A Note on Generated Systems of Sets*

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I - Introduction

The existence of a smallest algebra (σ-algebra) containing a given collection of subsets of a set X is an elementary and well known fact (cf. [3] p. 22). Considering a modified definition of algebra (σ-algebra) (see the definition below) where the unit is not necessary the whole X, the existence of the smallest algebra (σ-algebra) in this generalized sense can not be guaranteed. This fact was noted by A. B. Brown and G. Freilich, in [1], where a necessary and sufficient condition was given as follows:

THEOREM. *Let $\mathscr{S}$ be a collection of subsets of X such that $\bigcup \mathscr{S}\, Z \neq X$. A necessary and sufficient condition for the existence of a smallest σ-algebra* (in the sense of the below definition) *containing $\mathscr{S}$ is the existence of a countable collection $\{S_n\}$, $S_n \in \mathscr{S}$ such that $Z = \bigcup_{n=1}^{\infty} S_n$.*

Note that the condition $Z \neq X$ which is not stated in [1] can not be omitted.

The notations of algebra and σ-algebra are in the theory of quantum probability spaces frequently substituted by that of q-algebra and q-σ-algebra. The last are usually named s-class or σ-class respectively (see [2], [4], [5]).

The difference between the notions q-algebra and q-σ-algebra is in substituting the condition of closedness with respect to unions, by the condition of closedness with respect to disjoint unions (see the definition below). When the q-algebras and q-σ-algebras with a unit different of X are considered, evidently the existence of a smallest one needs not be guaranteed. It seems to be natural that a necessary and sufficient condition for the existence can be obtained if the condition $Z = \bigcup_{n=1}^{\infty} S_n$ in Theorem is substituted by the

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similar one where $\{S_n\}$ are required to be pairwise disjoint. But this is not the case as we show in this note.

II - Notations and notions

DEFINITION. A nonempty collection $\mathscr{A}$ of subsets of a set X is said to be *algebra* if

a) there exists $E \in \mathscr{A}$ such that $E \supset A$ for any $A \in \mathscr{A}$;
b) if $A \in \mathscr{A}$ then $E - A \in \mathscr{A}$;
c) if $A, B \in \mathscr{A}$ then $A \cup B \in \mathscr{A}$.

NOTE 1. E is called the *unit* of $\mathscr{A}$. In the case $E = X$ we get the definition in the usual sense (see [3]).

NOTE 2. To obtain the notion of σ-algebra we substitute c) by the usual condition of countable unions.

NOTE 3. The notions of ring and σ-ring are used in the same sense as in [3].

DEFINITION 2. A nonempty collection $\mathscr{A}$ of subsets of X is said to be a q-*ring* if

a) $A, B \in \mathscr{A}$, $A \subset B$ implies $B - A \in \mathscr{A}$;
b) $A, B \in \mathscr{A}$, $A \cap B = \phi$ implies $A \cup B \in \mathscr{A}$.

If moreover there is $E \in \mathscr{A}$ such tht $A \subset E$ for any $A \in \mathscr{A}$ then is said to be a q-algebra.

NOTE 4. The notions of q-σ-ring and q-σ-algebra are defined in a natural way substituting the conditions of the closedness with respect to finite disjoint unions by that countable disjoint unions in the q-anel and q-algebra respectively.

III - Results

THEOREM 1. *Let $\mathscr{S}$ be a collections of subsets of X such that $\bigcup \mathscr{S} = Z$.*

i) *A sufficient condition for the existence of a smallest q-algebra* (*q*-σ-algebra) *containing* $\mathscr{S}$ *is*

$$Z = \bigcup_{i=1}^{n} S_i \quad \textit{where} \quad S_i \in \mathscr{S},\ i = 1, 2, \dots n,\ S_i \cap S_j = \phi, \quad \textit{if} \quad i \neq j;$$

$$(Z = \bigcup_{i=1}^{\infty} S_i,\ S_i \in \mathscr{S},\ S_i \cap S_j = \phi,\ i, j = i, 2, \dots, \quad \textit{if} \quad i \neq j)$$

ii) *If $Z \neq X$ a necessary condition for the existence of a smallest* q-*algebra* (q-σ-*algebra*) *containing* $\mathscr{S}$ *is*

$$Z = \bigcup_{i=1}^{n} S_i, \quad S_i \in \mathscr{S},\ i = 1, \dots n.$$

$$(Z = \bigcup_{i=1}^{\infty}, \quad S_i \in \mathscr{S},\ i = 1, 2, \dots).$$

iii) *The condition* i) *is not necessary and the condition* ii) *is not sufficient.*

PROOF. We shall give a proof for q-algebras (the proof for q-σ-algebras is analogous.

i) Considering all the q-algebras with the fixed unit Z the usual approach gives the existence of the smallest one containing $\mathscr{S}$ belonging to this collection. Let $\mathscr{A}$ be this q-algebra. We shall prove that $\mathscr{A}$ is the smallest among all q-algebras containing $\mathscr{S}$, with or not the fixed unit Z. In fact, let $\mathscr{A}' \supset \mathscr{S}$ be any q-algebra. The condition

$$Z = \bigcup_{i=1}^{n} S_i, \quad S_i \in \mathscr{S}, \quad S_i \cap S_j = \phi \quad \text{if} \quad i \neq j,$$

gives $Z \in \mathscr{A}'$. Let $\mathscr{E} = \{A : A \in \mathscr{A},\ A \in \mathscr{A}',\ Z - A \in \mathscr{A}'\}$. The system $\mathscr{E}$ is a q-algebra with the unit Z because $A_1, A_2 \in \mathscr{E}$, $A_1 \cap A_2 = \phi$ implies $A_1 \cup A_2 \in \mathscr{A}$, $A_1 \cup A_2 \in \mathscr{A}'$ and $Z - (A_1 \cup A_2) = (Z - A_1) - A_2 \in \mathscr{A}'$ because of the fact $A_2 \subset Z - A_1$.

On the other hand $\mathscr{S} \subset \mathscr{E}$. Thus $\mathscr{A} \subset \mathscr{E}$. But $\mathscr{E} \subset \mathscr{A}'$ hence, $\mathscr{A} \subset \mathscr{A}'$.

ii) If Z is not a finite union of the elements of $\mathscr{S}$ then it is possible using the method of [1] to construct an algebra $\mathscr{A}$ such that $\mathscr{A} \supset \mathscr{S}$ and $Z \notin \mathscr{A}$. A smallest q-algebra $\mathscr{A}'$ containing $\mathscr{S}$ cannot exist. In fact, if it exists then it is easy to prove that $Z \in \mathscr{A}'$. But $\mathscr{A}$ being algebra is also a q-algebra, hence $\mathscr{A}' \subset \mathscr{A}$. Thus $Z \in \mathscr{A}$, what is a contraction.

iii) Let $X = \{1, 2, 3, \ldots, 8, 9\}$ and $Z = \{1, 2, \ldots, 8\}$.

Let $\mathscr{S}$ be the collection containing the set $\{1, 2, 3, 4\}$ and all three elements subsets of $\{1, 2, \ldots 8\}$. Evidently Z is not a disjoint union of the elements of $\mathscr{S}$. Nevertheless the smallest q-algebra containing $\mathscr{S}$ exists and it is the q-algebra of all subsets of Z. Hence the condition i) is not necessary.

To show that (ii) is not sufficient let

$$X = \{a, b, c, d\}, \quad \mathscr{S} = \{\{a, b\}, \{b, c\}\} \quad \text{and} \quad Z = \{a, b, c\}.$$

It is easy to see that $\mathscr{A} = \{\{a, b\}, \{b, c\}, \{a, b, c\}, \{c\}, \{a\}, \{a, c\}, \{b\}, \phi\}$ is a q-algebra containing $\mathscr{S}$ with unit $E = \{a, b, c\}$

Since $\mathscr{A}$ is the smallest q-algebra of subsets of Z with the unit Z containing $\mathscr{S}$ then, as we know, if there exists a smallest q-algebra $\mathscr{A}^*$ containing $\mathscr{S}$ it should coincide with $\mathscr{A}$. But $\mathscr{A}$ is not the smallest q-algebra containig $\mathscr{S}$. In fact if

$$\mathscr{B} = \{\{a, b\}, \{b, c\}, \{a, b, c, d\}, \{c, d\}, \{a, d\}, \phi\}$$

then $\mathscr{B}$ is a q-algebra which contains $\mathscr{S}$, but $\mathscr{A} \not\subset \mathscr{B}$.

NOTE. The part (iii) of the preceeding theorem shows that an analogy of the Theorem proved in [1] is not valid for q-algebra (q-σ-algebras). The following is a necessary and sufficient condition for the existence of a smallest q-algebra and can be formulated in the same manner also for algebras.

THEOREM 2. *Let $\mathscr{S}$ be a collection of subsets of X such that $\bigcup \mathscr{S} = Z \neq X$. Denote by $\mathscr{A}_Z$ the smallest* q-algebra *of subsets of Z which contains $\mathscr{S}$ and by $\mathscr{A}$ the smallest* q-ring *(which always exists) containing $\mathscr{S}$. A necessary and sufficient condition for the existence of the smallest* q-algebra *$\mathscr{A}_0$ containing $\mathscr{S}$ is $\mathscr{A} = \mathscr{A}_Z$.*

PROOF. Let $\mathscr{A} \neq \mathscr{A}_Z$. Then evidently $Z \notin \mathscr{A}$. Choose $\alpha \in X$, $\alpha \notin Z$ and put

$$\mathscr{B} = \{A : A \in \mathscr{A} \quad \text{or} \quad (Z \cup \{\alpha\}) - A \in \mathscr{A}\}.$$

$\mathscr{B}$ is a q-algebra *which contains $\mathscr{S}$. The fact $\mathscr{S} \subset \mathscr{B}$ is obvious.*

Now let $A, B \in \mathscr{B}$, $A \subset B$. If both $A, B \in \mathscr{A}$, $A \cap B = \phi$, then $A \cup B \in \mathscr{A}$, hence $A \cup B \in \mathscr{B}$. If under the same conditions $A \in \mathscr{A}$ and $Z \cup \{\alpha\} - B \in \mathscr{A}$, then $(Z \cup \{\alpha\}) - (A \cup B) = (Z \cup \{\alpha\} - B) - A \in \mathscr{A}$. The case $B \in \mathscr{A}$, $(Z \cup \{\alpha\}) - A \in \mathscr{A}$ is analogous. The case $(Z \cup \{a\}) - A \in \mathscr{A}$, $(Z \cup \{\alpha\}) - B \in \mathscr{A}$ is not possible because A, B are disjoint.

The fact that $A \subset (Z \cup \{\alpha\})$ for any $A \in \mathscr{B}$ is evident as well fact when $A \in \mathscr{B}$ also the complement $(z \cup \{\alpha\}) - A \in \mathscr{B}$. Hence $\mathscr{B}$ is a q-algebra with the unit $Z \cup \{\alpha\}$.

Evidently $\mathscr{S} \subset \mathscr{B}$. But $Z \notin \mathscr{B}$ because $Z \in \mathscr{A}$ and $(Z \cup \{\alpha\}) - Z = \{\alpha\} \notin \mathscr{A}$. Hence a smallest q-algebra $\mathscr{A}_0$, containing $\mathscr{S}$ doesn't exist. Suppose it exists; then we have $Z \in \mathscr{A}_0 \subset \mathscr{B}$, which is impossible.

Now let $\mathscr{A} = \mathscr{A}_Z$. Then $\mathscr{A}$ is a q-algebra which contains $\mathscr{S}$ with a unit Z. If $\tilde{\mathscr{A}}$ is any q-algebra containing $\mathscr{S}$ then $\tilde{\mathscr{A}}$ is a q-ring. This $\tilde{\mathscr{A}} \supset \mathscr{A} = \mathscr{A}_Z$. Hence $\mathscr{A}_0 = \mathscr{A}_Z = \mathscr{A}$ is the smallest q-algebra which contains $\mathscr{S}$.

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