Characterization of Compactness for Symplectic Manifolds*

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Introduction. In this paper we prove the following:

THEOREM. *Let* (M, ω) *be a symplectic manifold,* $\mathfrak{X}_H(M)$ *the Lie algebra of Hamiltonian vectorfields on* M, $C^\infty_H(M)$ *the Lie algebra of* C^∞ *functions on* M *with the Poisson brackets and* $\pi\colon C^\infty_H(M) \longrightarrow \mathfrak{X}_H(M)$ *the symplectic gradient mapping. Then the following statements are equivalent:*

1) M *is compact;*
2) π *has a right inverse in the category of Lie algebras;*
3) $C^\infty_H(M) \neq [C^\infty_H(M), C^\infty_H(M)]$;
4) $H^1(C^\infty_H(M)) \neq \{0\}$.

When $\mathscr{L}$ is a Lie algebra, his commutator $[\mathscr{L}, \mathscr{L}]$ is defined to be the subset of elements which can be expressed as a finite sum of brackets.

The cohomology we consider is the usual Lie algebra cohomology with real coefficients [3], [6].

It is a well known fact [3], [2] that the first cohomology group $H^1(\mathscr{L})$ is the linear space dual to $\mathscr{L}/[\mathscr{L}, \mathscr{L}]$, hence $3 \Leftrightarrow 4$.

The inclusions $1 \Rightarrow 2$ and $1 \Rightarrow 3$ have been proven by Arnold in [2].

We will prove $3 \Rightarrow 1$ in Proposition 4, and $2 \Rightarrow 3$ in Lemma 1.

All our manifolds are supposed to be C^∞, connected and without boundary, unless we mention it explicitly otherwise.

§1. Let us first recall some definitions and facts which can be found in all handbooks about symplectic geometry, e.g. [1].

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DEFINITION. A *symplectic manifold* (M^{2n}, ω) is a C^∞ manifold of dimension $2n$, together with a nondegenerate closed 2-form ω: i.e.

$$d\omega = 0$$
$$\omega^n = \omega \wedge \ldots \wedge \omega \quad \text{is a volume on } M.$$

We denote by α the volume $[(-1)^{n/2}/n!]\ \omega^n$.

By the theorem of Darboux [1] we know that in each point of M we can find a local chart (U, φ) such that $\omega \mid U$ gets the canonical expression

$$\omega \mid U = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

The local charts are called "symplectic charts" and the component functions (x^i, y^i) are called "canonical coordinates".

DEFINITION. A vectorfield X on (M, ω) is called *symplectic* if $L_X\omega = 0$, where L_X is Lie derivation with respect to X.

We denote by $\mathfrak{X}_\omega(M)$ or $\mathfrak{X}_\omega$ the Lie algebra of sympletic vectorfields with the usual Lie brackets for vectorfields

$$[X, Y] = XY - YX$$

DEFINITION. The *symplectic gradient* of $f \in C^\infty(M)$ is the vectorfield X_f with following property

$$\forall\, Y \in \mathfrak{X}^\infty(M)\colon\ \omega(X_f, Y) = df(Y).$$

Let us denote the symplectic gradient mapping by π. Clearly $\pi(C^\infty(M)) \subset \mathscr{X}_\omega$.

DEFINITION. We call a vectorfield *Hamiltonian* if it lies in the image of π and we denote $\pi(C^\infty(M))$ by $\mathscr{X}_H$.

For each $X \in \mathfrak{X}_H$ the elements of $\pi^{-1}(X)$ are called Hamiltonians for X and since M is supposed to be connected, two Hamiltonians for the same vectorfield only differ by a constant.

The Poisson brackets on $C^\infty(M)$ are defined as follows:

$$\{f, g\} = L_{X_f}\, g = -L_{X_g} f$$

It is easy to verify that $\pi(\{f, g\}) = [X_f, X_g]$ which means that π is a Lie algebra morphism.

§2. Before proving Proposition 4 we are now going to present some well known ingredients, under the form of propositions.

PROPOSITION 1. [4]. *If M is an open connected C^∞ manifold, then there exists an infinite sequence $\{K_i\}_{i=0}^{\infty}$ of compact connected C^∞ submanifolds with boundary such that*

$$M = \bigcup_{i=0}^{\infty} K_i \quad \text{and} \quad \forall\, i\colon K_i \subset \mathring{K}_{i+1}.$$

PROPOSITION 2. *Let (M^m, ω) be an open symplectic manifold. Each K_i as in the statement of Proposition 1 can be covered with open balls $\{B_{ij}\}_{i=1}^{m+1}\ {}_{j=1}^{p_i}$, such that each B_{ij} lies in the domain of a symplectic chart and such that $B_{ij_1} \cap B_{ij_2} = \phi$ for all i, j_1, j_2 with $j_1 \neq j_2$ and $1 \leq j_1, j_2 \leq p_i$.*

The proof is trivial if we use a triangulation of K_i [5] subordinated to a finite covering with symplectic charts.

PROPOSITION 3. *Let $I^{2n} \subset \mathbb{R}^{2n}$ denote the open unit cube of $\mathbb{R}^{2n}$, which we endow with the canonical symplectic structure*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

for some coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ of $\mathbb{R}^{2n}$. Then each f with the property

$$\text{support } (f) \subset I^{2n} \quad \text{and} \quad \int f\alpha = 0$$

can be expressed as a finite sum of Poisson brackets

$$f = \sum_{i=1}^{2n} \{g_i, h_i\}$$

where $\forall i$ $supp(\{g_i, h_i\}) \subset I^{2n}$.

The proof of this proposition is a straightforward calculation using the coordinate functions x^i, y^i as g_i and applying the theorem of Fubini.

§3. PROPOSITION 4. *Suppose* (M^{2n}, ω) *is an open symplectic manifold; then* $C^\infty_H(M) = [C^\infty_H(M), C^\infty_H(M)]$.

PROOF. 1) We first decompose each $f \in C^\infty(M)$ in an infinite sum of f_i such that for each f_i, *supp* f_i is compact and

$$\int f_i \ \alpha = 0.$$

We do this by induction on i using the sequence of K_i obtained in Proposition 1:

$$f_1 \quad \text{has properties} \quad f_1 \,|\, K_1 = f,\ f_1 \,|\, K_2^c = 0 \quad \text{and} \quad \int f_1 \ \alpha = 0,$$

where this last condition can be obtained by adapting any f_1' satisfying the first two properties in a small ball lying in $K_2 \backslash K_1$.

By induction, if we have f_i for $1 \le i \le k-1$, then we construct f_k as follows:

$$f_k \,|\, K_k = f - \sum_{i=1}^{k-1} f_i$$

$$f_k \,|\, K_{k+1}^c = 0$$

$$\int f_k \ \alpha = 0$$

Clearly $f = \sum_{i=1}^{\infty} f_i$ and $\forall\, i$: *supp* $f_i \subset K_{i+1} \backslash K_{i-1}$.

2) We now prove that each of these f_i can be decomposed as a finite sum of Poisson brackets.

Therefore we cover $K_{i+1} \backslash K_i$ with open balls as given in Proposition 2.

Using a C^∞ partition of unity subordinated to that covering, we can express f_i as a sum of functions $f_i = \sum_j h_{ij}$ such that each h_{ij} has his support inside an open ball on which ω can be given a canonical form.

Also is it possible to adapt the construction such that all h_{ij} satisfy $\int h_{ij} \ \alpha = 0$. This can f.e. be done by putting all open balls of our covering

in a finite sequence of open balls $(B_1, \ldots, B_p)$ such that

$$\forall\, \alpha \in \{1, \ldots, p-1\}: \ B_\alpha \cap B_{\alpha+1} \ne \phi$$

3) Using Proposition 3, we know that

$$\forall j\colon\ h_{ij} = \sum_{k=1}^{2n} \{l_{ijk}, m_{ijk}\}.$$

Since $\forall\, i$ the supports of the h_{ij} are contained in open balls $\{B_{ij}\}_{i=1}^{2n+1}{}_{j=1}^{p_i}$ such that $B_{ij_1} \cap B_{ij_2} = \phi \ \ \forall\, i, j_1, j_2$ with $j_1 \ne j_2$, we can write:

$$f_i = \sum_{\alpha=1}^{2n(2n+1)} \{u_{i\alpha}, v_{i\alpha}\}.$$

Now

$$f = \sum_{i\ even} \left(\sum_{\alpha=1}^{2n(2n+1)} \{u_{i\alpha}, v_{i\alpha}\} \right) + \sum_{i\ odd} \left(\sum_{\alpha=1}^{2n(2n+1)} \{u_{i\alpha}, v_{i\alpha}\} \right)$$

$$= \sum_{\alpha=1}^{2n(2n+1)} [\{ \sum_{i\ even} u_{i\alpha}, \sum_{i\ even} v_{i\alpha} \} + \{ \sum_{i\ odd} u_{i\alpha}, \sum_{i\ odd} v_{i\alpha} \}]$$

Hence

$$f = \sum_{\beta=1}^{4n(2n+1)} \{w_\beta, z_\beta\}.$$

REMARKS 1. From the previous construction it is clear that in the compact case the commutant of $C^\infty_H(M)$ coincides with the sub-Lie algebra of functions having a zero average value.

The one and only Lie algebra-inverse for π is the mapping $X_f \longmapsto f + t_f$, where $t_f \in \mathbb{R}$ has the property

$$\int (f + t_f)\, \alpha = 0.$$

2. In all cases $\mathfrak{X}_H = [\mathfrak{X}_H, \mathfrak{X}_H]$ or equivalently $H^1(\mathfrak{X}_H) = \{0\}$. (See also [2]).

LEMMA 1. *If* $C^\infty_H(M) = [C^\infty_H(M), C^\infty_H(M)]$, *then* π *has no right inverse in the category of Lie algebras.*

PROOF. Each $f \in C^\infty_H(M)$ can be expressed as a finite sum of brackets.

Hense $f = \sum_{i=1}^{n} \{h_i, g_i\}$ and also $f+1 = \sum_{j=1}^{m} \{u_j, v_j\}$.

So

$$\pi(f) = \sum_{i=1}^{n} [\pi(h_i), \pi(g_i)] = \sum_{i=1}^{n} [X_{h_i}, X_{g_i}] = X_f$$

$$\pi(f+1) = \sum_{j=1}^{m} [\pi(u_j), \pi(v_j)] = \sum_{j=1}^{m} [X_{u_j}, X_{v_j}] = X_{f+1}$$

and $X_f = X_{f+1}$.

Suppose now that κ is a Lie-algebra inverse of π. Then

$$\kappa(X_f) = \kappa\left(\sum_{i=1}^{n} [X_{h_i}, X_{g_i}]\right) = \sum_{i=1}^{n} \{\kappa(X_{h_i}), \kappa(X_{g_i})\} = \sum_{i=1}^{n} \{h_i, g_i\},$$

since $\forall t,s \in \mathbb{R}$ and $\forall \xi, \eta \in C^\infty_H(M)$: $\{\xi + t, \eta + s\} = \{\xi, \eta\}$. This means that necessarily $\kappa(X_f) = f$, but also $\kappa(X_f) = \kappa(X_{f+1}) = f+1$, which contradicts the existence of κ.

§4. For this chapter we refer to the book of Souriau [7] or to the paper of Robbin [6] which is an excellent introduction to this book. Let us recall some facts.

When $\mathscr{L}$ is a Lie algebra and M a manifold, an action of $\mathscr{L}$ on M is a morphism of Lie algebras $\mathscr{L} \xrightarrow{h} \mathfrak{X}(M)$, $A \longmapsto A_M$ such that the evaluation map is smooth.

If we have a symplectic structure on M then we say an action is Hamiltonian if $h(\mathscr{L}) \subset \mathfrak{X}_H$.

In this case there exist linear mappings $J: \mathscr{L} \longrightarrow C^\infty(M)$, $A \longmapsto J_A$ such that $i_A\omega = dJ_A$.

Such a linear mapping is called a "moment" and it is shown in [6] how to associate to $(M, \omega, \mathscr{L}, h)$ a cohomology class $[f] \in H^2(\mathscr{L})$ which vanishes precisely when there exists a moment which is a morphism of Lie algebras. It is also shown that whenever ω is given by an $\mathscr{L}$-invariant 1-form θ (i.e. $\omega = -d\theta$ and $\forall A \in \mathscr{L}$: $L_{A_M}\theta = 0$) then $J: \mathscr{L} \to C^\infty_H(M)$, $A \longmapsto L_{A_M}\theta$ is a Lie algebra morphism. If we now consider the Hamiltonian

action $\mathfrak{X}_H \xrightarrow{id} \mathfrak{X}_H$ for some (M, ω) then the moments for this action are exactly the linear right inverses of π: $C^\infty_H(M) \longrightarrow \mathfrak{X}_H$. Hence using Proposition 4 and Lemma 1 we can state the following proposition.

PROPOSITION 5. *If (M, ω) is an open symplectic manifold, then $H^2(\mathfrak{X}_H) \neq \{0\}$.*

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