Adams Operations in KO(X) ⊕ KSp(X)*

JACQUES ALLARD

Introduction

Let $KO(X)$ and $KS_p(X)$ be the real and quaternionic K-theory of a finite CW-complex X. The tensor product and the exterior powers of vector bundles induce a $\mathbb{Z}_2$-graded λ-ring structure on $L(X) = KO(X) \oplus KS_p(X)$ (see [Bott]).

In this paper, it is shown that $L(X)$ is a special λ-ring. The Adams operations

$$\psi^k\colon\ L(X) \longrightarrow L(X) \qquad k = 1, 2, \ldots$$

associated to this λ-ring are therefore ring homomorphisms and satisfy the composition law:

$$\psi^k \circ \psi^l = \psi^l \circ \psi^k = \psi^{kl} \qquad k, l = 1, 2, \ldots$$

(see [Atiyah and Tall]).

The Adams operations on $L(X)$ were first used in the classification of H-spaces of rank 2 by [Sigrist and Suter]. This paper contains the proof of our result in the case where $L(X)$ has no torsion.

This paper has five sections. §1 contains the definitions and properties of λ-semi-rings that we need.

*Recebido pela SBM em 24 de maio de 1974.

Essentially, §2 points out that the direct sum of the real and quaternionic representation rings of a compact Lie group is a special λ-ring (hence its Adams operations are ring homomorphisms and satisfy the composition law described above). In §3, we prove the main result stated above, and §4 relates the ψ^k's on $L(X)$ with the real and complex Adams operations. Finally, in §5 we compute $L(X)$ and the ψ-operations for the complex and quaternionic projective spaces $\mathbb{CP}^n$ and $\mathbb{HP}^n$.

§1. λ-semi-rings

In this section, we recall a few facts about λ-semi-rings and we propose a definition of special λ-semi-rings. A detailed treatment of λ-rings can be found in [Atiyah and Tall], part I. Only minor adaptations are necessary for semi-rings:

1.1. Definition. A *λ-semi-ring* R is a commutative semi-ring with identity together with a set of maps $\lambda^n : R \longrightarrow R$, $n = 0, 1, \ldots$, that satisfy the following three properties for all $x, y \in R$:

(1) $\lambda^0(x) = 1$

(2) $\lambda^1(x) = x$

(3) $\lambda^n(x + y) = \sum_{r=0}^{n} \lambda^r(x)\,\lambda^{n-r}(y)$.

A *λ-ring* R is a λ-semi-ring that is also a ring.

If R and R' are λ-semi-rings, a λ-homomorphism $f : R \longrightarrow R'$ is a semi-ring homomorphism commuting with the maps λ^n, $n = 0, 1, \ldots$ (We use the same symbol for the maps λ^n on all λ-semi-rings).

Let $a_1, \ldots, a_q$ and $b_1, \ldots, b_r$ be indeterminates and let s_i and σ_i be the i^{th} elementary symmetric functions in $a_1, \ldots, a_q$ and $b_1, \ldots, b_r$ respectively. Take $q \geq max(n, mn)$, $r \geq n$ and let $P_n(s_1, \ldots, s_n; \sigma_1, \ldots, \sigma_n)$ be the coefficient of t^n in $\prod_{i,j}(1 + a_i b_j t)$; let $P_{m,n}(s_1, \ldots, s_{mn})$ be the coefficient of t^m in in $\prod_{i_1 < \cdots < i_m} (1 + a_{i_1} \ldots a_{i_n} t)$. P_n and $P_{m,n}$ are uniquely defined polynomials with coefficients in $\mathbb{Z}$. Also, the terms in each of these polynomials are of constant weight. They split into:

$$P_n = P_n^+ - P_n^-$$
$$P_{m,n} = P_{m,n}^+ - P_{m,n}^-$$

where P_n^+(resp. $P_{m,n}^+$) is the sum of the terms of P_n(resp. $P_{m,n}$) having positive coefficient, and P_n^-(resp. $P_{m,n}^-$) is minus the sum of the terms of P_n(res. $P_{m,n}$) having negative coefficient.

1.2. Definition. A λ-semi-ring R is a *special λ-semi-ring* if the following identities hold $\forall\, x, y \in R$:

(4) $$P_n^-(\lambda^1(x), \ldots, \lambda^n(x);\ \lambda^1(y), \ldots, \lambda^n(y)) + \lambda^n(xy) = P_n^+(\lambda^1(x), \ldots, \lambda^1(x);\ \lambda^n(y), \ldots, \lambda^n(y))$$

(5) $$P_{m,n}^-(\lambda^1(x), \ldots, \lambda^{mn}(x)) + \lambda^m(\lambda^n(x)) = P_{m,n}^+(\lambda^1(x), \ldots, \lambda^{mn}(x)).$$

Identity (4) relates $\lambda^n(xy)$ to $\lambda^i(x)$, $\lambda^i(y)$ $(i = 1, \ldots, n)$; identity (5) relates $\lambda^m(\lambda^n(x))$ to $\lambda^i(x)$ $(i = 1, \ldots, mn)$. From now on, we will write these identities as:

(4′) $$F_n(x, y) = G_n(x, y)$$

(5′) $$F_{m,n}(x) = G_{m,n}(x),$$

where $F_n(x, y)$ denotes the left hand side of (4), etc.

1.3. Lemma. *Let R_0 be a λ-semi-ring. Then the Grothendieck group R of R_0 has a unique λ-ring structure such that the canonical map $R_0 \longrightarrow R$ is a λ-semi-ring homomorphism. Moreover, if R_0 is a special λ-semi-ring, R is a special λ-ring.*

Proof. For the unique λ-ring structure on R, see [Husemoller] chap. 12. If R_0 is special, then R is special, because (4) and (5) are intrinsically compatible with (3) (see [Atiyah and Tall], lemma 1.5), and $Im(R_0 \longrightarrow R)$ generates additively R.

For $k = 1, 2, \ldots$ let Q_k be the unique polynomial with integer coefficients such that:

$$Q_k(s_1, \ldots, s_k) = a_1^k + \ldots + a_q^k$$

with the s_i's and a_j's as above, and $q \geq k$. Given a λ-ring R, the Adams operations on R are the maps $\psi^k : R \longrightarrow R$ defined by:

$$\psi^k(x) = Q_k(\lambda^1(x), \ldots, \lambda^k(x))$$

for $x \in R$, $k = 1, 2, \ldots$ We have

1.4. PROPOSITION. *If R is a special λ-ring, the maps ψ^k, $k = 1, 2, \ldots$ are* ring *homomorphisms, and*

$$\psi^k\psi^l = \psi^{lk} = \psi^l\psi^k \qquad l, k = 1, 2, \ldots$$
$$\psi^p(x) = x^p \pmod p, \quad p \text{ prime}, \quad x \in R.$$

For the proof, see [Atiyah and Tall], part I.

§2. Representations of Compact Lie Groups.

Let $\mathscr{G}$ be a compact Lie group. For $\Lambda = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we denote by $\mathscr{R}_\Lambda(\mathscr{G})$ the group of Λ-representations on $\mathscr{G}$. When $\Lambda = \mathbb{R}$ or $\mathbb{C}$, the tensor product and the exterior powers of representations induce a λ-ring structure on $\mathscr{R}_\Lambda(\mathscr{G})$. In fact, we have the following theorem which is due to [Adams, 1].

2.1 THEOREM. *$\mathscr{R}_{\mathbb{R}}(\mathscr{G})$ and $\mathscr{R}_{\mathbb{C}}(\mathscr{G})$ are special λ-rings.*

We will now turn our attention to the group $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \otimes \mathscr{R}_{\mathbb{H}}(\mathscr{G})$. Given real representations α, α' and quaternionic representations β, β' of $\mathscr{G}$, the representations $\alpha \otimes \alpha'$, $\beta \otimes \beta'$, $\lambda^k\alpha$, $\lambda^{2k}\beta$ $(k = 0, 1, \ldots)$ are real representations of g, and the representations $\alpha \otimes \beta$, $\lambda^{2k+1}\beta$ $(k = 0, 1, \ldots)$ are quaternionic representations of $\mathscr{G}$ (see [Bott]). Therefore a multiplication is defined in $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G})$ by

$$([\alpha], 0) \cdot ([\alpha'], 0) = ([\alpha \otimes \alpha'], 0)$$
$$([\alpha], 0) \cdot (0, [\beta]) = (0, [\alpha \otimes \beta])$$
$$(0, [\beta]) \cdot (0, [\beta']) = ([\beta \otimes \beta'], 0)$$

and extending linearly([α] denotes the class of α in $\mathscr{R}_{\mathbb{H}}(\mathscr{G})$, etc). We also define:

$$\lambda^k([\alpha], 0) = ([\lambda^k\alpha], 0)$$
$$\lambda^k(0, [\beta]) = \begin{cases} ([\lambda^k\beta], 0) & k \text{ even} \\ (0, [\lambda^k\beta]) & k \text{ odd} \end{cases}$$

$$\lambda^k([\alpha], [\beta]) = \sum_{r=0}^{k} \lambda^{k-r}([\alpha], 0) \cdot \lambda^r(0, [\beta]).$$

The elements of the form $([\alpha], [\beta])$ in $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G})$ form a λ-semi-ring

of which $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G})$ is the Grothendieck group. By lemma 1.3, $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G})$ is a λ-semi-ring. Actually we have

2.3. THEOREM. *For compact Lie group $\mathscr{G}$, $\mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G})$ is a special λ-ring.*

PROOF. Let us consider the group $\mathscr{R}_{\mathbb{C}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{C}}(\mathscr{G})$ with the following λ-ring structure: for u, v, u', $v' \in \mathscr{R}_{\mathbb{C}}(\mathscr{G})$, define:

$$(u, v) \cdot (u', v') = (uu' + vv', uv' + u'v)$$
$$\lambda^k(0, v) = \begin{cases} (\lambda^k v, 0) & k \text{ even} \\ (0, \lambda^k v) & k \text{ odd} \end{cases}$$

$$\lambda^k(u, v) = \sum_{r=0}^{k} \lambda^{k-r}(u, 0)\, \lambda^r(0, v).$$

$\mathscr{R}_{\mathbb{C}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{C}}(\mathscr{G})$ with this structure is a special λ-ring because $\mathscr{R}_{\mathbb{C}}(\mathscr{G})$ itself is. Let

$$(2.4) \qquad \begin{aligned} c &: \mathscr{R}_{\mathbb{R}}(\mathscr{G}) \longrightarrow \mathscr{R}_{\mathbb{C}}(\mathscr{G}) \\ c' &: \mathscr{R}_{\mathbb{H}}(\mathscr{G}) \longrightarrow \mathscr{R}_{\mathbb{C}}(\mathscr{G}) \end{aligned}$$

be the maps induce by "complexification" of real and quaternionic representations respectively (see [Adams, 2], 3.5, 3.26). These maps are compatible with the λ-operations and

$$a = c \oplus c' : \mathscr{R}_{\mathbb{R}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{H}}(\mathscr{G}) \longrightarrow R_{\mathbb{C}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{C}}(\mathscr{G})$$

is a λ-ring monomorphism (see [Adams 2], 3.27). Since the λ-structure on $\mathscr{R}_{\mathbb{C}}(\mathscr{G}) \oplus \mathscr{R}_{\mathbb{C}}(\mathscr{G})$ is special, the theorem follows.

§3. The λ-ring $L(X) = KO(X) \oplus KSp(X)$

For $\Lambda = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and for a finite CW-complex X, let $Vect_\Lambda(X)$ be the abelian semi-group of isomorphism classes of Λ-vector bundles over X. We use the same symbol for a vector bundle and its isomorphism class. Let

$$V(X) = Vect_{\mathbb{R}}(X) \oplus Vect_{\mathbb{H}}(X)$$

and denote by $KO(X)$, $KU(X)$, $KSp(X)$ and $L(X) = KO(X) \oplus KSp(X)$ the Grothendieck groups of $Vect_{\mathbb{R}}(X)$, $Vect_{\mathbb{C}}(X)$, $Vect_{\mathbb{H}}(X)$ and $V(X)$ respectively. We wish to show that $V(X)$ is a special λ-semi-ring.

We will use the following notation throughout §3 and §4. Let $U(t, \Lambda)$ be the t-dimensional orthogonal group ($\Lambda = \mathbb{R}$), unitary group ($\Lambda = \mathbb{C}$) or symplectic group ($\Lambda = \mathbb{H}$). Let ξ be a Λ-vector bundle and η be a Λ'-vector bundle over X with $dim_\Lambda \xi = r$ and $dim_{\Lambda'} \eta = s$. Since X is compact, ξ and η have structure group $U(r, \Lambda)$ and $U(s, \Lambda')$ respectively. This means that there is an open covering $\{V_i\}_{i \in I}$ of X such that ξ and η are determined by the system of transition functions:

$$\{g_{ij} : V_i \cap V_j \longrightarrow U(r, \Lambda)\}_{i,j \in I}$$
$$\{h_{ij} : V_i \cap V_j \longrightarrow U(s, \Lambda')\}_{i,j \in I}.$$

Let

$$\pi_1 : U(r, \Lambda) \times U(s, \Lambda') \longrightarrow U(r, \Lambda)$$
$$\pi_2 : U(r, \Lambda) \times U(s, \Lambda') \longrightarrow U(s, \Lambda')$$
$$I : U(r, \Lambda) \longrightarrow U(r, \Lambda)$$

be the representations of compact Lie Groups defined by the two projection maps and the identity map, and let (Λ, Λ') be one of the pairs $(\mathbb{R}, \mathbb{R})$, $(\mathbb{C}, \mathbb{C})$, $(\mathbb{R}, \mathbb{H})$, $(\mathbb{H}, \mathbb{H})$.

In view of the previous section, the representations $\pi_i \otimes \pi_2$ and $\lambda^n(I)$ are defined for each pair. Therefore we can define the vector bundles $\xi \otimes \eta$ and $\lambda^n(\xi)$ as the vector bundles determined respectively by the following systems of transition functions:

$$\{\pi_1 \otimes \pi_2 \circ (g_{ij}, h_{ij})\}_{i,j \in I}$$
$$\{\lambda^k(I) \circ g_{i,j}\}_{i,j \in I}$$

where $(g_{ij}, h_{ij}) : V_i \cap V_j \longrightarrow U(r, \Lambda) \times U(s, \Lambda')$ is the obvious map.

When $(\Lambda, \Lambda') = \mathbb{R}, \mathbb{R})$ or $(\mathbb{C}, \mathbb{C})$, the usual tensor product and exterior powers are obtained. When $(\Lambda, \Lambda') = (\mathbb{R}, \mathbb{H})$, $\xi \otimes \eta$ is a quaternionic vector bundle; when $(\Lambda, \Lambda') = (\mathbb{H}, \mathbb{H})$, $\xi \otimes \eta$ is a real vector bundle and $\lambda^n(\xi)$ is real if k is even and quaternionic if k is odd.

From now on, we identify $Vect_{\mathbb{R}}(X)$ and $Vect_{\mathbb{H}}(X)$ with their canonical images in $V(X) = Vect_{\mathbb{R}}(X) \oplus Vect_{\mathbb{H}}(X)$. Let $(\alpha, \beta), (\alpha', \beta') \in V(X)$ and define:

$$(\alpha, \beta) \cdot (\alpha', \beta') = (\alpha \otimes \alpha' + \beta \otimes \beta', \alpha \otimes \beta' + \beta \otimes \alpha')$$
$$\lambda^k(\alpha, \beta) = \sum_{r=0}^{k} \lambda^{k-r}(\alpha, 0) \cdot \lambda^r(0, \beta).$$

Note that $\lambda^r(0, \beta) = (\lambda^r \beta, 0)$ for r even and $\lambda^r(0, \beta) = (0, \lambda^r \beta)$ for r odd. $V(X)$ is then a ($\mathbb{Z}_2$-graded) λ-semi-ring. Moreover,

3.2. Lemma. *$V(X)$ is a special λ-semi-ring*

Proof. (4') and (5') need to be verified. It is sufficient to check these identities for elements of $Vect_{\mathbb{R}}(X) \cup Vect_{\mathbb{H}}(X) \subset V(X)$ since this set generates additively $V(X)$ (cf proof of 1.3.). Let us check (5'). We have to show that $F_{m,n}(\xi)$ and $G_{m,n}(\xi)$ are isomorphic vector bundles ($\Lambda = \mathbb{R}$ or $\mathbb{H}$). Since the terms of $F_{m,n}$ are of constant weight, $F_{m,n}(\xi)$ and $G_{m,n}(\xi)$ are both quaternionic or both real. They are determined by the systems of transition functions

$$\{F_{m,n}(I) \circ g_{ij}\}_{i,j \in I} \qquad (3.3)$$
$$\{G_{m,n}(I) \circ g_{ij}\}_{i,j \in I}$$

Since $\mathscr{R}_{\mathbb{R}}(U(r, \Lambda)) \oplus \mathscr{R}_{\mathbb{H}}(U(r, \Lambda))$ is a special λ-ring (th. 2.2), $F_{m,n}(I)$ and $G_{m,n}(I)$ are equivalent representations, and there is an element $M \in U(t, \Lambda'')$ (t, Λ'' depending on Λ and $\binom{r}{mn}$), such that

$$M \circ F_{m,n}(I)(g) = G_{m,n}(I)(g) \circ M \qquad \forall g \in U(r, \Lambda).$$

Therefore, the set of constant maps $\{r_i : V_i \longrightarrow U(t, \Lambda'')\}_{i \in I}$ defined by $r_i(x) = M$ determines an equivalence between the systems of transition functions (3.3). Thus $F_{m,n}(\xi)$ and $G_{m,n}(\xi)$ are isomorphic vector bundles (see [Husemoler] ch. 5), and (5') is verified. (4') is verified similarly, and this completes the proof of the lemma.

3.3. Theorem. *For a finite CW-complex X, $L(X) = KO(X) \oplus Ksp(X)$ is a special λ-ring.*

The proof consists of lemmas 3.2 and 1.3.

Hence, we have proved that $L(X)$ is a contravariant functor from the category of finite CW-complexes to the category of ($\mathbb{Z}_2$-graded) special λ-rings. Given another finite CW-complex Y and a continuous map $f : Y \longrightarrow X$, we denote by $f^! : L(X) \longrightarrow L(Y)$ the ($\mathbb{Z}_2$-graded) λ-homomorphism induced by f.

The Adams operations associated to the λ-ring $L(X)$ are a family of natural tranformations of the functor $L(X)$ to itself. They have the two basic properties of proposition 1.4. Notice that ψ^k is a ring homomorphism but that it does *not* respect the $\mathbb{Z}_2$-grading. In fact, $\psi^k(\xi)$ is real if ξ is real or k even, and quaternionic if ξ is quaternionic and k is odd.

If X is given a base point x_0, all the above maps pass to the reduced functor

$$\tilde{L}(X) = ker(L(X) \longrightarrow L(x_0)),$$

and one has a natural splitting

$$L(X) = L(x_0) \oplus \tilde{L}(X).$$

If X^+ denotes the disjoint union of X and a point taken as base point, we have also

$$\tilde{L}(X^+) = L(X).$$

Finally, for a point x_0,

$$L(x_0) = \mathbb{Z} \oplus \varepsilon\,\mathbb{Z}$$

with $\varepsilon^2 = 4$ and $\psi^k(\varepsilon) = \begin{cases} \varepsilon & k \ \text{odd} \\ 2 & k \ \text{even} \end{cases}$,

ε being represented by the trivial 1-dimensional quaternionic bundle over $\{x_0\}$.

§4. The Relation between the Adams Operations on $L(X)$ and the classical Adams Operations.

We first look at the relation of the ψ^k's on $L(X)$ to the classical complex Adams operations on $KU(X)$. Let us endow $KU(X) \oplus KU(X)$ with a $\mathbb{Z}_2$-graded special λ-ring structure in the same way as for $\mathcal{R}_{\mathbb{C}}(\mathcal{G}) \oplus \mathcal{R}_{\mathbb{C}}(\mathcal{G})$ (cf. §2). We write $LU(X) = KU(X) \oplus KU(X)$ and we denote $\psi^k_{\mathbb{C}}$ the Adams operations on $LU(X)$; they are completely determined by the Adams operations on $KU(X)$. Let ξ, $\{V_i\}_{i \in I}$, $\{g_{ij}\}_{i,j \in I}$ be as in §3. If $\Lambda = \mathbb{R}$ (resp. $\Lambda = \mathbb{H}$) $c\xi$ (resp. $c'\xi$) is the complex vector bundle given by the system of transition functions $\{c(I) \circ g_{ij}\}_{i,j \in I}$ (resp. $\{c'(I) \circ g_{ij}\}_{i,j \in I}$) with $c(I)$ (resp. $c'(I)$) given by (2.4). The induced group homomorphisms

$$c: KO(X) \longrightarrow KU(X)$$
$$c': KSp(X) \longrightarrow KU(X)$$

make the map

$$a = c \oplus c': L(X) = KO(X) \oplus KSp(X) \longrightarrow KU(X) \oplus KU(X) = LU(X)$$

a λ-homomorphism. Hence, we have a commutative diagram:

$$\begin{array}{ccc} L(X) & \xrightarrow{c \oplus c'} & LU(X) \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle \psi^k_{\mathbb{C}}} \\ L(X) & \xrightarrow[c \oplus c']{} & LU(X) \end{array}$$

If $L(X)$ is torsion-free, $c \oplus c'$ is a monomorphism and $L(X)$ can be viewed as a natural sub-λ-ring of $LU(X)$. In this case, the complex Adams operations on $KU(X)$ determine the operations on $L(X)$ (see [Sigrist and Suter]).

Concerning the real Adams operations, recall that the Bott isomorphism gives a group isomorphism

$$Id \oplus B\colon \tilde{L}(X) \longrightarrow \tilde{K}O^{\circ}(X) \oplus \tilde{K}O^{-4}(X).$$

This is actually a ring isomorphism, with $\tilde{K}O^{\circ}(X) \oplus \tilde{K}O^{-4}(X)$ being considered as a subring of $\tilde{K}O^*(X)$. We remark first that the restriction of ψ^k: $L(X) \longrightarrow L(X)$ to $KO(X)$ gives the real Adams operation on $KO(X)$. Furthermore, using the Bott isomorphism $\tilde{K}Sp(X) \simeq \tilde{K}O(X \wedge S^4) = \tilde{K}O^{-4}(X)$, we state.

4.1. Proposition. *The following diagrams are commutative:*

$$k \ \ \text{odd} \qquad \begin{array}{ccc} KSp(X) & \xrightarrow[\simeq]{Bott} & KO(X \wedge S^4) \\ \downarrow{\scriptstyle k^2 \cdot \psi^k} & & \downarrow{\scriptstyle \psi^k_{\mathbb{R}}} \\ KSp(X) & \xrightarrow[Bott]{\simeq} & KO(X \wedge S^4) \end{array}$$

$$
\begin{array}{cccc}
 & KSp(X) & \xrightarrow[\simeq]{\text{Bott}} & KO(X \wedge S^4) \\
 & \downarrow \psi^k & & \\
k \ \text{even} & KO(X) & & \downarrow \psi^k_{\mathbb{R}} \\
 & \downarrow \frac{k^2}{2}\varepsilon & & \\
 & KSp(X) & \xrightarrow[\text{Bott}]{\simeq} & KO(X \wedge S^4)
\end{array}
$$

where $\frac{k^2}{2}\varepsilon$ *denotes the map* $\alpha \longmapsto \frac{k^2}{2}\varepsilon \cdot \alpha$, *and* ε *is the trivial* 1*-dimensional quaternionic vector bundle.*

The proof in [Sigrist and Suter], where $L(X)$ is assumed to be torsion free, remains true in general without change.

§5. Computation for $\mathbb{HP}^n$ and $\mathbb{CP}^n$

We will describe the Adams operations on $L(\mathbb{HP})$ and $L(\mathbb{CP}^n)$. Only an outline of the method of calculation is given, since only standard techniques are used. The results are given in terms of the polynomials $T_k \in \mathbb{Z}[x]$ such that

$$T_k(Z + \frac{1}{Z} - 2) = Z^k + \frac{1}{Z^k} - 2.$$

There is a unique such polynomial for each positive integer k.

Let ε be the trivial 1-dimensional quaternionic vector bundle, and ξ the canonical quaternionic line bundle over $\mathbb{HP}^n$.

5.1. THEOREM. *The ring* $L(\mathbb{HP}^n)$ *is generated by* 1, ε *and* $\tau = c'\xi - \varepsilon$ *with relations* $\varepsilon^2 = 4$ *and* $\tau^{n+1} = 0$. *Moreover, the Adams operations are given by*:

$$\psi^k(\tau) = \begin{cases} \frac{\varepsilon}{2} T_k\left(\frac{\varepsilon}{2}\tau\right) & k \ \text{odd} \\ T_k\left(\frac{\varepsilon}{2}\tau\right) & k \ \text{even.} \end{cases}$$

The proof is as follows. First one shows easily that $L(\mathbb{HP}^n)$ is torsion free (induction on n using the long exact sequence of the cofibration $\mathbb{HP}^{n-1} \longrightarrow \mathbb{HP}^n \longrightarrow S^{4n}$). Then, by §4, $L(\mathbb{HP}^n) \subset LU(\mathbb{HP}^n)$. The Adams operations on $LU(\mathbb{HP}^n)$ can be computed using the map $f^!: LU(\mathbb{HP}^n) \longrightarrow LU(\mathbb{CP}^{2n+1})$ induced by the canonical map $f: \mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n$, and the results of [Adams, 1]. Then it suffices to study the (λ-ring) inclusion $L(\mathbb{HP}^n) \subset LU(\mathbb{HP}^n)$ to get the result.

Let η be the canonical complex line bundle over $\mathbb{CP}^n$, and let $\mu = \eta - 1 \in KU(\mathbb{CP}^n)$. Let γ be a bundle such that $[\gamma]$ generates $\tilde{K}U(S^2) = \mathbb{Z}$. Let μ_0 be the real vector bundle obtained from μ by forgetting the complex structure, and let μ_2 be the real vector bundle over $\mathbb{CP}^n \wedge S^4 \approx \mathbb{CP}^n \wedge S^2 \wedge S^2$ obtained from $\mu \cdot \gamma^2$ by forgetting the complex structure. Finally, let $\nu_2 = B^{-1}(\mu_2) \in \tilde{K}Sp(\mathbb{CP}^n)$ be given by the Bott isomorphism B.

5.2. THEOREM. (i) *Let* n *be even.* $\tilde{L}(\mathbb{CP}^n)$ *is the free abelian group generated by* $\mu_0, \mu_0^2, \ldots, \mu_0^{n/2-1}, \nu_2, \nu_2\mu_0, \ldots, \nu_2\mu_0^{n/2-1}$. *The multiplicative structure is completed by* $\nu_2^2 = \mu_0^2$.

(ii) *Let* $n = 2t + 1$. $\tilde{L}(\mathbb{CP}^n)$ *is the direct sum of the free abelian group generated by* $\mu_0, \mu_0^2, \ldots, \mu_0^{(n-1)/2}, \nu_2, \nu_2\mu_0, \ldots, \nu_2\mu_0^{(n-3)/2}$ *and the cyclic group of order two generated by* μ_0^{t+1} *if* t *is odd or* $\nu_2\mu_0^{t+1}$ *if* t *is even. The multiplicative structure is completed by* $\nu_2^2 = \mu_0^2$.

The Adams operations on $L(\mathbb{CP}^n)$ *are given by*:

$$\psi^k(\mu_0) = T_k(\mu_0) \qquad k = 1, 2, \ldots$$

$$\psi^k(\nu_2) = \begin{cases} T_k(\mu_0) & k = 2, 4, \ldots \\ \frac{\nu_2}{\mu_0} T_k(\mu_0) & k = 1, 3, \ldots \end{cases}$$

S. Araki (see [Fujii]) computed the ring $KO^*(\mathbb{CP}^n)$. This gives the structure of $L(\mathbb{CP}^n) = KO^\circ(\mathbb{CP}^n) \oplus KO^{-4}(\mathbb{CP}^n)$. For n even, $L(\mathbb{CP}^n)$ is torsion free, and therefore the Adams operations ψ^k are determined by the complex Adams operations $\psi^k_{\mathbb{C}}$ on $LU(\mathbb{CP}^n)$; the latter were computed in [Adams, 1]. For n odd, one observes that the natural inclusion $\mathbb{CP}^n \longrightarrow \mathbb{CP}^{n+1}$ induces an epimorphism $\tilde{L}(\mathbb{CP}^{n+1}) \longrightarrow \tilde{L}(\mathbb{CP}^n)$. The ψ^k's on $\tilde{L}(CP^n)$ are then obtained by naturality.

5.3. REMARK. The cofibration $\mathbb{CP}^l \longrightarrow \mathbb{CP}^{n+l} \longrightarrow \dfrac{\mathbb{CP}^{n+l}}{\mathbb{CP}^l}$ induces an embedding

$$\pi^! : \tilde{L}(\mathbb{CP}^{n+l}/\mathbb{CP}^l) \longrightarrow \tilde{L}(\mathbb{CP}^{n+l}).$$

From theorem 5.2 and this remark, one gets the Adams operations on $L(\mathbb{CP}^{n+l}/\mathbb{CP}^l)$.

The author wishes to thank Dr. U. Suter for the numerous conversations leading to the completion of this paper.

BIBLIOGRAPHY

[Adams, 1] J. F. ADAMS, *Vector Fields on Spheres*, Annals of Math., Vol. **75** (1962), pp. 603-632.

[Adams, 2] J. F. ADAMS, *Lectures on Lie Groups*, W. A. Benjamin, inc., New York (1969).

[Atiyah and Tall] M. A. ATIYAH and D. O. TALL, *Group Representations, λ-Rings and the J-Homomorphism*, Topology, Vol. **8** (1969), pp. 253-297.

[Bott] R. BOTT, *Quelques remarques sur les theoremes de périodicité*, Bull. Soc. Math. France, Vol **87** (1959), pp. 293-310.

[Fujii] M. FUJII, *KO-Groups of Projective Spaces*, Osaka J. Math., Vol. **4** (1967), pp. 141-149.

[Husemoller] D. HUSEMOLLER, *Fiber Bundles*, McGraw-Hill, New York (1966).

[Sigrist and Suter] F. SIGRIST and U. SUTER, *Eine Anwendung der K-theorie in der theorie der H-Raume*, Commentarii Math. Helvetici, Vol. **47** (1972), pp. 36-52.

University of British Columbia
Vancouver — Canada