On the Existence of Local Solutions of Pseudodifferential Equations*

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0. Introduction. In this paper we mention some recent results on the local solvability of the pseudodifferential equation $\mathbb{P}u = f$. Closely related questions of hypoellipticity and analytic-hypoellipticity of the same equation will not be discussed here. We are concerned with pseudodifferential operators $\mathbb{P}$ that have real characteristics of multiplicity r, $r \geq 1$. In particular, we consider operators of the type

$$\mathbb{P} = P(x, D) \sim \sum_{j=0}^{+\infty} P_{m-j}(x, D), \tag{1}$$

where for each $j = 0, 1, \ldots, P_{m-j}(x, \xi) \in C^\infty(\Omega \times \mathbb{R}_N \backslash \{0\})$, ($\Omega$ an open subset of $\mathbb{R}^N$), is positive homogeneous of degree $m-j$ with respect to ξ and $\sim$ is the standard relation in the theory of pseudodifferential operators. The reader who does not feel confortable with pseudodifferential operators may think of $\mathbb{P}$ as the differential operator

$$\mathbb{P} = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha \left(D = -\sqrt{-1}\, \frac{\partial}{\partial x} \right), \tag{2}$$

where $c_\alpha(x) \in C^\infty(\Omega)$ are complex-valued functions. We recall that $P_m(x, \xi)$ is the *principal symbol* of $P(x, D)$. It is to be regarded as a (complex-valued) function on the cotangent bundle $T^*(\Omega)$ over Ω or, rather, on the complement of the zero section in $T^*(\Omega)$, complement which we denote by $\dot{T}^*(\Omega)$. No such intrisic meaning can be assigned to the $P_{m-j}(x, \xi)$ for $j > 0$. We require that $P_m(x, \xi)$ can be factored (microlocally) as

$$P_m(x, \xi) = Q(x, \xi)\{L(x, \xi)\}^r, \ r \geq 1, \tag{3}$$

in a conic neighborhood $\mathscr{U}$ (i.e. $\mathscr{U}$ is invariant under the dilations $(x, \xi) \longrightarrow (x, \rho\xi)$ when $\rho > 0$) of a point (x_0, ξ^0) of $\dot{T}^*(\Omega)$. The factors Q and L are C^∞ functions in $\mathscr{U}$, positive homogeneous of degree $m-r$ and 1, respectively, with respect to ξ, and satisfy the following conditions

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$$(4) \qquad L(x_0, \xi^0) = 0; \qquad \operatorname{grad}_\xi L(x_0, \xi^0) \neq 0; \qquad Q(x_0, \xi^0) \neq 0.$$

If we let $A = Re(L)$ and $B = Im(L)$, then (4) implies that $\operatorname{grad}_\xi A$ and $\operatorname{grad}_\xi B$ do not both vanish at (x_0, ξ^0). Assume that $\operatorname{grad}_\xi A(x_0, \xi^0) \neq 0$. After a possible shrinking of $\mathscr{U}$, we may further assume that

$$(5) \qquad \operatorname{grad}_\xi A \neq 0 \text{ in } \mathscr{U}$$

and that

$$(6) \qquad Q \neq 0 \text{ (i.e. elliptic) in } \mathscr{U}$$

With a real symbol such as $A(x, \xi)$ we may associate a vector field in $\dot{T}^*(\Omega)$, the Hamiltonian of A:

$$(7) \qquad H_A = \sum_{j=1}^{N} \frac{\partial A}{\partial \xi_j} \frac{\partial}{\partial x^j} - \frac{\partial A}{\partial x^j} \frac{\partial}{\partial \xi_j}.$$

The integral curves of H_A are the bicharacteristic strips of A; they are defined by the system of $2n$ ordinary (non-linear) differential equations (the Hamilton-Jacobi system)

$$(8) \qquad \dot{x} = \operatorname{grad}_\xi A(x, \xi), \qquad \dot{\xi} = -\operatorname{grad}_x A(x, \xi).$$

Note that, along such a strip, $A(x, \xi) = \text{const.}$. We shall refer to those along which $A = 0$ (e.g. the one through (x_0, ξ^0)) as null bicharacteristic strips of A. As a result of (5), the bicharacteristic strips of A in $\mathscr{U}$ are true curves and their projections in Ω are also true curves (not just a point). Observe that the principal symbol of the commutator $[A(x, D), B(x, D)]$, which is a pseudodifferential operator of order $\leq 2m-1$, is nothing else but $(H_A B)(x, \xi) = -(H_B A)(x, \xi) = \{A, B\}(x, \xi)$ (the Poisson bracket of A and B). We shall now define the property under study, namely local solvability assuming that P is a differential operator [see (2)]; for pseudodifferential operators, the definition has to be slightly modified to take into account the pseudolocal character of these operators.

DEFINITION 0.1. The differential operator $\mathbb{P}$ is said to be locally solvable at the point $x_0 \in \Omega$ if there is an open neighborhood U of x_0 in Ω such that given any function $f \in C_c^\infty(U)$, there is a distribution $u \in \mathscr{D}'(U)$ such that $Pu = f$ in U.

The operator $\mathbb{P}$ is locally solvable in a subset $S \subset \Omega$ if it is locally solvable at every point of S; note that it is then locally solvable in some open subset of Ω containing S. All differential operators with constant coefficients are locally solvable. This is a trivial corollary of the Malgrange-Ehrenpreis theorem on the existence of fundamental solutions for such operators.

All *elliptic* differential operators are locally solvable; it follows from the existence of parametrices for such operators. We recall that $\mathbb{P}$ is elliptic if the *characteristic variety* of $\mathbb{P}$ i.e. the set V_{P_m} of zeros of the principal symbol $P_m(x, \xi)$ in $\dot{T}^*(\Omega)$, is empty. After extending $L(x, \xi)$ to the complement of $\mathscr{U}$, we can consider it to be the symbol of a first order pseudodifferential operator $L(x, D)$ of *principal type*. Results on the local solvability of such operators are well known and we shall recall them (they correspond to $r = 1$ in (3)).

1. Operators of Principal Type

DEFINITION 1.1. The operator $\mathbb{P}$ is said to be of principal type in Ω if, given any $x \in \Omega$ and any $\xi \in \mathbb{R}_N$, $\xi \neq 0$, $\operatorname{grad}_\xi P_m(x, \xi) \neq 0$. Observe that, by Euler's homogeneity relation, $mP_m(x, \xi) = \xi \cdot \operatorname{grad}_\xi P_m(x, \xi)$ and, therefore, any zero of $\operatorname{grad}_\xi P_m(x, \cdot)$ would be at least a double zero of $P_m(x, \cdot)$. In other words, to say that $\mathbb{P}$ is of principal type is to say that its real characteristics are simple. All elliptic differential operators and all hyperbolic differential operators are of principal type; the parabolic ones are not. All first-order differential operators whose principal part does not vanish identically at any point are also of principal type. The operators of principal type are, in the sense of the multiplicity of their real characteristics, the simplest after the elliptic operators. The first important results on local solvability for differential operators of principal type were obtained by L. Hörmander and H. Lewy. In his thesis L. Hörmander proved that if $P_m(x, \xi)$ is *real* then $\mathbb{P}$ is locally solvable in Ω. Around 1957, H. Lewy surprised the specialists in the field by giving his celebrated example of a simple differential operator, in $\mathbb{R}^3$, which is not solvable at any point of $\mathbb{R}^3$. In fact, he proved that for "most" $f \in C^\infty(\mathbb{R}^3)$ the equation (with nonvanishing principal type, hence certainly with simple real characteristics)

$$(9) \qquad -iD_1 u + D_2 u - 2(x_1 + ix_2) D_3 u = f$$

does not have any (distribution) solution in any open nonvoid subset of $\mathbb{R}^3$. Its coefficients are very smooth: all of them constant, except one, linear! Of course, the coefficients in its principal part are not all real. An extension of this example, due to Hörmander (1959) gives a necessary condition for a differential equation $\mathbb{P} u = f$ to be locally solvable:

(10) *Suppose the differential equation is locally solvable in* Ω. *Then* $\{Re\, P_m, Im\, P_m\}(x, \xi) = 0$ *whenever* $P_m(x, \xi) = 0$, $(x, \xi) \in \dot{T}^*(\Omega)$.

Clearly this condition can be reformulated as follows:

(11) *Suppose that, at some characteristic point* $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ *the following holds: the restriction of* $Im\, P_m(x, \xi)$ *to the (null) bicharacteristic strip of* $Re\, P_m(x, \xi)$ *through* (x_0, ξ^0) *vanishes at this point whereas its first derivative along this curve does not. Then the equation* $\mathbb{P} u = f$ *is not locally solvable at* x_0.

Hörmander's condition reveals the rôle of the commutator of the principal part of the operator with its complex conjugate (note that $\{Re\, P_m, Im\, P_m\} \equiv \frac{1}{\sqrt{-1}} \{P_m, \bar{P}_m\}$); it also points to the significance of the bicharacteristics. Since then, thanks to the combined efforts of R. Beals, C. Fefferman, L. Nirenberg and F. Treves, much has been done to characterize operators of principal type in so far as local solvability is concerned. In particular, these studies show that Lewy's operator is in the limit of nonsolvability (e.g. an example such as Lewy's could not have been found in two independent variables). There is however one important drawback in the statement (11). It is clear that the property under study here, local solvability, is not only invariant under changes of coordinates in Ω (which means to say that $\mathbb{P}$ is locally solvable at x_0 if and only if $F^{-1}\mathbb{P}F$ is locally solvable at $\psi(x_0)$ where F is an elliptic Fourier Integral Operator associated with the canonical transformation ψ in $T^*(\Omega)$ induced by the change of coordinates in Ω), but also under multiplication of the operator $\mathbb{P}$ by a nonvanishing complex C^∞ function (or more generally, by an elliptic operator). Any condition which intends to characterize local solvability ought to possess such an invariance, if it is to be of use to us (in such a case there will be no ambiguity concerning our choices for the real and imaginary parts of P_m). Assuming that $\mathbb{P}$ is a pseudodifferential operator [see (1)] we state:

DEFINITION 1.2. We say that Property (ψ) holds at a point $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ if there is an open neighborhood $\mathcal{O}$ of (x_0, ξ^0) in $\dot{T}^*(\Omega)$ such that, for any complex number z satisfying:

(12) *grad* $Re(zP_m)(x, \xi) \neq 0$ for every $(x, \xi) \in \mathcal{O}$, the following is true:

(13) if the restriction of $Im\, zP_m$ to any null bicharacteristic strip of $Re\, zP_m$ in $\mathcal{O}$ is negative at some point, it remains nonpositive at all further points along the (oriented) bicharacteristic strip.

When the pseudodifferential operator is *antipodal*, i.e.,

$$P_m(x, -\xi) = (-1)^m P_m(x, \xi)$$

(in particular a differential operator) Property (ψ) becomes Property ($\mathscr{P}$).

DEFINITION 1.3. We say that Property ($\mathscr{P}$) holds at a point $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ if there is an open neighborhood $\mathcal{O}$ of (x_0, ξ^0) in $\dot{T}^*(\Omega)$ such that, given any complex number z satisfying (12), the following is true:

(14) the restriction of $Im\, zP_m$ to any null bicharacteristic strip of $Re\, zP_m$ in $\mathcal{O}$ does not change sign.

Properties (ψ) and ($\mathscr{P}$) are acceptable from the view point of invariance but lead to new difficulties, in so far as (13) and (14) must be checked, in principle, for all complex z satisfying (12). Fortunately, it suffices to check them for only one z such that (12) holds: they are then automatically true for all other such z's.

We may now state the conjectures (due to L. Nirenberg and F. Treves). Of course the operator $\mathbb{P}$ is assumed of principal type. If it were not so, these statements would not make (in general) any sense (notice that in general the concept of bicharacteristic is meaningless for higher multiplicities of the characteristics).

CONJECTURE 1. *The pseudodifferential operator* $\mathbb{P}$ [see (1)] *is locally solvable in* Ω *if and only if Property* (ψ) *holds at every point in* $\dot{T}^*(\Omega)$.

CONJECTURE 2. *The differential operator* $\mathbb{P}$ [see (2)] *is locally solvable in* Ω *if and only if Property* ($\mathscr{P}$) *holds at every point in* $\dot{T}^*(\Omega)$.

The sufficiency of Property ($\mathscr{P}$) in Conjecture 2 was proved in two different ways by R. Beals and C. Fefferman (before that, L. Nirenberg and F. Treves had proven the same result in the following situations: 1) $m=1$; 2) $N=2$; 3) P_m analytic). The first proof relies on a recent result of Calderon-Vaillancourt on the L^2 boundedness of pseudodifferential operators of order zero and type $(\frac{1}{2}, \frac{1}{2})$. For the second proof, they embedded the symbol P_m in a more sophisticated symbolic calculus (the class of symbols $S^{M,m}_{\Phi,\phi}$) and followed Nirenberg-Treves' ideas in the analytic case reducing the whole problem to proving the following lemma:

LEMMA 1.1. *Let* $p_t(x, \xi)$ *be a first-order classical symbol depending smoothly on the real parameter* t, *and suppose that for each fixed* (x, ξ), *the function* $t \longrightarrow p_t(x, \xi)$ *does not change sign. Then, the corresponding operator* $p_t(x, D)$ *may be written in the form* $p_t(x, D) = A_t B + C_t$, *where* B *is a fixed (unbounded) self-adjoint operator,* A_t *is self-adjoint bounded and non-negative,* C_t *is a bounded error, and* $[A_t, B]$, $[[A_t, B], B]$ *are bounded.*

Ideally, one should prove the lemma simply by writing the symbol $p_t(x, \xi)$ in the form:

(15) $p_t = a_t o b + c_t$ where $a_t \geq 0$ and a_t, b, c_t are classical symbols of order 0, 1, 0, respectively.

If p_t could be so expressed, the conclusion of Lemma 1.1 would follow instantly from the classical symbolic calculus and sharp Gärding inequality. Nirenberg and Treves carried this out in the case of real-analytic symbols p_t, but unfortunately p_t cannot, in general, be written in the form (15) using classical symbols (the unpublished counterexample is due to J. Mather).

Yu V. Egorov announced that he proved the sufficiency of Property (ψ) in Conjecture 1.

Under the additional hypothesis (FZ) of zeros of finite order (only at the point where a change of sign can take place), Yu V. Egorov, L. Nirenberg

and F. Treves proved that (ψ) is also necessary for the local solvability of $\mathbb{P}$.

DEFINITION 1.4. We say that Property (FZ) holds at a point $(x_0, \xi^0) \in \dot{T}^*(\Omega)$ where P_m vanishes, if whatever $z \in \mathbb{C}$ such that $\underset{\xi}{grad}\,(zP_m) \neq 0$ at (x_0, ξ^0), the restriction of $Im\, zP_m$ along the (null) bicharacteristic strip of $Re\, zP_m$ through (x_0, ξ^0), vanishes of finite order k at this point.

It can be shown that k is independent of all such z's. We shall outline briefly some ideas in the proof of Conjecture 1. Let $(x_0, \xi^0) \in$ Caract. $\mathbb{P}$ (outside this set $\mathbb{P}$ is elliptic hence locally solvable). The property *principal type* allows one to obtain a factorization, in a conic neighborhood $\mathscr{U}$ of (x_0, ξ^0), of the type:

$$P_m(x, \xi) = Q(x, \xi)\,(\xi_N - \lambda(x, \xi')) \tag{16}$$

(we assume $\dfrac{\partial P_m}{\partial \xi_N}(x_0, \xi^0) \neq 0$), where Q is elliptic (in $\mathscr{U}$) of order $m-1$, and $\lambda \in C^\infty(\mathscr{U})$, homogeneous of degree one in $\xi' = (\xi_1, \ldots, \xi_{N-1})$. By microlocalization, then, one is led to study the first order operator

$$L = D_N - \lambda(x, D') \tag{17}$$

which by a canonical transformation can be put in the form:

$$\tilde{L} = \frac{1}{i}\frac{\partial}{\partial t} - ib(x, t, D_x) \tag{18}$$

where $b(x, t, \xi)$ is *real*. We have changed completely the notation: x^N became t, $x = (x^1, \ldots, x^n)$ $(n = N-1)$ stands for the other variables in Ω (after the canonical transformation), τ and $\xi = (\xi_1, \ldots, \xi_n)$ are the co-variables associated to t and x respectively. We consider the ordinary differential equation (in t, depending on the parameters (x, ξ))

$$\frac{du}{dt} + b(x, t, \xi)u = f(x, t, \xi). \tag{19}$$

We suppose that t varies in a closed interval $|t| \leq T$, and (x, ξ) in a conic neighborhood $\mathscr{U}' = U \times \mathscr{U}_0$ of (x_0, ξ^0) in $\mathbb{R}^n \times (\mathbb{R}_n \backslash \{0\})$ where U is an open subset of $\mathbb{R}^n$ containing x_0 and $\mathscr{U}_0$ an open cone in $\mathbb{R}_n \backslash \{0\}$ containing ξ^0 (U and $\mathscr{U}_0$ with smooth boundaries). Let $\mathscr{E}$ be the space of functions

$f(x, t, \xi)$ which are C^∞ in $\bar{U} \times [-T, T] \times \bar{\mathscr{U}}_0$ and fastly decreasing with respecto to ξ, at infinity (in $\mathscr{U}_0$); let D' be the space of distributions in a neighborhood of $\bar{U} \times [-T, T]$ tempered (and measurable) with respect to $\xi \in \mathscr{U}_0$. We introduce the property:

(20) $\forall f \in \mathscr{E}, \quad \exists u \in D'$ satisfying (19).

We write

$$B(x, t, t', \xi) = \int_{t'}^{t} b(x, s, \xi) ds$$

and, for $-T \leq T(x, \xi) \leq T$,

(21) $$u(x, t, \xi) = \int_{T(x, \xi)}^{t} e^{-B(x, t, t', \xi)} f(x, t', \xi) dt'$$

It is easy to verify that (20) holds if and only if we can choose $T(x, \xi)$ (for each $(x, \xi) \in \mathscr{U}'$) such that the following is true:

(22) $\forall t \in [-T, T], \forall t'$ *in the interval with extremities* t *and* $T(x, \xi)$,

$$B(x, t, t', \xi) \geq 0.$$

The reader will convince him self that (22) is equivalent to the following property:

(22′) *If, for some* $|t_0| \leq T, b(x, t_0, \xi) < 0$, *then* $b(x, t, \xi) \leq 0$ *for all* $t, t_0 < t \leq T$.

It is not difficult to recognize (22′) as Property (ψ) for the first-order factor $\tilde{L}$. Thus an attractive feature of the solvability theory in the simple characteristics case is that it reduces to the study of the ordinary differential equation (19). One may ask then the analogous question for equations with multiple characteristics. Of course, the first-order *ODE* (19) should be replaced by an equation whose order is equal to the order of the multiplicity of the characteristics at the "central" point $(x_0, t_0, \xi^0, \tau_0)$. Finding out what the higher-order "basic" *ODE's* (if they exist) should be and proving the equivalence of its solvability (in the sense of solutions which are tempered in the variable ξ) with the local solvability of the pseudodifferential equation under consideration are two of the aims of the general

theory. This, however, can be very difficult as one realizes by examining the situation of the linear *ODE* theory of order > 1, from the view point of the asymptotic properties of solutions with respect to a parameter (in our case ξ) converging to $+\infty$. Fortunately we are interested in conditions not only for $(x_0, \rho\xi^0)$, $\rho \sim +\infty$, but for the entire cone $(x, \rho\xi)$, $\rho \sim +\infty$, $(x, \xi) \in$ an open subset of the cosphere-bundle $S^*(\Omega)$, and this imposes important restrictions to the type of *ODE* we have to study.

2. Operators with Multiple Characteristics

We recall that the characteristic variety of $\mathbb{P}$ [see (1)] is the set V_{P_m} of zeros of the principal symbol $P_m(x, \xi)$ in $\dot{T}^*(\Omega)$. We shall denote by W_{P_m} the set of singularities of V_{P_m} (in the sense of Analysis), that is to say, the set of points (x, ξ) in $\dot{T}^*(\Omega)$ which satisfy the equations:

(23) $$P_m^{(\alpha)}(x, \xi) = 0, \ |\alpha| \leq 1.$$

Let (x_0, ξ^0) be some point in W_{P_m}. Let us make the following assumption

(24) there is an N-tuple α, $|\alpha| = 2$, such that $P_m^{(\alpha)}(x_0, \xi^0) \neq 0$. Possibly after a linear change of the coordinates x^j we may assume that $\alpha = (0, \ldots, 0, 2)$.

By virtue of the Weierstrass-Malgrange preparation theorem, we may find a factorization of the following kind:

(25) $P_m(x, \xi) = Q(x, \xi)\{(\xi_N - \xi_N^0)^2 + a_1(x, \xi')(\xi_N - \xi_N^0) + a_2(x, \xi')\}$, valid in some open neighborhood $\mathscr{U}$ of (x_0, ξ^0) in $\dot{T}^*(\Omega)$. We have used the notation $\xi' = (\xi_1, \ldots, \xi_{N-1})$. The functions Q and a_j, $j = 1, 2$, are C^∞ in $\mathscr{U}$; $Q(x, \xi)$ does not vanish at any point of $\mathscr{U}$; $a_j(x_0, \xi'^0) = 0$, $j = 1, 2$. Furthermore, in view of (23):

(26) $$\underset{\xi}{grad}\ a_2(x_0, \xi'^0) = 0.$$

One remark is in order: We can (and shall) assume that $\mathscr{U}$ is conic and that Q and $a_j, j = 1, 2$ are positive homogeneous functions of ξ of degrees $m-2$ and $j = 1, 2$ respectively.

DEFINITION 2.1. We say that $(x_0, \xi^0) \in W_{P_m}$ is a *regular double characteristic point* of $\mathbb{P}$ if the factorization (25) holds in an open (conic) neighborhood $\mathscr{U}$ of (x_0, ξ^0) and if, furthermore,.

(27) $$\Delta(x, \xi') = \frac{1}{4}\, a_1^2(x, \xi') - a_2(x, \xi') = 0 \text{ in } \mathscr{U}.$$

It will be observed that we do not require a_1 to be real, and, therefore, we do not require the variety V_{P_m} to have points in $\mathscr{U}$ other than (x_0, ξ^0). When (27) holds, we may write $P_m(x, \xi)$ as in (3) with $r = 2$. In such a case, F. Cardoso and F. Treves have proved the following result:

THEOREM 2.1. *Assume that (FZ) holds for the first-order factor L and that (ψ) is violated by L at (x_0, ξ^0). Then $\mathbb{P}$ is not locally solvable at x_0.*

The noteworthy feature in Theorem 2.1 is that the lower-order terms of $\mathbb{P}$ do not influence the conclusion.

COROLLARY 2.1. *Suppose that $\mathbb{P}$ is a differential operator and that (FZ) holds at (x_0, ξ^0) with an odd integer k. Then neither $\mathbb{P}$ nor ${}^t\mathbb{P}$ (the transpose of $\mathbb{P}$) is locally solvable at x_0.*

The starting point in the proof of Theorem 2.1 is the same as always in this kind of question: the remark of Hörmander as to the functional-analytic consequence of local solvability: if $\mathbb{P}$ were locally solvable at x_0 there would be two neighborhoods $V \subset U$ of x_0 in Ω, a compact subset K of U, an integer $M \geq 0$, a constant $C > 0$ such that

(28) $$\left|\int f v dx\right| \leq C \sup_k \Big(\sum_{|\alpha| \leq M} |D^\alpha f| \Big) \cdot \sup_k \Big(\sum_{|\alpha| \leq M} |D^\alpha({}^t\mathbb{P} v)| \Big)$$

for every $f, v \in C_c^\infty(V)$. In order to show that (28) cannot hold, in the present situation, whatever the choice of U, V, K, M and C, one takes

(29) $$v = e^{i\rho\omega}\phi, \ \rho \sim +\infty$$

with the complex valued (phase) function $\omega \in C^\infty(\Omega)$ and the (amplitude) $\phi \in C_c^\infty(V)$ chosen in such a way that v is an "approximate" solution of the homogeneous equation ${}^t\mathbb{P}v = 0$. An important role in the investigation of problems of the type which we are concerned here is played by the following asymptotic expansion in powers of ρ, about $\rho \sim +\infty$ (*grad* ω at x_0 is equal to ξ^0):

(30) $$e^{-i\rho\omega}\,\mathbb{P}(e^{i\rho\omega}\phi) \sim \sum_{j=0}^{+\infty} \rho^{m-j}\,\mathscr{P}_j(\omega;\ x, D_x)\phi,$$

where, for each $j = 0, 1, \ldots, \mathscr{P}_j(\omega; x, D_x)$ is a *differential operator* in Ω, of order $\leq j$ whose coefficients depend on ω and on its derivatives. It is readily seen that

(31) $$\mathscr{P}_0(\omega; x, D_x)\phi = P_m(x,\ grad\ \omega)\phi.$$

A somewhat lengthier, but straightforward, computation shows that

(32) $$\mathscr{P}_1(\omega; x, D_x)\phi = \sum_{|\alpha|=1} P_m^{(\alpha)}(x,\ grad\ \omega)D^\alpha + \tilde{P}_{m-1}(x,\ grad\ \omega,\ (grad)^2\ \omega)\phi,$$

where $P_m^{(\alpha)}(x, \xi) = \left(\dfrac{\partial}{\partial \xi}\right)^\alpha P_m(x, \xi)$, and where

(33) $$\tilde{P}_{m-1}(x,\ grad\ \omega,\ (grad)^2\ \omega) = P_{m-1}(x,\ grad\ \omega) + i \sum_{|\beta|=2} \frac{1}{\beta!} P_m^{(\beta)}(x,\ grad\ \omega)D^\beta\ \omega.$$

The functional $\tilde{P}_{m-1}$ defines an invariant associated with the pseudodifferential operator $\mathbb{P}$. The correct way of looking at it is as a function on the bundle $J_2^*(\Omega)$ of jets of degree two over Ω (the cotangent bundle $T^*(\Omega)$ is nothing else but the bundle of jets of degree *one* over Ω). Suppose that ω has been chosen so as to satisfy

(34) $$P_m^{(\alpha)}(x,\ grad\ \omega) = 0 \ \textit{in}\ \Omega,\ |\alpha| = 1.$$

Notice that this implies that ω satisfies also the *characteristic equation*

(35) $$P_m(x,\ grad\ \omega) = 0 \ \textit{in}\ \Omega.$$

It then follows that the differential operator $\mathscr{P}_1(\omega;\ x, D_x)$ reduces to its zero-order term, which can be written:

(36) $$\mathscr{M}(x,\ grad\ \omega) = P_{m-1}(x,\ grad\ \omega) - \frac{1}{2}\sum_{|\alpha|=1} D_x^\alpha P_m^{(\alpha)}(x,\ grad\ \omega).$$

The remarkable fact is that (36) is a function on the *cotangent bundle*. Under this form it has been used by Mizohata and Ohya in the study of the Cauchy problem. It should also be remarked that when applying the asymptotic expansion we need (since ω is complex valued) a "good" analytic approximation, in the covariable ξ, of the symbol $P(x, \xi)$ of $\mathbb{P}$. Later, when combining the asymptotic expansion with (28) we perform also an analytic approximation in the variable x. All of this, of course, introduces some errors which must be estimated when disproving (28). We can assume in the proof of Theorem 2.1 that $\mathbb{P}$ is of the following form:

$$(37) \qquad \mathbb{P} = \left(\frac{\partial}{\partial t} + b(t)D_x\right)^2 - c(t)D_x + d(t), \quad N = 2, \quad x_0 = (0, 0)$$

where b, c, d are analytic functions of t, $|t| < T$ and $b(t)$ is *real.* Roughly speaking, one can "reduce" the general situation to one close to (37), by performing "admissible operations" such as: use of *canonical transformations* to straighten up bicharacteristics and to flatten transversal "pieces" of the characteristic variety perpendicular to the straightened bicharacteristics; moving away (along the characteristic variety) from the original point (x_0, ξ^0) to a new point (x_0', ξ'^0), *arbitrarily* close to (x_0, ξ^0) and *conveniently* chosen, so that the problem of disproving solvability of $\mathbb{P}$ at x_0 is transferred to that of the non-solvability of $\mathbb{P}$ at x_0' (in this step it is *very important* that the order k of the zero in Theorem 2.1 is necessarily odd); "division" by the elliptic factor Q. It is also important to mention that although the Mizohata-Ohya invariant

$$(39) \qquad \mathscr{M}(x, \xi) = P_{m-1}(x, \xi) - \frac{1}{2}\sum_{|\alpha|=1} \partial_\xi^\alpha D_x^\alpha P_m(x, \xi)$$

plays no role in the statement of Theorem 2.1, it is very much present in its proof which subdivides into two parts, according to whether the lower order terms have or do not have a "strong influence". The precise measurement of this influence is achieved by means of the imaginary part of the square-root of the invariant in question, defined in the characteristic variety.

The next case to investigate is when the factor L (still satisfying (FZ)) satisfies the condition (ψ) for local solvability. In such a case obstruction to solvability may come from the Mizohata-Ohya invariant (39), specifically from the fact that it does not decay to zero fast enough as (x, ξ) converges to (x_0, ξ^0) along certain bicharacteristic strips in the cotangent bundle. Results in this direction are fragmentary. A particular (but decisive) situation is that of the operator

$$(40) \qquad \mathbb{P} = (\partial_t + b(t, D_x))^2 + c(t, D_x)$$

where $b(t, \xi)$ is a *real valued* C^∞ function of (t, ξ) in $(-T, T) \times (\mathbb{R}_n \backslash \{0\})$, $T > 0$, positive homogeneous with respect to ξ of degree *one* and

$$c(t, \xi) = c_1(t, \xi) + \sum_{j=0}^{+\infty} c_{-j}(t, \xi),$$

where each $c_i(t, \xi)$, $i = 1, 0, -1, -2, \ldots$, is a C^∞ function of (t, ξ) in $(-T, T) \times (\mathbb{R}_n \backslash \{0\})$ positive homogeneous of degree i with respect to ξ. If we assume that for some $\xi^0 \in S^*(\Omega)$:

$$(41) \qquad b(t, \xi^0) = b_0 t^k (1 + 0(t)), \ k \ odd, \ b_0 > 0,$$

then Theorem 2.1 implies that ${}^t\mathbb{P}$ is not locally solvable at the points $(x, 0)$ in $\mathbb{R}^{n+1}$. In the cases corresponding to solvability of $\partial_t + b(t, D_x)$ i.e.:

(42) *for every* $\xi^0 \in S^*(\Omega)$, $b(t, \xi^0) = b_0 t^k (1 + 0(t))$, *$k$ even or else k odd and* $b_0 < 0$,

we might have nonsolvability for $\mathbb{P}$ due to the influence of the lower order terms. In fact, if we suppose k *even* (and without loss in generality $b \geq 0$), J. Barros Neto and F. Treves have shown that the condition

$$(43) \qquad \left(\int_{t'}^{t} \left| Im \sqrt{c_1(s, \xi)} \right| ds\right)^2 \leq \text{const.} \int_{t'}^{t} b(s, \xi) ds,$$

for all $t > t'$ in a neighborhood of zero and for all $\xi = \dfrac{\xi}{|\xi|}$ in a neighborhood of ξ^0, implies that $\mathbb{P}$ is hypoelleptic at $(x, 0)$, and, hence, that ${}^t\mathbb{P}$ is locally solvable at $(x, 0)$ (They have also shown that $\mathbb{P}$ is analytic hypoelliptic if this is true of $\partial_t + b(t, D_x)$). F. Cardoso and F. Treves have been trying to show (so far unsuccessfully) that (43) is also necessary for the solvability of ${}^t\mathbb{P}$ at $(x, 0)$ (and hence also for the hypoellipticity of $\mathbb{P}$ at $(x, 0)$).

If we now go back to the factorization (25), without constant multiplicity [see Def. 2.1] and assume that P_m factorizes smoothly as a product:

$$(44) \qquad P_m(x, \xi) = Q(x, \xi)\, L_1(x, \xi)\, L_2(x, \xi)$$

where L_1 and L_2 are first order symbols, then new phenomena can be expected as has been shown by A. Gilioli and F. Treves. They studied completely (by the method of "concatenation"), from the view point of local solvability, the second-order operator in $\mathbb{R}^2$,

$$\mathbb{P}_0 = \left(\frac{\partial}{\partial t} - at^k |D_x|\right)\left(\frac{\partial}{\partial t} - bt^k |D_x|\right) + ct^{k-1} |D_x| \tag{45}$$

where k is an integer, $a, b \in \mathbb{R}$, $c \in \mathbb{C}$. According to the theory of operators of principal type, if k is even, the operator $X = \frac{\partial}{\partial t} - at^k |D_x|$ is both locally solvable and hypoelliptic (in fact, subelliptic), if $a \neq 0$. When k is odd, X is hypoelliptic but non-solvable if $a > 0$ and solvable but not hypoelliptic if $a < 0$. The interesting case is therefore when

$$k \text{ is odd}, \quad a > 0, \quad b < 0 \tag{46}$$

(the other cases are easy to classify). They proved the following result

THEOREM 2.2. *The following conditions are equivalent*

a) $\mathbb{P}_0$ *is not locally solvable at the origin*

b) $\frac{c}{a-b}$ *is a positive integer congruent to* 0 *or* 1 modulo $k+1$.

This shows that phenomena of a discrete type occur (they suggest spectral properties of second order equations). This had already been shown in the study of hypoellipticity by Grushin and others. Theorem 2.2 has been extended by Gilioli in his thesis. When $k=1$, J. Sjöstrand, F. Treves and L. Boutet de Monvel have obtained more powerful results. We return now to the situation described in the Introduction [see (1), (3) and (4)] with $r \geq 3$. R. Goldman proved in his thesis the following

THEOREM 2.3. *Assume that (FZ) holds for the first order factor L and that* (ψ) *is violated by L at* (x_0, ξ^0). *Furthermore assume that the Mizohata-Ohya invariant* $\mathcal{M}(x, \xi)$ [see (39)] *does not vanish at* $(x_0, -\xi^0)$. *Then* $\mathbb{P}$ *is not locally solvable at* x_0.

We remind the reader that the principal symbol of ${}^t\mathbb{P}$ is $P_m(x, -\xi)$ and its "subprincipal symbol" or Mizohata-Ohya invariant is $\mathcal{M}_{{}^tP}(x, \xi) = \mathcal{M}_P(x, -\xi)$. Therefore the extra condition in Theorem 2.3 means that $\mathcal{M}_{{}^tP}(x_0, \xi^0) \neq 0$. It was conjectured by F. Treves that Theorem 2.3 is true without this extra condition [see Theorem 2.1].

COROLLARY 2.2. *Suppose* $r=3$ *and that (FZ) holds at* (x_0, ξ^0) *with* $k=1$. *If* (ψ) *is violated by L at* (x_0, ξ^0), *then* $\mathbb{P}$ *is not locally solvable at* x_0.

PROOF. If $\mathcal{M}(x_0, -\xi^0) \neq 0$ the Corollary is merely a restatement of the theorem. In the case $\mathcal{M}(x_0, -\xi^0) = 0$ we apply a recent result of Sjöstrand to obtain the conclusion.

COROLLARY 2.3. *Suppose* $\mathbb{P}$ *is a differential operator in* Ω *and that (FZ) holds at* (x_0, ξ^0) *with an odd integer k. Then if* $\mathcal{M}(x_0, \xi^0) \neq 0$, *neither* $\mathbb{P}$ *nor* ${}^t\mathbb{P}$ *is locally solvable at* x_0.

Finally we mention that Treves proved that Property (ψ) for an arbitrary symbol p without critical points is equivalent with the fact that p is the limit, in the local C^1 topology, of subelliptic symbols p_j of order $\frac{1}{2}$ i.e. p_j satisfy the following condition:

$$\frac{1}{i}\{p_j, \bar{p}_j\} > 0 \text{ at all characteristic points of } p_j \tag{47}$$

($\bar{p}_j$ is the complex conjugate of p_j).
Such a result points to a new definition of (ψ) which is totally independent of the concept of *bicharacteristic* and thus lends itself perfectly to generalization to arbitrary symbols, with an arbitrary multiplicity of the characteristics. This, of course, led Treves to a new general conjecture on the necessity of (ψ), redefined as indicated, for local solvability of any linear differential, or pseudodifferential equation. Of course, the sufficiency of (ψ) in this case is out of question since we know that lower-order terms in a differential operator can affect its solvability properties.

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