# On Klingenberg's Theorem*

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1. In the classical problem of classifying the structure of a riemannian manifold $M$ from the properties of the seccional curvature $K_M$ of $M$, the following well known results were obtained, when $K_M$ is bounded below by a positive constant. Berger proved in [1] and [2] that a complete, simply connected and even dimensional riemannian manifold with $\frac{1}{4} \leq K_M \leq 1$ is homeomorphic to a sphere, or otherwise $M$ is isometric to one of the compact symmetric spaces of rank one. For arbitrary dimensional riemannian manifolds, Klingenberg proved in [8] that a complete and simply connected riemannian manifold with $\frac{1}{4} < K_M \leq 1$ is homeomorphic to a sphere. Moreover in the odd dimensional case with $\frac{1}{4} \leq K_M \leq 1$, Klingenberg proved in [9] that $M$ is still homeomorphic to a sphere.

The above results were proved, using the following Klingenberg's theorem [7].

THEOREM A. *Let $M$ be a compact, simply connected, n-dimensional riemannian manifold, such that $\frac{1}{4} \leq K_M \leq 1$. Then $\forall\, m \in M$, the distance from $m$ to its cut locus $C(m)$, satisfies*

$$d(m, C(m)) \geq \pi.$$

This theorem follows from the following

LEMMA A. *Let $M$ be a compact, simply connected n-dimensional riemannian manifold, such that $0 < K_M \leq 1$. Then $\forall\, m \in M$*

$$d(m, C(m)) \geq \min\left\{\pi, \frac{1}{2} l\right\},$$

*where $l$ denotes the length of the smallest closed geodesic on $M$. If $n$ is even, then $l \geq 2\pi$. If $n$ is odd and $\frac{1}{4} \leq K_M \leq 1$, then $l \geq 2\pi$.*

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Berger in [3] gave an example showing that Klingenberg's theorem for odd-dimensional manifolds cannot be proved for $0 < K_M \leq 1$. The example is a three-dimensional riemannian manifold, whose seccional curvature satisfies $\frac{1}{9} \leq K_M \leq 1$ and it has closed geodesics with length less than $2\pi$.

In this paper we consider three and five dimensional manifolds $M$, with $\delta \leq K_M \leq 1$ where $\delta < \frac{1}{4}$, and we prove results analogous to Klingenberg's theorem, with hypothesis on the diameter and volume $v(M)$ of $M$. The hypothesis on the diameter is based on the following result proved by Berger [3]. If $M$ is a complete, simply connected $n$-dimensional riemanniam manifold, such that

$$0 < \delta \leq K_M \leq 1 \quad \text{and} \quad diam\ M > \frac{\pi}{2\sqrt{\delta}},$$

then $M$ is homeomorphic to the $n$-dimensional sphere $S^n$ where $n \neq 3, 4$.

Let $S^n_\delta$ denote the $n$-dimensional sphere with constant seccional curvature $\delta$. In §3 we prove the following results.

THEOREM 1. *Let $M$ be a complete, three-dimensional riemannian manifold such that $0 < \delta \leq K_M \leq 1$, $\delta \leq \frac{4}{25}$,*

$$diam\ M \leq \frac{\pi}{2\sqrt{\delta}} \quad \text{and} \quad v(M) > \frac{9}{20}\ v(S^3_\delta).$$

*Then every closed geodesic on $M$ has lenght $> 2\pi$.*

THEOREM 2. *Let $M$ be a complete, five-dimensional riemannian manifold such that*

$$0 < \delta \leq K_M \leq 1,\ \delta \leq \left(\frac{23}{60}\right)^2,\ diam\ M \leq \frac{\pi}{2\sqrt{\delta}} \ \text{and} \ v(M) > \frac{9}{20}\ v(S^5_\delta).$$

*Then every closed geodesic on $M$ has length $> 2\pi$.*

From Lemma *A*, we get

COROLLARY. *If $M$ is a simply connected, riemannian manifold satisfying the conditions of theorem 1 or 2, then $\forall\, m \in M$*

$$d(m,\ C(m)) \geq \pi.$$



We remark that once the value of $\delta$ is fixed it is possible to improve the condition on the volume, for example we can prove that in the 3-dimensional case, if

$$\delta = \frac{1}{9},\ diam\ M \leq \frac{3\pi}{2} \quad \text{and} \quad v(M) > \frac{\sqrt{6}}{6}\ v(S^3_\delta)$$

then every closed geodesic on $M$ has length $> 2\pi$. It is not difficult to see that Berger's example [3] mentioned above does not satisfy our conditions on the diameter of $M$.

In §2 we introduce the main tool used in the proof of theorems 1 and 2. It is a generalization of a result obtained in the author's doctoral thesis. In §3 we prove our main results.

§2. In ([10] Theorem 2 and Theorem 3) we obtained a method which gives a lower bound for the length of the closed geodesics on a complete, riemannian manifold $M$, such that $K_M \geq 1$ and $v(M) > V$. This method was obtained, considering $diam\ M \leq \pi$ which follows from Myers theorem.

Since in this paper, we have an extra hypothesis on the diameter of $M$, we are going to generalize the result mentioned above, when $diam\ M \leq d$.

Cheeger [4] proved the following

THEOREM B. *Let $M$ be a complete, $n$-dimensional riemannian manifold, such that $K_M \geq H$, $diam\ M \leq d$ and $v(M) > V$, where $d,\ V > 0$, $H \in \mathbb{R}$. Then there exists a constant $c_n(d, V, H) > 0$ such that every closed geodesic on $M$ has length $> c_n(d, V, H)$.*

Based essentially on the proof of theorem B, we obtain values for $c_n(d, V, H)$, when $K_M \geq H > 0$. Without loss of generality we can consider $K_M \geq 1$.

THEOREM 3. *Let $M$ be a complete, $n$-dimensional riemannian manifold, such that $K_M \geq 1$, $diam\ M \leq d$ and $v(M) > V$. Let $\theta < \frac{\pi}{2}$ and $r < d$ be respectively determined by the following equations*

$$(1) \qquad 2 \int \cdots \int \sin^{n-1}\alpha_1 \sin^{n-2}\alpha_2 \ldots \sin\alpha_{n-1}\, d\alpha_1 \ldots d\alpha_n = V_1$$

*where* $0 \leq \alpha_1 \leq d$, $\theta \leq \alpha_2 \leq \frac{\pi}{2}$, $0 \leq \alpha_i \leq \pi$, $i = 3, \ldots, n-1$, $0 \leq \alpha_n \leq 2\pi$, *and*

$$(2) \qquad \int_0^r \sin^{n-1}\alpha_1 \, d\alpha_1 = \frac{V_2 \int_0^d \sin^{n-1}\alpha_1 \, d\alpha_1}{2 \int \ldots \int \sin^{n-1}\alpha_1 \sin^{n-2}\alpha_2 \ldots \sin\alpha_{n-1} \, d\alpha_1 \ldots d\alpha_n - V_1}$$

*where* $0 \leq \alpha_1 \leq d$, $0 \leq \alpha_2 \leq \frac{\pi}{2}$, $0 \leq \alpha_i \leq \pi$, $i = 3, \ldots, n-1$, $0 \leq \alpha_n \leq 2\pi$, *and* $V_1$, $V_2$ *are positive numbers such that* $V_1 + V_2 = V$. *Let* $c_n(d, V, 1)$ *be any positive real number* $< 2 \tan^{-1}(\cos\theta \cdot \tan r)$ *and* $\leq \pi$. *Then every closed geodesic on* $M$ *has length* $> c_n(d, V, 1)$.

PROOF. Let $S^n$ be the $n$-dimensional unit sphere in $\mathbb{R}^{n+1}$ centered at $p = (0, \ldots, 0, 1)$. Consider the following parametrization

$$\begin{aligned} x_1 &= \sin\alpha_1 \cos\alpha_2 \\ x_2 &= \sin\alpha_1 \sin\alpha_2 \cos\alpha_3 \\ &\;\;\vdots \\ x_{n-1} &= \sin\alpha_1 \sin\alpha_2 \ldots \sin\alpha_{n-1} \cos\alpha_n \\ x_n &= \sin\alpha_1 \sin\alpha_2 \ldots \sin\alpha_{n-1} \sin\alpha_n \\ x_{n+1} &= 1 - \cos\alpha_1, \end{aligned}$$

where $0 < \alpha_i < \pi$, $i = 1, \ldots, n-1$ and $0 < \alpha_n < 2\pi$. $\mathscr{A}_{t,\theta}(u)$ will denote the set of vectors that form an angle $\leq \theta$ with $u$ or $-u$ and with length $\leq t$. If $m \in S^n$ we denote by $D_r(m)$ the set of vectors $v \in T_m S^n$ such that $\| v \| \leq r$. Fix $0 \in S^n$ and the vector $u = (1, 0, \ldots, 0) \in T_0 S^n$, with the above parametrization. It is not difficult to see that equations (1) and (2) are respectively equivalent to

$$v(\exp_0(D_d(0) - \mathscr{A}_{d,\theta}(u))) = V_1$$

and

$$v(\exp_0(\mathscr{A}_{r,\theta}(u))) = V_2.$$

Let $c_n(d, V, 1)$ be any positive real number less than $2\tan^{-1}(\cos\theta \cdot \tan r)$. Since $0 < \theta < \frac{\pi}{2}$, if follows from the first variation formula and the relations on spherical triangles, that if $\tau(t)$ and $\sigma(s)$ are geodesics on $S^n$ such that $\tau(0) = \sigma(0)$ and $(\tau'(0), \sigma'(0)) \leq \theta$, then



$$(3) \qquad \begin{aligned} &d(\sigma(r), \tau(t)) < r \text{ for all } t, \\ &0 < t \leq c_n(d, V, 1) < 2\tan^{-1}(\cos\theta, \tan r). \end{aligned}$$

We can now prove, that the existence of a closed geodesic on $M$ with length $\leq c_n(d, V, 1)$, would imply $v(M) \leq V$. What will follow is Cheeger's proof [4] of theorem B, which is included for the sake of completeness. Let $\gamma$ be a closed geodesic on $M$ with length $L \leq c_n(d, V, 1)$. If we prove that the set

$$W(\gamma(0)) = \{\omega \in T_{\gamma(0)} M;\ d(\exp_{\gamma(0)} \omega, \gamma(0)) = \|\omega\|\}$$

is contained in

$$(W(\gamma(0)) - \mathscr{A}_{d,\theta}(\gamma'(0)) \cup \mathscr{A}_{r,\theta}(\gamma'(0)),$$

then it will follow from Rauch Comparison theorem that

$$\begin{aligned} v(M) &= v(\exp_{\gamma(0)}(W(\gamma(0)) - \mathscr{A}_{d,\theta}(\gamma'(0)))) + \\ &\quad + v(\exp_{\gamma(0)}(W(\gamma(0)) \cap \mathscr{A}_{r,\theta}(\gamma'(0)))) \leq \\ &\leq v(\exp_0(D_d(0) - \mathscr{A}_{d,\theta}(u))) + v(\exp_0(\mathscr{A}_{r,\theta}(u))) = \\ &= V_1 + V_2 = V. \end{aligned}$$

We now prove the inclusion mentioned above. Let $w \in W(\gamma(0))$; since $\mathit{diam}\, M \leq d$, it follows that

$$w \in W(\gamma(0)) - \mathscr{A}_{d,\theta}(\gamma'(0)) \quad \text{or} \quad w \in \mathscr{A}_{d,\theta}(\gamma'(0)).$$

If $w \in \mathscr{A}_{d,\theta}(\gamma'(0))$, then

$$\exp_{\gamma(0)} t \frac{w}{\| w \|}$$

is not minimal for $t \geq r$. In fact, suppose it is minimal up to $r$; then it follows from Toponogov's theorem [5] and the fact that $c_n(d, V, 1)$ satisfies (3) and $c_n(d, V, 1) \leq \pi$ that

$$d(\exp_{\gamma(0)} r \frac{w}{\| w \|}, \gamma(L)) < r.$$

Since $\gamma$ is a closed geodesic,

$$d(\exp_{\gamma(0)} r \frac{w}{\| w \|}, \gamma(0)) < r$$

i.e. $\exp_{\gamma(0)} t \dfrac{w}{\| w \|}$ is not minimal for $t \geq r$.

Hence, $w \in \mathscr{A}_{r,\theta}(\gamma'(0))$, which completes the proof.

Theorem 3 is closely related to a result of C. Heim [6].

§3. Before proving theorems 1 and 2, we remark that if $M$ is a riemannian manifold such that $\delta \leq K_M \leq 1$, we can multiply the metric by $\delta$, so that in the new metric $1 \leq K_M \leq \frac{1}{\delta}$. Hence theorem 1 is equivalent to the following.

THEOREM 1. *Let $M$ be a complete, 3-dimensional riemannian manifold such that $1 \leq K_M \leq \frac{1}{\delta}$, $\delta \leq \frac{4}{25}$, diam $M \leq \frac{\pi}{2}$ and $v(M) > \frac{9}{20}\, v(S^3)$. Then every closed geodesic on $M$ has length $> 2\pi\sqrt{\delta}$.*

PROOF. It follows from theorem 3. We initially remark that in theorem 3 when $d = \frac{\pi}{2}$, equations (1) and (2) are respectively equal to

$$\text{(3)} \qquad \frac{v(S^n)}{2} - 2\int \dots \int \sin^{n-1} \alpha_1 \sin^{n-2} \alpha_2 \dots \sin \alpha_{n-1}\, d\alpha_1 \dots d\alpha_n = V_1$$

where $0 \leq \alpha_1 \leq \frac{\pi}{2}$, $0 \leq \alpha_2 \leq \theta$, $0 \leq \alpha_i \leq \pi$, $i = 3, \dots, n-1$, $0 \leq \alpha_n \leq 2\pi$, and

$$\text{(4)} \qquad \int_0^r \sin^{n-1} \alpha_1\, d\alpha_1 = \frac{V_2}{\dfrac{v(S^n)}{2} - V_1} \int_0^{\frac{\pi}{2}} \sin^{n-1} \alpha_1\, d\alpha_1.$$

Let $V_1$ and $V_2$ be respectively equal to 3/5 and 2/5 of the lower bound of $v(M)$, i.e.

$$V_1 = \frac{27}{100}\, v(S^3) \quad \text{and} \quad V_2 = \frac{9}{50}\, v(S^3).$$

We obtain $0 < \theta < \frac{\pi}{2}$ from equation (3), which gives

$$\text{(5)} \qquad \cos\theta = \frac{27}{50}$$

we get $0 < r < \frac{\pi}{2}$ from equation (4) i.e.

$$\int_0^r \sin^2 \alpha_1\, d\alpha_1 = \frac{18}{23}\, \frac{\pi}{4}.$$

It is not difficult to see that $r$ satisfies

$$\tan r > 5.70037.$$



We conclude using (5) and the hypothesis on $\delta$, that

$$2 \tan^{-1} (\cos\theta \cdot \tan r) > 2 \cdot \frac{2}{5}\, \pi > 2\sqrt{\delta}\, \pi.$$

Finally, using Theorem 3, we get that every closed geodesic on $M$ has length $> 2\pi\sqrt{\delta}$. q.e.d.

In a similar way we can prove Theorem 2, which is equivalent to the following:

THEOREM 2. *Let $M$ be a complete, 5-dimensional riemannian manifold such that $1 \leq K_M \leq \frac{1}{\delta}$, $\delta \leq (\frac{23}{60})^2$, diam $M \leq \frac{\pi}{2}$ and $v(M) > \frac{9}{20}\, v(S^5)$. Then every closed geodesic on $M$ has length $> 2\pi\sqrt{\delta}$.*

REMARKS. 1. Results analogous to the above theorems can be obtained for higher dimensions.

2. If $M$ is an $n$-dimensional, complete riemannian manifold, given $0 < \delta < \frac{1}{4}$ it would be interesting to find a function $f(\delta)$, such that if

$$\delta \leq K_M \leq 1, \quad \text{diam } M \leq \frac{\pi}{2\sqrt{\delta}} \quad \text{and} \quad v(M) > f(\delta)\, v(S^n_\delta),$$

then every closed geodesic on $M$ has length $> 2\pi$. Clearly this function will depend on the dimension $n$. In the three dimensional case, when

$$0{,}016 \leq \delta < \frac{1}{4}, \quad f(\delta) = \frac{\sqrt[4]{4\delta}}{2}$$

could be such a function.

3. Suppose $M$ is a simply connected, riemannian manifold satisfying the conditions of theorem 1, with

$$\delta = \frac{4}{25}, \quad \text{and hence} \quad \text{diam } M \leq \frac{5\pi}{4}.$$

Let $B_r(p) = \{m \in M;\ d(m, p) < r\}$. Is it possible to obtain $M = B_r(p) \cup B_r(q)$, where $p, q \in M$ and $r < \pi$? If the answer is affirmative, then $M$ is homeomorphic to a sphere. Similarly, one may ask an analogous question where $M$ satisfies theorem 2 with $\delta = (\frac{23}{60})^2$.

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