have always henselian valuation rings (namely, the canonical valuation ring and those which contain it) but not necessarily of rank 1.

On the other hand, we conclude that a field which admits a henselian non-archimedean valuation cannot have an archimedean ordering. This can also be obtained as a consequence of the following result, which is due to Prestel:

PROPOSITION 6. *Let A be a henselian valuation ring of K, $A \neq K$, and let $\prec$ be an ordering of K. Then A is $\prec$-convex* (i.e., *if $0 \prec a \prec b$ and $b \in A$ then $a \in A$*) *and the ordering $\prec$ is non-archimedean.*

PROOF. To prove $\prec$-convexity, it suffices to show that $0 \prec a \prec b$ implies $\frac{a}{b} \in A$. Suppose $\frac{a}{b} \notin A$; then $\frac{b}{a} \in \mathfrak{M}$ (unique maximal ideal of A) and

therefore $X^2 + X + \frac{b}{a} \equiv X \cdot (X+1) \mod \mathfrak{M}$. By Hensel's condition (cf [3], (16.6)) there exist $c, d \in A$ such that $X^2 + X + \frac{b}{a} = (X+c) \cdot (X+d)$. We conclude that $1 = c + d$, $1 \prec \frac{b}{a} = c \cdot d$, hence $0 \prec c \prec 1$ and $0 \prec d \prec 1$, hence $c \cdot d \prec 1$, a contradiction. Assume that the ordering $\prec$ is archimedean and let $a \in \mathfrak{M}$, $0 \prec a$. Then $1 \prec n\,a$ for some $n \in N$, hence $0 \prec (n\,a)^{-1} \prec 1$ and therefore $(n\,a)^{-1} \in A$; this is impossible since $n\,a \in \mathfrak{M}$. □

Note that, for an arbitrary valuation ring A of K, any ordering $\prec$ of K such that A is $\prec$-convex induces an ordering of the residue field $A/\mathfrak{M}$, and any ordering of $A/\mathfrak{M}$ is obtained in this way (cf [8] for a detailed discussion of the relationship between the orderings of K and those of $A/\mathfrak{M}$). In particular, we get as a consequence of Proposition 6:

COROLLARY. *Let A be a henselian valuation ring of K. Then $A/\mathfrak{M}$ is formally real if and only if K is formally real.*

Let (K, φ) be a valued field such that φ is *not* henselian. We claim that its henselization is an infinite extension (unless K is real-closed) whose normal closure is equal to Ω. In fact, this is a consequence of the following theorem[4]:

THEOREM 2. *Let (L, ψ) be an extension of (K, φ) such that $L \subseteq \Omega$ and assume that φ is non-henselian and ψ is henselian. Then:*
a) *The normal closure of $L|K$ is equal to Ω.*
b) *If $[L:K] < \infty$ then K is real closed, $K \neq L = K(\sqrt{-1}) = \Omega$, and φ is antihenselian.*

PROOF. a) Let N be a normal extension of K such that $L \subseteq N \subseteq \Omega$ and χ be the unique valuation of N which extends ψ. Since φ is non-henselian, there is a valuation χ' of N, different from χ, which also extends φ. Since N is a normal extension of K, it follows from Corollary 1 of Proposition 1 that $\chi' = \chi \circ \sigma$ for some K-automorphism σ of N, and since χ is henselian, so is χ'. We claim that $N = \Omega$. In fact, otherwise N is not separably closed and therefore, by Corollary 1 of Proposition 4, the valuations χ, χ' are equivalent. Since both are extensions of φ, they are even equal, contradicting the choice of χ'.

[4]For non-archimedean valuations this theorem was proved in [1].

b) If L is a finite extension of K then so is the normal closure of $L|K$; therefore $[\Omega:K] < \infty$, by a). Using a slight generalization of Artin-Schreier's theorem (cf [1], Lemma), we conclude that K is real-closed and $\Omega = K(\sqrt{-1})$. Since $K \subset L \subseteq \Omega$ and $[\Omega:K] = 2$, we have $L = \Omega$. Since φ is not henselian and Ω is the only finite extension of K, φ satisfies condition (ii) of Proposition 2 and is therefore antihenselian. □

COROLLARY. *Assume that K has no henselian valuation and is not real--closed. Then any $H \in \mathfrak{H}$ is an infinite extension of K and Ω is the normal closure of $H|K$.*

The preceding Corollary as well as Proposition 5 show that all henselizations $H \in \mathfrak{H}$ are, roughly speaking, "large" subextensions of $\Omega|K$ (unless $\mathfrak{H} \subseteq \{K, \Omega\}$). Our next aim is to show that even each intersection of finitely many fields $H_1, \ldots, H_k \in \mathfrak{H}$, belonging to non-equivalent valuations $\varphi_1, \ldots, \varphi_k$ of K, is large in the sense that almost all of its valuations (up to equivalence) are antihenselian. This statement will be obtained from the following much more general result, the first part of which is essentially due to Neukirch [7].

THEOREM 3. *Let $\omega_1, \ldots, \omega_k$ be valuations of Ω and $\bar{L}_1, \ldots, \bar{L}_k$ be subfields of Ω such that $\omega_1|\bar{L}_1, \ldots, \omega_k|\bar{L}_k$ are henselian and $\omega_1|L, \ldots, \omega_k|L$ are pairwise nonequivalent, where $L = \bar{L}_1 \cap \ldots \cap \bar{L}_k$. Then*

a) *$(\bar{L}_j, \omega_j)$ is a henselization of (L, ω_j), for $j = 1, \ldots, k$.*[5]
b) *Any valuation ψ of L which is non-equivalent to $\omega_1|L, \ldots, \omega_k|L$ is antihenselian.*

PROOF. a) Let (H_j, ω_j) be the unique henselization of (L, ω_j) contained in $(\bar{L}_j, \omega_j)$ $(j = 1, \ldots, k)$. It suffices to show that $\bar{L}_1 \subseteq H_1$. Let $\alpha_1 \in \bar{L}_1$ and let $P_1 \in H_1[X]$ be the minimal polynomial of α_1 over H_1, of degree n (say). For $j = 2, \ldots, k$ let P_j be a product of n distinct linear factors $X - \gamma$ in $H_j[X]$. Since $\omega_1|L, \ldots, \omega_k|L$ are pairwise non-equivalent and L is dense in (H_j, ω_j), for $j = 1, \ldots, k$, we may approximate $P_1, \ldots, P_k$ simultaneously by monic polynomials $F \in L[X]$ of degree n. By the continuity

[5]We write (L, ω) instead of $(L, \omega|L)$ when ω is a valuation of Ω and L is a subfield of Ω.

of polynomial roots (cf [3], (24.4)), F may be chosen such that, for $j = 1, \ldots, k$, a 1-1 correspondence between the roots (in Ω) γ of F and the roots α of P_j is given by $\omega_j(\gamma - \alpha) < \varepsilon$, where ε is sufficiently small. Let γ_0 be that root of F for which $\omega_1(\gamma_0 - \alpha_1) < \varepsilon$; then $H_1(\alpha_1) = H_1(\gamma_0)$ by Krasner's lemma (cf [3], (24.1)). We claim that $\gamma_0 \in H_j$ for $j = 2, \ldots, k$. In fact, let $\alpha_j \in H_j$ be that root of P_j for which $\omega_j(\gamma_0 - \alpha_j) < \varepsilon$; then for any H_j-automorphism σ of Ω we have $\omega_j(\sigma\gamma_0 - \alpha_j) = (\omega_j \circ \sigma)(\gamma_0 - \alpha_j) < \varepsilon$, since $\omega_j \circ \sigma = \omega_j$ and $\alpha_j = \sigma\alpha_j$, and therefore $\sigma\gamma_0 = \gamma_0$. We conclude that $\gamma_0 \in H_1(\alpha_1) \cap H_2 \cap \ldots \cap H_k \subseteq \bar{L}_1 \cap \ldots \cap \bar{L}_k = L$ and therefore $\alpha_1 \in H_1(\alpha_1) = H_1(\gamma_0) = H_1$.

b) Let ω_{k+1} be a valuation of Ω which extends ψ and let $\bar{L}_{k+1} = \Omega$; then the assumptions of this theorem hold for $\omega_1, \ldots, \omega_{k+1}$, $\bar{L}_1, \ldots, \bar{L}_{k+1}$ and $L = \bar{L}_1 \cap \ldots \cap \bar{L}_k = \bar{L}_1 \cap \ldots \cap \bar{L}_{k+1}$. By a), (Ω, ω_{k+1}) is a henselization of (L, ψ). □

The assumptions of Theorem 3 are clearly satisfied whenever the fields $\bar{L}_1, \ldots, \bar{L}_k$ are extensions of henselizations $\bar{K}_1, \ldots, \bar{K}_k$ of K with respect to non-equivalent valuations $\varphi_1, \ldots, \varphi_k$. Therefore, Theorem 3 yields the construction of a field extension L of K with prescribed henselizations, in the following sense:

COROLLARY 1. *Let $\varphi_1, \ldots, \varphi_k$ be pairwise non-equivalent valuations of K. For $j = 1, \ldots, k$ let $(\bar{K}_j, \bar{\varphi}_j)$ be a henselization of (K, φ_j) and $\bar{L}_j$ be a field between $\bar{K}_j$ and Ω. Then the field $L = \bar{L}_1 \cap \ldots \cap \bar{L}_k$ has the following properties:*

a) *For each $j \in \{1, \ldots, k\}$ there is a valuation ψ_j of L which extends φ_j and such that $(\bar{L}_j, \bar{\psi}_j)$ is a henselization of (L, ψ_j) (where $\bar{\psi}_j$ is the unique extension of $\bar{\varphi}_j$ to $\bar{L}_j$).*

b) *Any valuation ψ of L which is non-equivalent to $\psi_1, \ldots, \psi_k$ is antihenselian* (i.e., *has henselization* (Ω, ω), *where ω extends ψ*).

From this corollary, together with [3], Exercise IV-18, one can obtain an analogous result about the prescription of value groups and residue fields in the case of non-archimedean valuations, which has been proved in a direct way in [3], (28.1). Therefore, this corollary fits in the context of [3], Chapter IV, which for this purpose has to be modified by substituting henselizations for completions. Such a substitution is convenient whenever infinite extensions of valued fields are involved, since infinite algebraic extensions of henselian fields are henselian whereas infinite separable extensions of complete fields are never complete.

By setting $\bar{L}_1 = \bar{K}_1, \ldots, \bar{L}_k = \bar{K}_k$ in Corollary 1, we get the following result which was already announced above:

COROLLARY 2. *Let $(\bar{K}_j, \bar{\varphi}_j)$ be a henselization of (K, φ_j), where $\varphi_1, \ldots, \varphi_k$ are pairwise non-equivalent valuations of K. Then any valuation of $\bar{K}_1 \cap \ldots \cap \bar{K}_k$ which is non-equivalent to the restrictions of $\bar{\varphi}_1, \ldots, \bar{\varphi}_k$ is antihenselian.*

A weaker form of Corollary 1 was already stated in [4]. Under some additional hypotheses, Corollary 2 was proved in [6] and [4].

References

[1] ENDLER, O., *A note on henselian valuation rings*, Canad. Math. Bull. 11, (1968), 185-189.

[2] ENDLER, O., *Über henselsche Erweiterungen bewerteter Körper und die Fortsetzung von Bewertungen*, J. reine angew. Math. 239/240, (1970), 306-320.

[3] ENDLER, O., Valuation theory, Universitext, Springer-Verlag Berlin-Heidelberg-New York, 1972.

[4] ENDLER, O., *Sobre valorizações henselianas*, Atas do 9.º Colóquio Brasileiro de Matemática (to appear).

[5] GEYER, W. D., *Unendliche algebraische Zahlkörper, über denen jede Gleichung auflösbar von beschränkter Stufe ist*, Journal of Number Theory 1, (1969), 346-374.

[6] KRULL, W., *Eine Bemerkung zur Bewertungstheorie*, Anais Acad. Brasil. Ci. 35, (1963), 457-481.

[7] NEUKIRCH, J., *Über gewisse ausgezeichnete unendliche algebraische Zahlkörper*, Bonner Math. Schriften 25, Bonn (1965).

[8] PRESTEL, A., *On Formally real fields*, To appear in Monografias de Matemática, IMPA, Rio de Janeiro.

Instituto de Matemática Pura e Aplicada
Rio de Janeiro — BRASIL

Mathematisches Institut der Universität
Bonn 53 Bonn — Germany