Recent Developments on the Time-Dependent Approach to the three Dimensional Inverse Scattering Problem*

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I. Introduction. We shall present here recent developments in the so-called time-dependent approach for the Inverse Scattering Problem (ISP) Roughy speaking this ISP deals with all the possible information that we can obtain about the dynamics of a system by knowing its asymptotic behavior. By now, we are more interested in examples rather than abstract theorems, thus we will restrict ourselves in these notes to a pair of important models in mathematical-physics, namely

(1) $$\Box u + q(x)u = 0 \quad \text{in} \quad \Omega = \mathbb{R}^3, \quad -\infty < t < \infty$$

$\left(\Box = \dfrac{\partial^2}{\partial t^2} - \Delta,\ \Delta = \sum_{j=1}^{3} \dfrac{\partial^2}{\partial x_j^2}\right)$, where $q(x)$ denotes the potencial energy and $u(x,t)$ the state of the system.

And, the nonlinear wave equation

(2) $$\Box u + q(x)u^3 = 0 \quad \text{in} \quad \Omega = \mathbb{R}^3, \quad -\infty < t < \infty.$$

As we mention above, our main interest here is to use the time-dependent approach to show that the scattering operator S (one for each equation (1) and (2) respectively) determines uniquely the scatterer, at least when $q(x)$ satisfies certain reasonable conditions at infinity which we will specify later.

Before concluding this introduction we would like to make a few comments on the literature. First, in the case in which $q(x)$ is spherically symmetric then the ISP for equation (1) reduces to one-dimensional case and has been intensely studied through a stationary approach by using the methods of Gel'fand-Levitan-Marchenko, see [3]. In 1955, J. Berezanskii [2] gave a uniqueness result, through a stationary method, for the three-

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dimensional (and two-dimensional) Schrödinger equation, i.e., $iu_t = \Delta u + q(x)u$, although $q(x)$ it was assumed to be C^2 and with compact support. In [11], T. Schonbek gives a result on inverse scattering for the Klein-Gordon equation (i.e., $\Box u + m^2 u + q(x)u = 0 \quad m > 0$) concluding that for suitably small potentials $q_1(x)$, $q_2(x)$ there is uniqueness of the inverse problem provided that $q_1(x) - q_2(x)$ is either non-negative or non-positive (for all x's) with compact support. Recently L. D. Faddeev [4] outlined a generalization of the Gel'fand-Levitan method to the three-dimensional Schöedinger equation. (See also [7].) In [8] ([9]) G. Perla Menzala presents a uniqueness result concerning equation (1) and a sketch of his method will be shown in (II).

In what concerns equation (2) W. Strauss in [12] presented the scattering properties for the case when $q(x)$ is "small at infinity" and again W. Strauss in [13] gives an abstract time-dependent approach for the inverse (nonlinear) scattering problem from which equation (2) is a particular case. In (III) we will sketch his method. It is perhaps surprising, as W. Strauss pointed out in [13], that the nonlinear ISP is much easier than the corresponding linear one.

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II. The linear case. We consider the initial value problem

(II.1) $$\Box u + q(x)u = 0$$

in $\Omega = \mathbb{R}^3, -\infty < t < \infty$ with C^∞-data of compact support (i.e. $C_0^\infty(\mathbb{R}^3)$) at $t = 0$. We assume that the real-valued function $q(x)$ is non-negative and, in order to simplify our arguments we will assume that $q(x) = 0(|x|^{-3-\varepsilon})$ as $|x|$ approaches infinity ($\varepsilon > 0$). Furthermore we assume that $q(x)$ is continuous.

From now on we shall refer to solutions u of

(II.2) $$\Box u = 0$$

in $\Omega = R^3, -\infty < t < \infty$ with C_0^∞-data at $t = 0$ as *free solutions.* In the space of such free solutions we define the norm (energy) $\| \ \|_{E_0}$ by

$$\|u\|_{E_0}^2 = \frac{1}{2}\int_{\mathbb{R}_3} [|grad_x u|^2 + |u_t|^2]\, dx$$

where $grad_x u$ denotes the gradient of u (in x) and

$$|grad_x u|^2 = \sum_{j=1}^{3} |u_{x_j}|^2.$$

In the space of solutions of (I.1) with such initial data at $t = 0$ we define the (total) energy of u as

$$\|u\|_E^2 = \frac{1}{2}\int_{\mathbb{R}_3} [|grad_x u|^2 + |u_t|^2 + q(x)|u|^2]\, dx.$$

It is not difficult to show that $\| \ \|_{E_0}$ and $\| \ \|_E$ are constant in t i.e. we are dealing with two conservative equations. A much more deep result is the following: For each solution u of (II.1) with C_0^∞-data at $t=0$ there exists a unique pair $u_\pm$ of free solutions wuch that

$$\|u - u_\pm\|_E \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty.$$

Furthermore, the operator which relates $u_- \longrightarrow u_+$ is unitary. Such operator is called the scattering operator associated with (II.1) and denote by $S = S(q)$.

REMARK. The existence and unitarity of S can be proved under much weaker conditions on $q(x)$. See T. Kato [5] and S. Agmon [1] for example. As pointed out (and proved) in [8] for our needs it is better to write S as

(II.3) $$Su_+(x,t) = u_-(x,t) + \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} R(x-y, t-s)q(y)u(y,s)dyds$$

where $u_\pm$ are the corresponding free solutions for u (solution of (II.1)), $x \in \mathbb{R}^3$, $t \in \mathbb{R}$ and R denotes the Riemann function associated with (II.2).

In the special case in which $S = I$ = identity operator then (II.3) gives us

(II.4) $$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} R(x-y, t-s)q(y)u(y,s)dyds = 0$$

for each "perturbed" solution u and all $x \in \mathbb{R}^3$, $t \in \mathbb{R}$.

From (II.4) we would like to show that $q \equiv 0$, however some technical difficulties arose and we had to use classical limiting process together with the energy method. In fact, we can write

$$u(y, s) = u_-(y, s) + P u(y, s) \tag{II.5}$$

where

$$Pu(y, s) = -\frac{1}{4\pi} \int_{\mathbb{R}_3} |\xi|^{-1} q(y + \xi) u(y + \xi, s - |\xi|) d\xi.$$

We state now a lemma which was proved in [8]

LEMMA II.1. *Let u be a solution of (II.1) with C_0^∞-data at $t = 0$ then*

a) $$\int_{Ch} q(x) |u(x, t)|^2 \, dS_{Ch} \leq const. \| u \|_E^2$$

where Ch denotes any characteristic cone and dS_{Ch} the surface measure on Ch, and

b) $$\sup_{y,s} |Pu(y, s)| \leq const \| q \|_{L^1}^{1/3} \| q \|_{L^\infty}^{1/3} \| u \|_E .$$

Now, by substituting (II.5) in (II.4) and using the above lemma we get

$$\| q \| \leq const \| q \|_{L^1}^{5/6} \| q \|_{L^\infty}^{2/3} \| u \|_E \tag{II.6}$$

where

$$\| q \| = \sup_{\substack{\|u_-\|_{E_0} = 1 \\ x, t}} \left| \int_{-\infty}^{+\infty} \int_{\mathbb{R}_3} R(x - y, t - s) q(y) u_-(y, s) dy ds \right|$$

It is really shown by using the continuity of $q(x)$ that $\| \;\; \|$ is a norm (See [8]).

Let us choose a solution u (of II.1) such that $\| u \|_E = 1$ and let us substitute q by εq ($\varepsilon > 0$) in (II.6) therefore we would obtain that the ratio

$$\| \varepsilon q \| / \| \varepsilon q \|_{L^1}^{5/6} \| \varepsilon q \|_{L^\infty}^{2/3}$$

is bounded by a positive constant. But as $\varepsilon \longrightarrow 0$ this cannot be possible unless $q \equiv 0$. This result can be summarized in the

THEOREM II.2. *Let $q(x)$ satisfying all hypotheses given in the beginning of this section. If $q \not\equiv 0$ then there exists a positive number ε_0 such that for $0 < \varepsilon < \varepsilon_0^2$ we must have $S(\varepsilon q) \neq I = $ identity. Here $S(\varepsilon q)$ denotes the scattering operator associated with $\Box u + \varepsilon q u = 0$.*

REMARKS

1) The case in which we have two spatial potentials $q_1(x)$, $q_2(x)$ can be treated in a similar way. The reason is not because the scattering operator depends linearly on the potentials which in general is *false* but because (II.3) and the way that our method was carried out.

2) The method that we describe here for the ISP for small potentials, seems to work for many other equations. For example, we have carried out the computations for the Klein-Gordon equation with an external potential in $\mathbb{R}^3$ and it still works, (See [10]). This also improves Schonbek's result, [11].

III. The nonlinear case. In this section we consider classical solutions of

$$\Box u + q(x) u^3 = 0 \tag{III.1}$$

in $\Omega = \mathbb{R}^3$, $-\infty < t < \infty$ which have initial data C_0^∞ at $t = 0$. In order to garantee the existence of the scattering operator S we assume for example that $0 \leq q(x) \leq const |x|^{-1-\varepsilon}$, $q \in L^1(\mathbb{R}^3)$ and that $q(x)$ has first and second derivatives in $L^\infty(\mathbb{R}^3)$. The following argument that we will use to solve the inverse problem will require also the continuity of $q(x)$, however, by using Fourier transformation, $q(x)$ can be determined provided it is integrable.

Let u and v solutions of (II.1) with C_0^∞-data at $t = 0$. We consider the bilinear form

$$W(u, v)(t) = \int_{\mathbb{R}^3} (u_t v - u v_t) dx$$

Formally, we differentiate with respect to t and use of (III.1) give us

$$\frac{d}{dt} W(u, v)(t) = \int_{\mathbb{R}^3} q(x)(u v^3 - u^3 v) dx \tag{III.2}$$

Integration of (III.2) from $-T$ to T gives

(III.3) $$W(u,v)(T) - W(u,v)(-T) = \int_{-T}^{T}\int_{\mathbb{R}^3} q(x)(uv^3 - u^3v)dxdt$$

Observe that

$$W(u,v)(\pm T) - W(u_\pm, v_\pm)(\pm T) = W(u-u_\pm, v_\pm)(\pm T) + \\ + W(u-u_\pm, v_\pm)(\pm T) + W(u_\pm, v-v_\pm)(\pm T)$$

and because each term on the right hand side tends to zero as $T \longrightarrow \pm\infty$ then

(III.4) $$W(u,v)(\pm T) - W(u_\pm, v_\pm)(\pm T) \longrightarrow 0 \quad \text{as} \quad T \longrightarrow \pm\infty$$

Letting $T \longrightarrow \infty$ in (III.3) and using (III.4) we get

(III.5) $$\lim_{T\to\infty} W(u_+, v_+)(T) - W(u_-, v_-)(T) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} q(x)(uv^3 - u^3v)dxdt.$$

The right side converges because of the left side. In fact, to see this it is enough to observe the bilinear form W is invariant under free solutions, because as easily shown $\frac{d}{dt} W(u_0, v_0)(t) = 0$ if u_0, v_0 are solutions of $\Box u = 0$ with C_0^∞-data at $t = 0$. Thus (III.5) give us

(III.6) $$W(u_+, v_+)(0) - W(u_-, v_-)(0) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} q(x)(uv^3 - u^3v)dxdt.$$

For solutions of (II.1) with C_0^∞-data at $t = 0$ we consider the norm

$$|||u||| = \sup_{-\infty<t<\infty} [\|u(\,,t)\|_E + (1 + |t|)\|u(\cdot,t)\|_{L^\infty}].$$

We can write $u = u_- + Pu$, where

$$Pu(y,s) = -\int_{-\infty}^{s}\int_{\mathbb{R}^3} R(y-z, s-r)\, q(z)\, u^3(z,r)\, dzdr$$

i.e. Pu is essentially the Riemann function (R) convoluted with qu^3.

Let φ any nontrivial free solution (in the domain of S) and choose $u_- = \varepsilon\varphi$ and $v_- = 2\varepsilon\varphi$ ($\varepsilon > 0$).

It was shown by W. Strauss [13] that the corresponding perturbed solution u satisfies

a) $|||u||| = O(\varepsilon)$ and b) $|||Pu||| \leq const\, |||u|||^3$

By substituting this on (III.6) we obtain

$$W(S(\varepsilon\varphi), S(2\varepsilon\varphi))(0) - W(\varepsilon\varphi, 2\varepsilon\varphi)(0) = \\ = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} q(x)[(\varepsilon\varphi)(2\varepsilon\varphi)^3 - (\varepsilon\varphi)^3(2\varepsilon\varphi)]dxdt + O(\varepsilon^6)$$

as $\varepsilon \longrightarrow 0$. That is

(III.7) $$\lim_{\varepsilon\to 0} \frac{1}{6\varepsilon^4}[W(s(\varepsilon\varphi), s(2\varepsilon\varphi))(0) - W(\varepsilon\varphi, 2\varepsilon\varphi)(0)] = \\ \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} q(x)\varphi^4(x,t)dxdt = I(\varphi)$$

for all free solutions φ (of $\Box\varphi = 0$) with C_0^∞-data at $t = 0$. Observe that (III.7) shows that the scattering operator S determines

$$I(\varphi) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3} q(x)\,\varphi^4(x,t)dxdt.$$

Now we use the scaling argument to determine $q(x)$ from S. In fact, let $x_0 \in \mathbb{R}^3$ and suppose we would like to determine $q(x_0)$. We choose the free solution φ_λ with the initial datum $\varphi_\lambda(x,0) = g(\lambda(x-x_0))$, where $\lambda > 0$ and g is any nice function (C_0^∞ for example). Also, it is enough to choose $\frac{\partial\varphi_\lambda}{\partial t}(x,0) = 0$. If we change variables

$$y = \lambda(x - x_0), \qquad \lambda > 0$$
$$s = \lambda t$$

and $\Phi(y,s) = \varphi_\lambda(x,t)$ then Φ is the free solution in the new variables with initial datum $\Phi(y,0) = g(y)$. Thus

(III.8) $$I(\varphi_\lambda) = \int_{-\infty}^{\infty}\int_{\mathbb{R}_3} q\left(\frac{y}{\lambda} + x_0\right)\Phi^4(y,s)\frac{dyds}{\lambda^4}$$

As $\lambda \longrightarrow \infty$, the right hand side of (III.8) approaches to

$$q(x_0) \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \Phi^4(y, s) dy ds,$$

which shows that $q(x_0)$ is determined by S and therefore $q(x)$ for any x.

REMARKS

1) The method describe above works also for the nonlinear Schrödinger equation $u_t = i(-\Delta u + q(x)|u|^{p-1} u)$ $x \in \mathbb{R}^n, p =$ integer ≥ 3 ($p > 3$ if $n = 2$, $p > 4$ if $n = 1$) with the appropriate norms and conditions on $q(x)$. See [13]. Recently, C. Morawetz – W. Strauss generalize the method to an interaction term $q(x, \varphi) = \sum_{j \geq 3} q_j(x)\varphi^j$ analytic in a neighborhood of $\varphi = 0$. See [6].

2) Generalizations of the above method to nonlinear symmetric hyperbolic systems

$$u_t = \sum_{j=1}^{n} A_j u_{x_j} + P(u)$$

where $x \in \mathbb{R}^n$, $u(x, t) \in \mathbb{R}^m$, A_j are real symmetric $(m \times m)$ matrices and $P(u)$ is a smooth vector function vanishing to a certain order at $u = 0$, remain, far as we know still open.

3) By using essentially the same idea as in (III), I and Prof. L. A. Medeiros considered the equation $u_t + u_x - u_{xxt} + \gamma u^m u_x = 0$ in $-\infty < x, t < \infty$, $\gamma =$ const. and $m =$ even > 7 and showed that the scattering operator determines uniquely the constant γ.

References

[1] AGMON, S., Actes du congress international des mathematiciens, Nice, France, Vol. 2, 679-683, Guthier-Villars, Paris (1970).

[2] BEREZANSKII, JU., *The uniqueness theorem in the inverse problem of spectral analysis for the Schrödinger equation.* Trans. AMS. Vol. **35**, 167-235.

[3] FADDEEV, L. D., *The inverse problem in the quantum theory of scattering*, J. Math. Phys. Vol. **4**, N.º 1, (1963), 72-104.

[4] FADDEEV, L. D., *Three-dimensional inverse problem in the quantum theory of scattering* (Preprint), ITP-71-106E Kiev (1971)

[5] KATO, T., *Scattering theory with two Hilbert spaces*, J. Funct. Anal., **1**, (1967), 342-369.

[6] MORAWETZ, C., *Notes on time decay and scattering for some hyperbolic problems.* Proceedings of the Reg. Conf. on Exterior Initial boundary value problems for hyperbolic differential equations held at SUNY at Buffalo, June 3-9 (1973).

[7] NEWTON, R., *Notes on scattering theory.* Proceedings of NATO Conference on Scattering Theory (1973), Univ. of Denver, Colorado.

[8] PERLA MENZALA, G., *On the inverse problem for three-dimensional potential scattering*, Ph. D. Thesis dissertation, Brown University (1974).

[9] PERLA MENZALA, G., *On the inverse problem for three-dimensional potential scattering;* J. Diff. Equations (to appear).

[10] PERLA MENZALA, G., *The S-matrix for the Klein-Gordon equation with an external potential* (to appear).

[11] SCHONBEK, T., *On inverse scattering for the Klein-Gordon equation*, Trans. AMS, Vol. **166**, April (1972); 101-123.

[12] STRAUSS, W. A., *Decay of solutions of hyperbolic equations with localized nonlinear terms*, Symposia Mathematica VII, Ist. Naz. Alta Mat., Rome, (1971), 339-355.

[13] STRAUSS, W. A., *Nonlinear scattering theory*, Proceedings of the NATO Advanced Study Institute held at Denver, Colo. USA, June 11-29 (1973).

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