"Threshold of Singularity" for an Equation of non Linear Evolution*

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1. Introduction. We consider here a master equation, connected with the Theory of Turbulence. The master equation is referred to an historical introduction in the physical thesis of M. Lesieur, in which he and U. Frisch give a markovianized version of the Kraichnan's Random Coupling Model (MRCM).

When the MRCM is applied to the Bürgers equation, we obtain the master equation:

$$\frac{\partial u}{\partial t} + \frac{\partial^2}{\partial x^2}\{(u(.\,,0) - u)^2\} = 0. \tag{1.1}$$

(One can also apply the MRCM to three dimensional homogeneous isotropic turbulence with or without "helicity"; see U. Firsch and M. Lesieur).

It is known in the "inviscid" limit that the Bürgers model leads to the formation of shocks after a finite time. In the same way, let us mention Onsager's conjecture (1949), that Euler's equation for perfect incompressible fluid can have "turbulent" irregular solutions for which the energy is not conserved (see also M. Lesieur and U. Frisch).

In order to prove this result on equation (1.1), we study the regularity of solutions and we establish the existence of a "limit" of regularity, or better a "threshold" of singularity. We wish to call the attention of the reader for such an important particular property: the master equation is essentially parabolic in the cone of positive-type functions, but it necessarily generates irregular solutions, after a finite time, and it happens even if we have smooth initial data.

In formulating this property we first prove a negative theorem. Then we divide this paper into several theorems giving a result of existence

*Recebido pela SBM em 21 de fevereiro de 1975.

of a solution to equation (1.1) and results of regularity of solutions. We conclude showing some numerical results.

All results presented here constitute part of the author's doctoral dissertation and part of a work of C. M. Brauner, R. Teman and the author.

2. Non-existence of regular solutions

We want to rewrite equation (1.1) in a more convenient form which we will use:

$$\frac{\partial u}{\partial t}(t,x) - \frac{\partial}{\partial x}\alpha(u)(t,x)\frac{\partial u}{\partial x}(t,x) = 0 \tag{2.1}$$

where

$$\alpha(u)(t,x) = 2(u(t,0) - u(t,x)) \tag{2.2}$$

The positive-type of u insures that $\alpha(u)(t,.) \geq 0$ for all $t \geq 0$, u attaining its maximum at $x = 0$.

We look for a function u defined in $[0, \infty[\times \mathbb{R}$, solution to equation (2.1) and subject to the following initial and boundary conditions

$$\begin{aligned} &u(.,x) \longrightarrow 0 \quad \text{when} \quad x \longrightarrow \pm\infty \\ &u(0,.) = u_0 \end{aligned} \tag{2.3}$$

THEOREM 2.1. *The master equation does not have solutions of positive-type in* $C^\infty([0, \infty[\times \mathbb{R})$, *nor in* $C^4([0, \infty[\times \mathbb{R})$ *either.*

PROOF. We assume that the master equation has a solution u in $C^4([0, \infty[\times \mathbb{R})$. Let us differentiate equation (2.1) with respect to x, we have

$$\frac{\partial^3 u}{\partial t \partial x^2} - \alpha(u)\frac{\partial^4 u}{\partial x^4} + 8\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + 6\left(\frac{\partial^2 u}{\partial x^2}\right)^2 = 0 \tag{2.4}$$

and let us denote $\frac{\partial^2 u}{\partial x^2}(t,0)$ by $D(t)$.

Then, u being a positive-type function,

$$\frac{d}{dt}D(t) + 6D^2(t) = 0. \tag{2.5}$$

Denoting the initial condition by D_0 ($D_0 < 0$), one can easily verify that

$$D(t) = \frac{D_0}{1 + 6D_0 t} \tag{2.6}$$

$$D(t) \longrightarrow -\infty \quad \text{as} \quad t \longrightarrow -\frac{1}{6D_0}. \tag{2.7}$$

This contradicts the assumption for regular solutions.

3. A result on existence of solutions

THEOREM 3.1. *For all positive-type initial data* u_0 *given in* $H^1(\mathbb{R})$, *there exists a solution* u *to the problem* (2.1) (2.3) *such that*

(3.1) $u(t,.)$ *has the positive-type for all* $t \geq 0$,

(3.2) $u \in L^\infty([0, \infty[; H^1(\mathbb{R}))$,

(3.3) $\Phi_0(u) = \alpha(u)\frac{\partial u}{\partial x}$ *and* $\Phi_1(u) = \alpha(u)\frac{\partial^2 u}{\partial x^2} \in L^2([0, \infty[\times \mathbb{R})$.

Under the assumption that $\frac{du_0}{dx} \in L^\infty(\mathbb{R})$, *we also have:*

(3.4) $\frac{\partial u}{\partial x} \in L^\infty([0, \infty[\times \mathbb{R})$,

(3.5) $\Phi_0(u) \in L^2([0, \infty[; H^1(\mathbb{R})) \cap L^\infty([0, \infty[\times \mathbb{R})$,

(3.6) $\frac{\partial u}{\partial t} \in L^2([0, \infty[\times \mathbb{R})$.

For a proof we refer the reader to [6] [7] Theorems I.1, II.1, III.1 and IV.2. So, if we denote by u_ν the solution to problem (2.1) (2.3) with viscosity,

$$\frac{\partial u_\nu}{\partial t} - \frac{\partial}{\partial x}\alpha(u_\nu)\frac{\partial u_\nu}{\partial x} = \nu\frac{\partial^2 u_\nu}{\partial x^2} \tag{3.7}$$

we can observe that:

There exist a sequence $\nu' \longrightarrow 0$ and a function u solution to problem (2.1) (2.3) such that

(3.8) $u_{\nu'} \longrightarrow u$ in $L^\infty([0, \infty[; L^2(\mathbb{R}))$ and $L^\infty([0, \infty[\times \mathbb{R})$ weak-star;

(3.9) $\dfrac{\partial u_{\nu'}}{\partial x} \longrightarrow \dfrac{\partial u}{\partial x}$ in $L^\infty([0, \infty[; L^2(\mathbb{R}))$ and $L^\infty([0, \infty[\times \mathbb{R})$ weak-star.

4. "Threshold" of singularity

THEOREM 4.1. *Let u be a solution to problem (2.1) (2.3), given by Theorem 3.1. Besides let us assume that* $u_0 \in H^2(\mathbb{R})$ *and that the Fourier transform in x,* $\mathscr{F}\dfrac{d^2u_0}{dx^2} \in L^1(\mathbb{R})$. *Then there exists* $t_* > 0$ *such that*

$$u \in L^\infty([0, t_*[; H^2(\mathbb{R})). \tag{4.1}$$

In order to prove this result, we shall first assume that u_0 belongs to $H^4(\mathbb{R})$. We shall prove a lemma which is very useful later on.

LEMMA 4.1. *Let* u_ν *be the solution to problem (3.7). Let c be a real parameter such that* $0 < c < -\dfrac{1}{\dfrac{d^2u_0}{dx^2}(0)}$. *Then there exists* $t_* = t_*(u_0, c)$ *such that*

$$0 > \frac{\partial^2 u_\nu}{\partial x^2}(t, 0) > -\frac{1}{c} \text{ for all } t,\ 0 \le t \le t_*. \tag{4.2}$$

PROOF. Denoting $\dfrac{\partial^2 u_\nu}{\partial x^2}$ by $u_\nu^{(2)}$ and $\dfrac{\partial^4 u_\nu}{\partial x^4}$ by $u_\nu^{(4)}$, we have

$$\frac{d}{dt}u_\nu^{(2)}(.\,,0) + 6\{u_\nu^{(2)}(.\,,0)\}^2 - \nu u_\nu^{(4)}(.\,,0) = 0 \tag{4.3}$$

$$u_\nu^{(2)}(0,0) = \frac{d^2u_0}{dx^2}(0) < 0. \tag{4.4}$$

By using the positive-type of u_ν, it follows from (4.3) that

$$\frac{-1}{u_\nu^{(2)}(t,0)} > \frac{-1}{\dfrac{d^2u_0}{dx^2}} - 6t \tag{4.5}$$

Hence we conclude (4.2) for all $t \le t_* = \dfrac{1}{6}\left\{-\dfrac{1}{\dfrac{d^2u_0}{dx^2}(0)} - c\right\}$.

By means of results on regularity of u_ν (see [7] chap. 3), we can write the successive derived equations for $u_\nu^{(2)}, u_\nu^{(3)}, u_\nu^{(4)}$. We can justify particularly (4.3) (4.4) and besides we can prove the following à-priori estimates:

(4.6) $u_\nu^{(2)}, u_\nu^{(3)}, u_\nu^{(4)}$ remain in bounded sets of $L^\infty([0, t_*]; L^2(\mathbb{R}))$;

(4.7) $\nu^{\frac{1}{2}} u_\nu^{(4)}$ remains in a bounded set of $L^2([0, t_*]; C^0(\mathbb{R}))$.

For example we have

$$\frac{\partial u_\nu^{(2)}}{\partial t} - \frac{\partial}{\partial x}\left[(\nu + \alpha(u_\nu))\frac{\partial u_\nu^{(2)}}{\partial x}\right] + 6\frac{\partial}{\partial x}[u_\nu^{(1)} u_\nu^{(2)}] = 0. \tag{4.8}$$

We multiply (4.8) by $u_\nu^{(2)}$ and we integrate over $\mathbb{R}$: (denoting by $|\cdot|$ the norm in $L^2(\mathbb{R})$),

$$\frac{1}{2}\cdot\frac{d}{dt}|u_\nu^{(2)}(t)|^2 + \int_{\mathbb{R}} (\nu + \alpha(u_\nu)(t,x))\left(\frac{\partial u_\nu^{(2)}}{\partial x}(t,x)\right)^2 dx + 3\int_{\mathbb{R}} (u_\nu^{(2)}(t,x))^3\, dx = 0. \tag{4.9}$$

Because of the negative-type of $u_\nu^{(2)}$, we have

$$u_\nu^{(2)}(t,x) \ge u_\nu^{(2)}(t,0) \text{ for all } x, \text{ for all } t \ge 0. \tag{4.10}$$

Then

$$\frac{d}{dt}|u_\nu^{(2)}(t)|^2 + 6u_\nu^{(2)}(t,0)\cdot|u_\nu^{(2)}(t)|^2 \le 0. \tag{4.11}$$

It follows from Lemma 4.1 and from Growall's Lemma:

$$|u_\nu^{(2)}(t)|^2 \le \left|\frac{d^2u_0}{dx^2}\right|^2 \cdot exp[-6\int_0^t u_\nu^{(2)}(\sigma,0)\,d\sigma] \text{ for all } t \ge 0$$
$$\le \left|\frac{d^2u_0}{dx^2}\right|^2 \cdot exp\left[\frac{6t_*}{c}\right] \text{ for } t \le t_*. \tag{4.12}$$

REMARK 4.1. We have also the inequality (cf. (4.9))

$$\int_0^{t_*}\int_{\mathbb{R}} (\nu + \alpha(u_\nu)(t,x))(u_\nu^{(3)}(t,x))^2 dx dt \le \frac{1}{2}\left|\frac{d^2u_0}{dx^2}\right|^2\cdot\left(1 + \frac{6}{c}exp\left[\frac{6t_*}{c}\right]\right).$$

We refer to [7] for the other a-priori estimates.

Now, we conclude the proof of Theorem 4.1:

Taking $\eta > 0$ appropriately, under the assumption $u_{0\eta}$ of positive-type in $H^4(\mathbb{R})$, we get

$$(4.13) \qquad 0 > u_{\nu\eta}^{(2)}(t, 0) > -\frac{1}{c}$$

for all t such that

$$0 \leq t \leq \tilde{t}_* = \frac{1}{6} \left| -\frac{1}{\frac{d^2 u_{0\eta}}{dx^2}(0)} - c \right|$$

and therefore by (4.12),

$$(4.14) \qquad |u_{\nu\eta}^{(2)}(t)|^2 \leq |u_{0\eta}^{(2)}|^2 \, exp\left[\frac{6\tilde{t}_*}{c}\right], \quad 0 \leq t \leq \tilde{t}_* .$$

We choose $u_{0\eta}$ such that, as $\eta \longrightarrow 0$

$$(4.15) \qquad u_{0\eta} \longrightarrow u_0 \quad \text{in} \quad H^2(\mathbb{R});$$

$$(4.16) \qquad \mathscr{F} u_{0\eta}^{(2)} \longrightarrow \mathscr{F} u_0^{(2)} \quad \text{in} \quad L^1(\mathbb{R}).$$

For instance we take $u_{0\eta} = \varphi_\eta * u_0$ where $\varphi_\eta(x) = \frac{1}{\eta} exp\left[-\frac{x^2}{\eta^2}\right]$.

Then by means of Lebesgue's theorem (4.16) is obvious. Indeed $u_{0\eta}$ has the positive-type and $|\mathscr{F} u_{0\eta}^{(2)}|_{L^1(\mathbb{R})} = -u_{0\eta}^{(2)}(0)$. Then (4.14) holds also with $u_{\nu\eta}^{(2)}$ replaced by $u_\nu^{(2)}$, for all t such that $0 \leq t \leq t_* = \frac{1}{6}\left[\frac{1}{|\mathscr{F} u_0^{(2)}|_{L^1}} - c\right]$.

The result follows as $\nu \longrightarrow 0$.

5. Regularity C^∞ in $[0, T_*] \times \mathbb{R}$

THEOREM 5.1. *If u_0 belongs to $C_0^\infty(\mathbb{R})$, the solutions u to problem* (2.1) (2.3) *verify*:

$$(5.1) \qquad u \in C_0^\infty([0, T_*] \times \mathbb{R})$$

where

$$(5.2) \qquad T_* = \frac{-1}{6\frac{d^2 u_0}{dx^2}(0)}.$$

PROOF. The proof extends the results of the kind of (4.6) (4.7) and Remark (4.1), for all derived equations.

As $\nu \longrightarrow 0$, if $t_* = t_*(u_0, c)$ is given by Lemma 4.1, we can obtain for all integer $m \geq 4$:

(5.3) $u_\nu^{(m)} = \frac{\partial^m u_\nu}{\partial x^m}$ remains in a bounded set of $L^\infty([0, t_*]; L^2(\mathbb{R}))$;

(5.4) $\Phi_m(u_\nu) = \alpha(u_\nu)\frac{\partial u_\nu^{(m)}}{\partial x}$ and $\frac{\partial u_\nu^{(m)}}{\partial t}$ remain in bounded sets of $L^2([0, t_*] \times \mathbb{R})$.

We induce the regularity at the limit $\nu = 0$ by an iterative process in the successive derived forms of equation (2.1).

REMARK 5.1. One can easily verify the uniqueness of solution to problem (2.1) (2.3) in $[0, T_*[\times \mathbb{R}$.

REMARK 5.2. One can write an asymptotic development (see [7])

$$(5.5) \qquad u(t, x) = u(t, 0) + 2x^2 u^{(2)}(t, 0) + 0(x^4) \qquad 0 \leq t < T_* .$$

REMARK 5.3. Let u be a solution to problem (2.1) (2.3) under the assumptions of Theorem 3.1. Then for a.e. $t \geq 0$,

$$(5.6) \qquad \frac{\partial u}{\partial x}(t, .) \text{ is continuous in } \mathbb{R} - \{0\}$$

and we can obtain the following equation which precises this discontinuity at $x = 0$:

$$(5.7) \qquad \frac{\partial}{\partial t}[u(t, 0)] + 2\left[\frac{\partial u}{\partial x}(t, 0^+)\right]^2 = 0.$$

As long as u is differentiable at $x = 0$, therefore we have $\frac{\partial u}{\partial x}(t, 0) = 0$ and $\frac{\partial u}{\partial t}(t, 0) = 0$ for $0 \le t < T_*$.

6. Numerical results

Let u be a solution to the master equation; u is a function of covariance (see the model MRCM) and therefore its value at $x = 0$ exhibits a term of energy. Then we can interpret the results considering the energy: the energy is conserved for the differentiable solutions, in the interval $(0, T_*)$. Afterwards the energy becomes dissipated (see (5.7) and Fig. 1). The notion of "catastrophe of energy" is connected with the effective onset of turbulence.

The problem (2.1) (2.3) with $u_0(x) = exp[-\frac{x^2}{2}]$ has been discretised (with a variable step in space) and computed in the Laboratory of Dynamic Meteorology (ENS-Paris). We observe $T_* = 1/12$ and the singular point at $x = 0$.

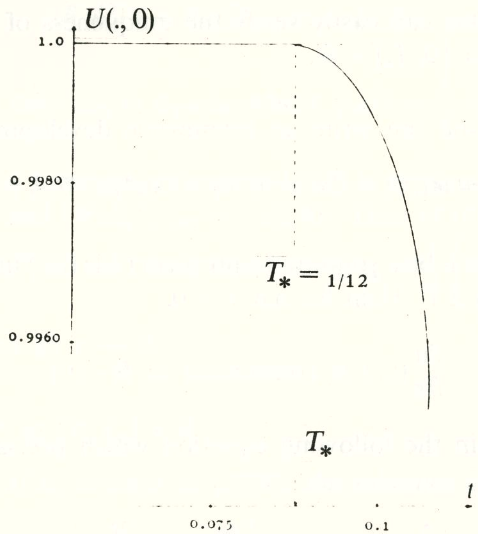

Fig. 1. Conservation of energy and "Catastrophe" of energy.

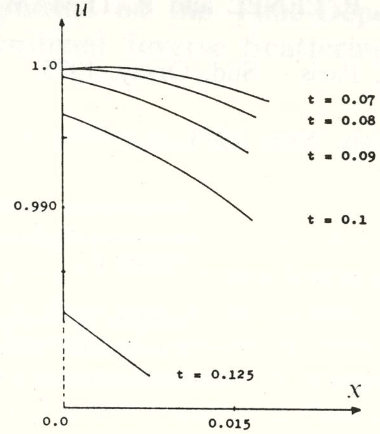

Fig. 2. Evolution of u solution to (2.1) (2.3), for $x \le 0.015$.

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Notes added in proof.

In relation to chap. 4 and remark 5.3, results on the energy dissipation have been obtained in:

C. FOIAS and P. PENEL, C.R.A.S., Paris, 280, série A, 1975, p. 629.

C. BARDOS, P. PENEL, U. FRISCH and P. L. SULEM, Modified dissipativity for a non linear evolution equation arising in turbulence (to appear).