Going down for Monoidal Transforms*

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1. Introduction. In this note I improve results of Dobbs [2]. The techniques are throughout of elementary nature and make no appeal to any heavy theorems.

I will stick to the following terminology: a *minimal prime over-ideal* of an ideal I in a commutative ring A is a prime ideal containing I and not containing properly any prime ideal that contains I. If A is noetherian or, more generally, if I admits a finite reduced primary representation, then the minimal prime over-ideals of I are exactly the isolated prime ideals of such a representation.

For the purpose of this work – and for lack of better terminology – I call *WBH ring* (short for "well behaved for hypersurfaces") a commutative ring such that every principal ideal has only finitely many minimal prime over-ideals and these are of height ≤ 1. Noetherian rings constitute the most important class of *WBH* rings. Another noteworthy class of *WBH* rings is formed by Krull domains (more generally, the so-called domains of finite real character).

Throughout $(a_1, \ldots, a_n)$ stands for the ideal of A generated by the elements $a_1, \ldots, a_n$. The Krull dimension of A is denoted *dim* A. Needless to insist, all rings are commutative with identity.

I wish to heartily thank David Dobbs, for a pleasant and enlightening correspondence on the subject.(1)

2. Monoidal transforms along a centre of codimension 2. Let me begin with the following existence result.

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(1) Results similar to the above have been obtained by Dobbs (Comm. in Algebra, 1 (1974) 439-458). Since the present work deals with other questions as well, I've decided to publish it in its entirety.

LEMMA 1.1. *Let A be a ring such that every principal ideal has only finitely many minimal prime over-ideals. If $\dim A \geq 2$ then A has a proper ideal (a, b) whose minimal prime over-ideals have height ≥ 2. Moreover, for any prime over-ideal P of (a, b) such that A_P is a WBH ring, the ideal $(a, b)A_P$ is not principal.*

PROOF. Start with any chain $P_0 \subsetneq P_1 \subsetneq P_2 \subset \ldots$ of prime ideals in A (P_0 is not necessarily assumed to be minimal). By assumption, (0) has only finitely many minimal prime over-ideals. Therefore, one can choose an element $a \in P_1$ belonging neither to P_0 nor to any minimal prime ideal of A. Then $\dim A/(a) \geq 1$.

I further claim that, for any choice of a as above, there exists $b \in A$ such that (a, b) fulfils the requirements of the lemma. For otherwise, any non-unit $\bar{b}$ of $\bar{A} = A/(a)$ is such that the ideal (a, b) of A has a minimal prime over-ideal of height ≤ 1; such an ideal is a fortiori a minimal prime over-ideal of (a) because of the way a has been chosen. Again, by hypothesis, $\bar{A}$ has only finitely many minimal prime ideals. It follows that every maximal ideal of $\bar{A}$ is contained is one single minimal prime ideal of A, thus implying that $\dim \bar{A} = 0$.

The last part of the lemma is immediate since for a minimal prime over-ideal P of (a, b), PA_P is (the unique) minimal prime over-ideal of $(a, b)A_P$.

REMARK. The first part of Lemma 1.1 admits the following easy extension: let A be a ring such that *Spec* A is a noetherian space. If $\dim A \geq n$ then A has proper ideal $(a_1, \ldots, a_n)$ whose minimal prime over-ideals have height $\geq n$. The proof is by induction on n and one can in fact start with any prime ideal of height $\geq n$. In case A itself is noetherian, the result is a well-known consequence of the "Primidealsatz" of Krull and carries several extra bonuses.

THEOREM 1.2. *Let A be a WBH domain with quotient field K. Then the following conditions are equivalent:*
(i) *For every $u \in K$ the natural injection $A \subsetneq A[u]$ satisfies going down.*
(ii) $\dim A \leq 1$.

PROOF. The implication (ii) $\Rightarrow$ (i) is easy. In fact, the following holds in general: if $A \subsetneq B$ is an injection of domains with $\dim A \leq 1$, then going-down is satisfied (Thus, at first sight the hypothesis of *WBH* is not needed to establish the implication (ii) $\Rightarrow$ (i). At second thought, however, this is not gaining much as there is only one single instance of a one-dimensional domain that is not *WBH*, namely, a non-noetherian one-dimensional G-domain possessing infinitely many maximal ideals: a rather weird tresspasser).

The implication (i) $\Rightarrow$ (ii) is a consequence of Lemma 1.1 and Chevalley's lemma. Namely, take an ideal (a, b) as in Lemma 1.1 and set $u = a/b \in K$. Then Chevalley's result [3, Theorem 55] warrants that $(a, b)A[u] \neq A[u]$ (say). Now, let P' be any minimal prime over-ideal of $(a, b)A[u]$. As $P' \cap A$ contains (a, b) it must have height ≥ 2. By assumption A is *WBH*, hence one can find a prime over-ideal P of (b) properly contained in $P' \cap A$. If $Q' \subset P'$ is a prime of $A[u]$ lying over P then Q' contains $(a, b)A[u] = bA[u]$, consequently $Q' = P'$ by minimality of P'. Thus, no Q' lies over P, showing that going-down fails for $A \subsetneq A[u]$.

Note that the above theorem truly generalizes the following results of [2]: Proposition 7, Corollary 9, part (a) of Theorem 3 and, to some extent, part (b) of the latter. I do however not know the exact relation (if any) between *WBH* domains and the *FC* domains mentioned by Dobbs.

In the noetherian case one can discard the hypothesis that the ring A have no proper zero-divisors, according to the following result.

PROPOSITION 1.3. *Let A be a noetherian ring with total quotient ring T. Consider the following conditions.*
(i) *For every $u \in T$, the injection $A \subsetneq A[u]$ satisfies going-down.*
(ii) *For every unit $u \in T$, $A \subsetneq A[u]$ satisfies going-down.*
(iii) $\dim A \leq 1$ *or* $A = T$.
Then (i) *implies* (iii) (*and, trivially,* (i) *implies* (ii)). *If A is besides normal then the three conditions are equivalent.*

PROOF. Firstly, in all generality, (i) implies (iii). Indeed, if $\dim A \geq 2$, choose (a, b) as in Lemma 1.1 in such a way that a is a non-zero-divisor of A (this is possible unless A coincides with its total quotient ring). Then $b/a \in T$ and one can define an A-homomorphism $A[X] \longrightarrow A[b/a]$ by assigning $X \longmapsto b/a$. Let N be the kernel. Clearly, $aX - b \in N$. As A is noetherian, (a,b) has height 2 exactly. Therefore, by [1, Proposition 1], the ideals N and $(AX-b)$ have the same radical. Given a prime over-ideal P of (a,b), one then has $N \subset PA[X]$, thus implying that $PA[b/a]$ is a prime ideal of $A[b/a]$. It follows that $(a,b)A[b/a]$ is a proper ideal and the rest of the proof proceeds exactly as in the proof of Theorem 1.2.

Now assume A is normal (i.e., locally normal) and suppose (iii) holds. At any rate, A is the product of its irreducible components, say, $A \simeq A/P_1 \times \ldots \times A/P_n$, where $P_1, \ldots, P_n$ are the minimal prime ideals of A. Accordingly, one has that $T \simeq A_{P1} \times \ldots \times A_{Pn}$ and A_{Pi} is the quotient field of A/P_i. Let $u = a/b \in T$. Then

$$A[u] \simeq (A/P_1 \times \ldots \times A/P_n)[(u, \ldots, u)] = \\ = A/P_i[u] \times \ldots \times A/P_n[u] \subset A_{P1} \times \ldots \times A_{Pn}$$

(as a slight check on this equality, note that $u \in A_{Pi}$ for every i since $b \notin \bigcup_{i=1}^{n} P_i$). Now, by the implication (ii) $\Rightarrow$ (i) of Theorem 1.2, one knows that going-down holds in $A/P_i \subsetneq A/P_i[u]$ for every i. This implies going-down in $A \subsetneq A[u]$. Precisely, given $P_i \subsetneqq M \subset A$ (M maximal) and given $M' \subset A[u]$ lying over M, one has that M' is identified with

$$A/P_i[u] \times \ldots \times M'_j \times \ldots \times A/P_n[u]$$

where M'_j is a prime ideal of $A/P_j[u]$ (only one such j), hence M is identified with $A/P_i \times \ldots \times M'_j \cap (A/P_j) \times \ldots \times A/P_n$. One must have $P_i = P_j$ because $P_i \subset M$ (note that P_j is identified with

$$A/P_i \times \ldots \times \underset{j}{(0)} \times \ldots \times A/P_n).$$

Then the prime ideal of $A[u]$ identified with

$$A/P_i[u] \times \ldots \times \underset{j}{(0)} \times \ldots \times A/P_n[u]$$

lies over $P_i = P_j$.

Finally, (ii) implies (iii) under the conditions that A is normal. For assume $\dim A \geq 2$. Since the principal ideals of A have now no embedded prime ideals, one can choose (a, b) in Lemma 1.1 in such a way that (a, b) is an A-sequence. In particular, a and b are non-zero-divisors in A, hence $b/a \in T$ is a unit. Now one proceeds exactly as in the implication (i) $\Rightarrow$ (iii).

REMARK. Proposition 1.3 is not very satisfactory. In particular, (iii) implies (i) provided A is a product of its irreducible components. Of course, this assumption automatically implies that A has no embedded prime ideals. Also, if A is local non Cohen-Macaulay, then (trivially) (iii) $\Rightarrow$ (i). In general, one can reduce the question to the same one with A having no embedded primes. As to implication *(ii)* $\Rightarrow$ *(iii)*, one needs less than normality for all that is required is that a and b be non-zero-divisors in A, whereas $\{a, b\}$ being A-sequence is stronger.

I close this section with an amusing result. It is along the line of Chevalley's lemma and tells grosso modo that, in some cases, most prime ideals survive in both monoidal transforms $A[a/b]$ and $A[b/a]$, provided the center of the transform is sufficiently "generic".

PROPOSITION 1.4. *Let A be an integrally closed WBH domain of dimension ≥ 2. Then there exists an element a/b in the quotient field of A such that, for every prime over-ideal P of the ideal (a, b), the ideals $PA[a/b]$ and $PA[b/a]$ are both prime.*

PROOF. Choose (a, b) as in lemma 1.1 and let $P \subset A$ be any prime over-ideal of (a, b). Let $N \subset A[X]$ be the kernel of the A-homomorphism $A[X] \longrightarrow A[a/b]$ defined by $X \longmapsto a/b$. One has an induced A-homomorphism $A_P[X] \longrightarrow A_P[a/b]$ whose kernel is N_P. If $N_P \subset PA_P[X]$ then clearly $N \subset PA[X]$, hence $PA[a/b]$ is a prime ideal of $A[a/b]$. Therefore, one may assume that A is quasi-local with maximal ideal P (note that the property of WBH is preserved under localization). In this case, let $f(X) \in N$ have a coefficient outside P. Then [3, Theorem 6] one must have $a \in (b)$ or $b \in (a)$, thus implying that (say) a has a minimal prime over-ideal of height ≥ 2. This contradicts the assumption that A is WBH. Since b/a has a symmetrical role, the proof is finished.

REMARK. Note that $\{a, b\}$ as given in the above proposition is an analytically independent set in the sense of Davis [1].

3. Going-down and grade. In a commutative ring A one can define grade(A) as the supremum of the grades of all proper ideals of A. One of the remarkable properties of (noetherian) normal and Cohen-Macauly rings is the so called (S_2) condition. Thus, for such rings one can in Lemma 1.1

choose (a, b) in such a way that $\{a, b\}$ form an A-sequence. In other words, for a noetherian normal (resp. Cohen-Macaulay) ring A of dimension ≥ 2 one has $grade(A) \geq 2$.

Not so for general rings: one needs only think of a non-noetherian valuation ring A, where $grade(A) \leq 1$ while $\dim A$ is arbitrary. This is why, in general, one looks at the *WBH* property in order for going-down to fail. However, the *WBH* property is not fine enough. To see this, recall that for a domain A, one says that *Spec A* is a tree if no two non-comparable (by inclusion) prime ideals of A admit a bigger prime ideal containing both of them. In connection with a question raised by Dobbs, there is the following result:

PROPOSITION 3.1. *Let A be a WBH domain of dimension ≥ 2. Then Spec A is not a tree.*

PROOF. Let $(0) \subsetneq P_0 \subsetneq P_1$ be any chain of prime ideals in A. Choose $a \in P_1 \setminus P_0$. Since A is *WBH*, there exists a prime ideal $P_2 \subsetneq P_1$ such that $a \in P_2$. Supposing *Spec A* is a tree one must have $P_0 \subsetneq P_2$. But it can be assume that there are no prime ideals properly between P_0 and P_1 [3, Theorem 11]. Therefore, one gets a contradiction.

Note that the above proof actually gives that, for a domain A satisfying condition (S_2) (i.e., a domain whose principal ideals have no embedded primes), *Spec A* is never a tree. Dobb's question is to the effect whether for a treed domain A of grade 1 one always has going-down. The above proposition shows that our work is not sharp enough to answer this question.

Let A be a domain with quotient field K. Call an element $u \in K$ *grade-regular* if u admits a representative whose terms a, b form an A-sequence. Note I do not exclude the possibility of a, b generating the unit ideal; in particular every non-zero element of A, and the inverse in K of every non-zero element of A, are grade-regular. In terms of grade-regular elements, Theorem 1.2 can be reestated as follows.

PROPOSITION 3.2. *Let A be a domain with quotient field K. If A is WBH then the following conditions are equivalent:*

(i) *For every grade-regular $u \in K$, $A \subsetneq A[u]$ satisfies going-down.*
(ii) *$grade(A) \leq 1$.*

The proof is certainly a lot easier than that of Theorem 1.2; in particular, Lemma 1.1 is not needed. Thus, the implication (ii) $\Rightarrow$ (i) holds for any domain of *grade* ≤ 1 since any grade-regular $u \in K$ then has the form a/b, with $(a, b) = A$ and this implies that $1/b \in A[a/b]$, i.e., $A[a/b] = A[1/b]$. Also, (i) implies (ii) for any domain such that the minimal prime over-ideals of a principal ideal have all height ≤ 1.

Observe that a grade-regular element $u \in K$ admits a unique (up to unit factors) representative whose terms form an A-sequence. In fact, one has the following result which seems to be scattered in the literature.

PROPOSITION 3.3. *Let A be an integral domain satisfying the ascending chain condition for principal ideals, let K be the quotient field of A. The following are equivalent conditions:*
(i) *Every non-zero element of K is grade-regular.*
(ii) *A is a UFD.*
(iii) *Every non-zero element $u \in K$ can be written $u = a/b$, where a, b are such that the A-homomorphism $A[X] \longrightarrow A[u]$ defined by $X \longmapsto u$ has kernel $(bX - a)$.*

Moreover, if $\dim A \geq 2$ (any of) these conditions imply the following:
(iv) *For every $u \in K$ there exist $a, b \in A$ such that $u = a/b$ and such that for every minimal prime over-ideal Q of $(a, b)\, A[a/b]$, $Q \cap A$ is a minimal prime over-ideal of (a, b) of height ≥ 2.*

PROOF. (i) $\Rightarrow$ (ii) It suffices to show that every irreducible element generates a prime ideal. Let $a \in A$ be a non-zero irreducible and suppose $bd \in (a)$, say, $bd = ac$. Let $a/b = \alpha/\beta$, with $\{\alpha, \beta\}$ an A-sequence. Then $a = a'\alpha$ and $d = d'\alpha$ for some $a', d' \in A$. Since a is irreducible, either a' or α is a unit. If a' is a unit, one gets $\alpha = a'^{-1}a$, so $d = d'a'^{-1}a \in (a)$. If α is a unit, one obtains $b = a\beta\alpha^{-1}$ by construction, so $b \in (a)$. Therefore, (a) is prime.

(ii) $\Rightarrow$ (iii). Let $u \in K$, $u \neq 0$. Write $u = a/b$, with a, b relatively prime. Then $\{a, b\}$ is clearly an A-sequence, hence the kernel of $A[X] \longrightarrow A[u]$, $X \longmapsto u$, is $(bX - a)$ as is well known.

(iii) $\Rightarrow$ (i). Let $0 \neq u \in K$. By assumption,

$$A[X]/(bX-a) \simeq A[u]$$

(via $X \longmapsto u$) for some $a, b \in A$ such that $u = a/b$. I claim that $\{a, b\}$ is a regular sequence. Assume $ac = bd$, $c, d \in A$. Since $c(a/b) = d$, then $cX - d$ vanishes at a/b, hence $cX - d = (bX - a) f(X)$ some $f(X) \in A[X]$. Clearly, one must have $f(X) = e \in A$. Therefore $c = be$, as required.

Now suppose $dim\, A \geq 2$ and A is a *UFD*. Let $u = a/b$, with a, b relatively prime (*i.e.*, $\{a, b\}$ an A-sequence) and let $Q \subset A[u]$ be a minimal prime over-ideal of $(a, b)\, A[a/b]$. Since $grade(a, b) \geq 2$ then $Q \cap A$ has height ≥ 2. On the other hand, let $(a, b) \subset P \subset Q \cap A$, P prime ideal. Then $(bX - a) \subset PA[X]$, so $PA[a/b]$ is a prime ideal of $A[a/b]$ containing $(a, b)A[a/b]$. By minimality of Q, $PA[a/b] = Q$. Therefore $PA[a/b] \cap A = Q \cap A$. Since $PA[X] \cap A = P$, also $PA[a/b] \cap A = P$. Thus, $P = Q \cap A$, thereby showing that $Q \cap A$ is a minimal prime of (a, b).

QUESTION. Does condition (iv) of Proposition 3.3 characterize a *UFD* among Krull domains of dimension ≥ 2? An answer to this question, even in the classical literature of birational morphisms with target a normal variety, is not clear.

COROLLARY 3.4. *Let A be a UFD. If $grade(A) \leq 1$ then $dim\, A \leq 1$.*

PROOF. By Proposition 3.3, every non-zero $u \in K$ is grade-regular. By Proposition 3.2 it then follows that $A \subsetneq A[u]$ has going-down for *every* $u \in K$. By Theorem 1.2, $dim\, A \leq 1$.

QUESTION. What are the *UFD*'s A such that, for every $n \in N$, $grade(A) \leq d \Rightarrow dim\, A \leq n$ (hence $dim\, A = grade\,(A)$)?

References

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