Poincaré-Bendixon Theorem for $\mathbb{R}^2$-actions*

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§1. Preliminaries

Let $\varphi\colon \mathbb{R}^k \times M \to M$ be a C^r, $r \geq 1$, action on a differentiable manifold M. This means that $\varphi(0, p) = p$ and $\varphi(r_1 + r_2, p) = \varphi(r_1, \varphi(r_2, p))$ for any $p \in M$ and $r_1, r_2 \in \mathbb{R}^k$. The orbit of a point $p \in M$ is the set $\mathcal{O}_p = \{\varphi(r, p) \mid r \in \mathbb{R}^k\}$. The isotropy group of p is defined as $G_p = \{r \in \mathbb{R}^k \mid \varphi(r, p) = p\}$. Fixing $p \in M$, the map $\mathbb{R}^k \to M$ given by $r \to \varphi(r, p)$ induces an injective immersion of the homogeneous space $\mathbb{R}^k/G_p$ into M whose image is $\mathcal{O}_p$. Thus any orbit of φ is the image by an injective immersion of $\mathbb{R}^m$ or $\mathbb{R}^m \times T^n$ where m, n are integers, $m \geq 0$, $n \geq 1$, $m + n \leq k$ and $\mathbb{R}^\circ$ denotes a point.

A minimal set of φ is a nonempty invariant closed subset $\mu \subset M$ such that no proper subset of μ has these three properties.

A version of the Poincaré-Bendixon theorem for flows ($k = 1$) states that if $M = \mathbb{R}^2$ or S^2 then any minimal set of φ is an orbit. We generalize this theorem as follows. An invariant subset $\Lambda \subset M$ is called a *locally finite collection of orbits* if for any point $p \in \Lambda$ there is a neighborhood $U \subset M$ of p *intersecting finitely many orbits in* Λ.

THEOREM. *Let $\varphi\colon \mathbb{R}^2 \times M \to M$ be an action on a simply connected, 3-manifold M, compact or not, satisfying.*

(i) Any orbit of dimension one is embedded in M.

(ii) The union of all orbits of dimension less than two is a locally finite collection of orbits.

*Then any minimal set of φ is an orbit.***

COROLLARY. *Let $\varphi\colon \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ be a locally free $\mathbb{R}^2$-action. Then no orbit of φ is dense in $\mathbb{R}^3$.*

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**In a paper of J. Plante [7] we just received, appears a version similar to the above of the Poincaré-Bendixon theorem for actions of nilpotent Lie groups on compact manifolds. His method however does not hold for noncompact manifolds.

Recall that G. Hector [4], proved that there are codimension one foliations of $\mathbb{R}^3$ such that all leaves are everywhere dense.

Another version of the Poincaré-Bendixon theorem for flows on $\mathbb{R}^2$ caracterizes the ω-limit set $\omega(p)$ of a point p as follows: Suppose $\omega(p)$ is bounded and contains at most finitely many fixed points of the flow. Then either $\omega(p)$ is a periodic orbit or it is an embedded graph whose vertices are fixed points and whose sides are 1-orbits.

One could be tempted to conjecture that the ω-limit set of a point is an invariant embedded complex of dimension ≤ 2 provided the action has finitely many orbits of dimension ≤ 1.

At the end of §2 we show that this is false. There we define an $\mathbb{R}^2$-action on S^3 satisfying these conditions and exhibiting orbits which are dense in an embedded solid torus.

§2. Proof of the Theorem

Given $p \in M$ define $\partial\mathcal{O}_p = \bigcap_{n=1}^{\infty} \overline{\mathcal{O}_p - K_n}$ where each $K_n \subset \mathcal{O}_p$ is a compact neighborhood of p in $\mathcal{O}_p$, $K_n \subset K_{n+1}$, and $\bigcup_{n=1}^{\infty} K_n = \mathcal{O}_p$. Clearly $\overline{\mathcal{O}}_p = \mathcal{O}_p \cup \partial\mathcal{O}_p$.

Let μ be a minimal set of φ and $p \in \mu$. If $\mu \neq \mathcal{O}_p$ then $\mathcal{O}_p$ is not embedded. In fact, $\partial\mathcal{O}_p = \mu = \overline{\mathcal{O}}_p \supset \mathcal{O}_p$. Therefore for $\dim \mathcal{O}_p > 2$, $\mu = \mathcal{O}_p$.

Suppose now that $\mu \neq \mathcal{O}_p$ and $\dim \mathcal{O}_p = 2$. Call Σ the union of orbits of dimension less than two. The 2-orbits of φ define a C^r foliation $\mathcal{F}$ of codimension one of the manifold $M - \Sigma$. It is clear that one can find a closed path α: $S^1 \to M - \Sigma$ passing through p transverse to the leaves of $\mathcal{F}$. Since M is simply connected there is a map Ψ: $D^2 \to M$ such that $\Psi/\partial D^2 = \alpha$. We can assume (see [2]), that Ψ is a C^s immersion $s \geq 2$. We prove now that arbitrarily C^s close to Ψ there is an immersion Φ: $D^2 \to M$ which is transverse to all orbits of dimension less than two.

Since the fixed points of φ are isolated we take Ψ such that $\Psi(D^2)$ does not contain fixed points of φ. The set of orbits of dimension one intersecting $\Psi(D^2)$ is finite and partially ordered by the relation: $\mathcal{O}_p > \mathcal{O}_q$ iff $\partial\mathcal{O}_p \supset \mathcal{O}_q$. Let $\mathcal{O}^1$ be a minimal element under this relation. We have then $\partial\mathcal{O}^1 \cap \Psi(D^2) = {=}\varnothing$. Thus by Thom's transversallity theorem, we can finde Ψ^1: $D^2 \to M$ arbitrarily C^s close to Ψ such that Ψ^1 is transverse to $\mathcal{O}^1$. Repeating the same argument for all minimal elements we take Ψ^1 transverse to all of them.

Call Σ^0 the union of all fixed points of φ and Σ^1 the union of Σ^0 with the 1-orbits minimal under the relation defined above. Clearly $\Sigma^1 \cap \Psi^1(D^2)$ is finite and since φ is C^1 there is a neighborhood $W^1 \supset \Sigma^1 \cap \Psi^1(D^2)$ such that for any 1-orbit $\mathcal{O}$ intersecting W^1 we have Ψ^1 transverse to $\mathcal{O} \cap W^1$. Let $\mathcal{O}^2$ be a 1-orbit such that $\mathcal{O}^2 \cap \Psi^1(D^2) \neq \varnothing$ and $\partial\mathcal{O} \subset \overline{\Sigma^1}$. This means that $\mathcal{O}^2$ is immediately above some orbit of Σ^1 in the order relation. Let $K^2 \subset \mathcal{O}^2$ be an embedded compact cell such that all points of intersection of $\mathcal{O}^2 - K^2$ with $\Psi^1(D^2)$ are in W^1. Modify Ψ^1 outside $(\Psi^1)^{-1}(W^1)$ to find Ψ^2: $D^2 \to M$ C^s close to Ψ^1 and transverse to K^2. This implies $K^2 \cap \Psi^2(D^2)$ is finite and so there is a neighborhood $W^2 \supset K^2 \cap \Psi^2(D^2)$ such that for any 1-orbit $\mathcal{O}$ touching W^2 one has Ψ^2 transverse to $\mathcal{O} \cap W^2$.

Repeating this construction process we find maps Ψ^1, Ψ^2, Ψ^3... Since the set of 1-orbits intersecting $\Psi(D^2)$ is finite this process comes to an end for some n yielding a map $\Phi = \Psi^n : D^2 \to M$ which is arbitrarily C^s close to Ψ and transverse to all 1-orbits of φ.

From the definition of Φ one obtains a neighborhood $N(\Phi)$ of Φ in the C^s topology such that any $\tilde{\Phi} \in (N)\Phi$ is transverse to all 1-orbits of φ.

Given $\tilde{\Phi} \in N(\Phi)$ the foliation $\mathcal{F}$ of $M - \Sigma$ induces via $\tilde{\Phi}$ a foliation with singularities of D^2. Call $\mathcal{F}(\tilde{\Phi})$ this foliation and $S(\tilde{\Phi}) \subset D^2$ the set of singularities of $\mathcal{F}(\tilde{\Phi})$. Then $S(\tilde{\Phi})$ can be written as $S(\tilde{\Phi}) = S_1(\tilde{\Phi}) \cup S_2(\tilde{\Phi})$ where $S_1(\tilde{\Phi})$ is the union of points of tangency of $\tilde{\Phi}$ with the leaves of $\mathcal{F}$ and $S_2(\tilde{\Phi})$ is the union of points of intersection of $\tilde{\Phi}$ with the orbits of dimension one. It follows from [2] page 316 that there is an immersion χ: $D^2 \to M$, $\chi \in N(\Phi)$ such that:

(a) Any point $q \in S_1(\chi)$ is a nondegenerate point of tangency, i.e. there is a system of coordinates (x_1, x_2) in a neighborhood W of $q = (0,0)$ such that the leaves of $\mathcal{F}(\chi)/W$ are the level curves of a function f of one of the types $f(x_1, x_2) = x_1^2 + x_2^2$, $f(x_1, x_2) = -x_1^2 + x_2^2$ or $f(x_1, x_2) = -x_1^2 - x_2^2$.

(b) $\chi(D^2)$ is tangent at most once to the same orbit. It is inmediate that $S_1(\chi)$ is finite and using partition of unity one shows there is a C^{r-1} vector field X on D^2 tangent to $\mathscr{F}(\chi)$ such that $X(q) = 0$ if and only if $q \in S(\chi)$.

Assume X is entering on ∂D^2 and fix $q_0 \in \partial D^2$ such that $\chi(q_0) = p \in \mu$. The ω-limit set of q_0, $\omega(q_0)$, cannot contain points of $S_2(\chi)$ because otherwise μ would contain a 1-dimensional orbit and so by minimality $\mu \subset \Sigma^1$ which is absurd.

Since $S_1(\chi)$ is finite we can assume by perturbing χ if necessary that $\omega(q_0)$ is not a point. By (a) and (b) and the theorem of Poincaré-Bendixon for flows $\omega(q_0)$ is then a periodic orbit of X or a graph with only one vertex which is a saddle point.

The set $\chi(\omega(q_0))$ is the image of a closed curve $\gamma\colon S^1 \to M$ lying on an orbit $\mathcal{O}$ of dimension two. The curve γ is not homotopic to a constant map because the element of the holonomy group of $\mathscr{F}$ associated to γ is not the identity. Therefore $\mathcal{O}$ is homeomorphic to $\mathbb{R} \times S^1$ or $S^1 \times S^1$. Since $\mathcal{O} \subset \mu$ and $\mathcal{O} \neq \mu$, $\mathcal{O}$ cannot be a torus. Thus $\mathcal{O}$ is an immersed cylinder $\mathbb{R} \times S^1$ and $\partial\mathcal{O} \supset \mathcal{O}$. We show now that this is impossible.

We follow arguments of Lima in [5]. Take $w \in \mathcal{O}$. Since $G_w \neq 0$ there are numbers a_1, a_2 such that the orbit Γ of $Z = a_1X_1 + a_2X_2$ passing through w is periodic, here X_1 and X_2 are commuting vector fields generating φ. Fixing a Riemannian metric on M call C the union of small integral curves passing through Γ of the vector field normal to $\mathscr{F}$. Since $\mu \neq \mathcal{O}$ the orbit $\mathcal{O}$ returns infinitely many times to the fence C leaving as intersection a sequence $\{\Gamma_n\}$ of closed simple curves in C arbitrarily close to Γ for large n.

Suppose the vector field X_1 is not tangent to Γ. Then there is a tubular neighborhood $T(\Gamma) \subset C$ of Γ such that X_1 is not tangent to C at points on $T(\Gamma)$. Let η be the flow induced by X_1. For $z \in \Gamma$ call $t^+(z)$ the smallest $t > 0$ such that $\eta(t, z) \in C$ and assume $\{\eta(t^+(z), z) \mid z \in \Gamma\} = \Gamma^+ \subset T(\Gamma)$. Similarly let $t^-(z) < 0$ be the largest $t < 0$ such that $\eta(t, z) \in C$ and suppose $\{\eta(t^-(z), z) \mid z \in \Gamma\} = \Gamma^- \subset T(\Gamma)$. Eventually Γ^+ or Γ^- is empty but not both.

Let $\delta > 0$ be the distance between Γ and $\Gamma^+ \cup \Gamma^-$. Given $\varepsilon = \delta/2$ there is $z_0 \in \Gamma$ and τ such that $\eta(\tau, z_0) \in C$ is ε-close to Γ.

Suppose $\tau > 0$ and call C_1 the part of C between Γ and Γ^+. Then the union of C_1 and the piece of $\mathcal{O}$ given by $D = \{\eta(t, z) \mid 0 \leq t \leq t^+(z), z \in \Gamma\}$ is a topological torus bounding an open set V of M. The vector field X_1 is tanget to D and at points on C_1 it points toward V. Thus we have $\eta(t, z_0) \in V$ for every $t > t^+(z_0)$. Therefore $\eta(\tau, z_0) \in C$ implies that the distance between $\eta(\tau, z_0)$ and Γ is greater than δ which is absurd.

One deals with the case $\tau < 0$ in a similar manner. This finish the proof of the theorem.

Example. Define on $D^2 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ the following vector fields:

$$X(x_1, x_2, e^{i\theta}) = (\rho(r)\, x_1 - \beta x_2, \beta x_1 + \rho(r)\, x_2, 0)$$
$$Y(x_1, x_2, e^{i\theta}) = (-\alpha x_2, \alpha x_1, 2\pi i\, e^{i\theta})$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\rho(r)$ is a C^∞ nonnegative function satisfying, $\rho(r) = 0$ iff $r = 1$, $\rho^{(n)}(1) = 0$ for all $n \geq 1$ and $\rho(r) = 1$ for r in a neighborhood of zero. $\beta = \sqrt{\alpha^2 + 4\pi^2}$ and α is rationally independent of π. The vector fields X, Y commute and so they define an $\mathbb{R}^2$–action on $D^2 \times S^1$. One easily verifies that $\partial(D^2 \times S^1)$ and $\{0\} \times S^1$ are orbits of this action. The remaining orbits are immersed planes dense in $D^2 \times S^1$.

Using the decompositon of S^3 in two solid tori we obtain from this example an $\mathbb{R}^2$-action on S^3.

§3. Remarks on Locally Free $\mathbf{R}^2$-actions

In what follows $\varphi\colon \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes a locally free action i.e. all orbits have dimension two.

(a) One shows that any orbit of φ is a closed subset of $\mathbb{R}^3$. Thus from a theorem of Haefliger [3] φ admits a first integral. Moreover any orbit of φ is embedded and homeomorphic to $\mathbb{R}^2$ (see [5] page 77).

(b) Let $\mathscr{L}\mathscr{F}(\mathbb{R}^2, \mathbb{R}^3)$ be the space of locally free C^r, $r \geq 1$, $\mathbb{R}^2$-actions on $\mathbb{R}^3$ endowed with the uniform C^r topology. In a forthcoming paper we show using techniques of [1] that the set of structurally stable $\mathbb{R}^2$-actions is dense in $\mathscr{L}\mathscr{F}(\mathbb{R}^2, \mathbb{R}^3)$ (see [6] for definitions).

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