Note on Dold-Weinzweig Fibrations and Duality*

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This note follows the notation of [3], for the sake of clarity however we define some terms. Let $p: E \longrightarrow B$ be a map (i.e. a continuous function) B^I the space of paths in B of length one, $\varepsilon_i^* : B^I \longrightarrow B$ the evaluation at the point i $(i = 0, 1)$ and $\delta^*: B \longrightarrow B^I$ the map which takes a point b to the constant path δ_b at b. Let $B^I \sqcap E = \{(\lambda, e) \in B^I \times E \mid \lambda(0) = p(e)\}$ be the mapping track of p thus the diagram

$$\begin{array}{ccc} B^I \sqcap E & \xrightarrow{\ \pi\ } & E \\ \Big\downarrow \bar{p} & & \Big\downarrow p \\ B^I & \xrightarrow[\ \varepsilon_0^*\ ]{} & B \end{array}$$

is a pullback (π and $\bar{p}$ are the obvious projections).

There are maps $\langle \delta^* p, 1 \rangle : E \longrightarrow B^I \sqcap E$ and $\theta_p = \varepsilon_1^* \bar{p} : B^I \sqcap E \longrightarrow B$ defined by $\langle \delta^* p, 1 \rangle(e) = (\delta_{p(e)}, e)$ and $\theta_p(\lambda, e) = \lambda(1)$. It is well known that $\langle \delta^* p, 1 \rangle$ is a homotopy equivalence, θ_p is a Hurewicz fibration and $p = \theta \langle \delta^* p, 1 \rangle$.

In [3] the author has shown that when p has the weak covering homotopy property [1], there is a "homotopy operation" of B^I on E, that is a map $\gamma : B^I \sqcap E \longrightarrow E$ which satisfies

HOI. $p\gamma = \theta_p$
HOII. $\gamma \langle \delta^* p, 1 \rangle \simeq 1_E$ *rel* p
HOIII. $\gamma(1 \sqcap \gamma) \simeq \gamma(m \sqcap 1)$ *rel* p: $B^I \sqcap B^I \sqcap E \longrightarrow E$.

The map m: $B^I \sqcap B^I \longrightarrow B^I$ in HOIII is the partial multiplication of paths in B, HOIII requires the compatibility of this multiplication with γ. Finally that the homotopies are *rel* p in HOII and III means that they are vertical homotopies.

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Weinzweig in [3] defined the triple (E, p, B) to be a fibre space if there exists a connection for p, that is, if there exists a map $\tau\colon B^I \sqcap E \longrightarrow E$ satisfying conditions that are equivalent to HOI and HOII. It is clear therefore that Weinzweig's condition is equivalent to requiring the existence of a left fibre homotopy inverse for $\langle \delta^* p, 1\rangle$. Dold in [1] gives a definition of fibre space equivalent to that of Weinzweig. The interesting point here is that both definitions are equivalent to the existence of a homotopy operation and more specifically condition HOIII is implied by conditions HOI and II. We state these results as a Theorem and its Corollary.

Theorem 1. *Let $p\colon E \longrightarrow B$ be a map. The following are equivalent:*

(i) (E, p, B) is a fibre map in the sense of Weinzweig.

(ii) p is a fibre map in the sense of Dold.

(iii) The map $\langle \delta^* p, 1\rangle$ is a fibre homotopy equivalence.

(iv) The map $\langle \delta^* p, 1\rangle$ has a left fibre homotopy inverse.

(v) There is a homotopy operation $\gamma\colon B^I \sqcap E \longrightarrow E$ in the sense of [3].

Corollary 2. *Any left fibre homotopy inverse γ' of $\langle \delta^* p, 1\rangle$ satisfies HOIII.*

Proof. We show that any left fibre homotopy inverse is homotopic rel p to a homotopy operation and the result then follows from Theorem 5.5 in [3]. If γ' is a left homotopy inverse of $\langle \delta^* p, 1\rangle$ then p is a Dold fibration, that is, satisfies the weak covering homotopy property. The author's work in [3] shows there is a homotopy operation $\gamma\colon B^I \sqcap E \longrightarrow E$ which by Theorem 5.5 is a fibre homotopy equivalence so

$$\begin{aligned} \gamma' &= \gamma' \langle \delta^* p, 1\rangle \gamma \quad \text{rel } p \\ &= \gamma' 1 \qquad\qquad\quad \text{rel } p \\ &= \gamma' \end{aligned}$$

Dualy we have: For any map $i\colon A \longrightarrow X$ there is a factorization of i as $A \xrightarrow{j} X \sqcup A \times I \xrightarrow{<1, i(1\times\delta)>} X$, where $X \sqcup A \times I$ is the mapping cylinder (i.e. pushout)

$$\begin{array}{ccc} A & \xrightarrow{\;1 \times \varepsilon_0\;} & A \times I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow{\qquad} & X \sqcup A \times I \end{array}$$

where $j = i'(1 \times \varepsilon_1)$ is a cofibration, $(1 \times \varepsilon_i) = (a, i)$, $(i - 0, 1)$, $(1 \times \delta)\,(a, t) = a$, and $\langle 1, i(1 \times \delta)\rangle$ is a homotopy equivalence.

Recall that a map has the weak covering homotopy property (WCHP) if it has the CHP with respect to all homotopies $H + r$, where r is constant and H arbitrary.

Theorem 3. *Let $i\colon A \longrightarrow X$ be a map the following are equivalent:*

(i) i has the WCHP.

(ii) The map $\langle 1, i(1 \times \delta)\rangle$ is a cofibre homotopy equivalence (between j and i).

(iii) The map $\langle 1, i(1 \times \delta)\rangle$ has a right cofibre homotopy equivalence.

(iv) There is a homotopy cooperation of $A \times I$ on X in the sense of [3], that is, there is a map $\beta\colon X \longrightarrow X \sqcup A \times I$ satisfying axioms HCOI to HCOIII dual to axioms HOI to HOIII.

Corollary 4. HCOI *and* HCOII *imply* HCOIII.

Remark. The above theorem and its corollary may be proved directly or, within a suitable topological category, using the techniques of [2].

BIBLIOGRAPHY

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