Existence, Uniqueness and Approximation of Fixed Points as a Generic Property*

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A *generic property* about points of a topological space is a property which holds for all points which compose a subsete of second category. Since second category is the topological analogue of almost every-where, a generic property is a property which is true for most points of the given space. Orlicz [9] proved trat uniqueness of solutions of the Cauchy problem for ordinary differential equations is a generic property in the space of bounded continuous mappings $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$. Alexiewicz-Orlicz [1] later proved the same result for hyperbolic equations, while recently Lasota-Yorke [8] proved (apparently with some deficiency) that existence, uniqueness and continuous dependence of solutions of the Cauchy problem for ordinary differential equations in a Banach space X a generic property in the space of continuous maps $I \times X \longrightarrow X$.

The aim of this paper is to prove a general theorem about fixed points of monlinear operators which includes all the above mentioned results as special cases. The general theorem not only achieves this goal, but also allows us to solve in full generality the uniqueness problem studied by Cafiero [4], and provides an application to fixed points of nonexpansive mappings.

The following two considerations may motivate the interest in a subset M^* of a given space M of nonlinear operators such that M^* is of second category in M and all $f \in M^*$ has a unique fixed point:

(i) In the application if the subspace M^* of M is also a linear topological space, then one may use all the category theorems of linear analysis (Banach-Steinhaus theorem, open mapping theorem, etc.).

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(ii) Actually, we prove that M^* is *a* G_δ-set in M which contains a given dense set M_0. Therefore, by a theorem of Mazurkiewicz, M^* admits a complete metric d^*. Consequently M^* contains the complement of M_0 under every metric finer than the restriction d_0 of d^* to $M_0 \times M_0$, and coincides with the completion of (M_0, d_0). This may perhaps help to solve the problems of characterizing the continuous maps f for which the Cauchy problem $x' = f(t,x)$, $x(a) = x_0$, has a solution in a given infinite dimensional Banach space, and of the existence of fixed points of nonexpansive mappings.

This paper is divided into four sections:

Section 1 proves the general theorems. Their basic assumption is the existence of a dense subset M_0 of M such that each member of M_0 has a unique fixed point and that there holds a sort of continuous dependence of the fixed points of the members of M_0.

Section 2 applies the general results to the still unsolved problem of the existence of fixed points of nonexpansive mappings. We prove the genericity of existence and uniqueness of fixed points, as well as of their approximation according to the theory of Browder [3].

Section 3 applies the general results to prove, in a much simpler and shorter way (mainly because of the topological viewpoint of the problem) the results of Orlicz [9] and Lasota-Yorke [8]. As another application, Cafiero's uniqueness problem [4] is solved for $\mathbb{R}^n$ with $n > 1$.

Section 4 shows how to carry over the methods of § 3 to other situations, as to integral or to hyperbolic equations. Moreover, an open problem is pointed out.

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§ 1. General results

The main result of this paper is the following:

THEOREM 1. *Let X be a complete metric space and M a set of continuous maps $X \longrightarrow X$ endowed with any metric topology finer than the topology of uniform convergence on X. If M_0 is any subset of M such that*

(a) *Every $f \in M_0$ has a unique fixed point x_f;*
(b) *If $f \in M_0$, $(f_n)_n$ is a sequence in M converging to f in M, and x_n is a fixed point of f_n, $n \in \mathbb{N}$, then $\lim_n x_n = x_f$;*

then there exists a G_δ-set M^* of M such that $M^* \supseteq M_0$, every $f \in M^*$ has a unique fixed point x_f and $f \longrightarrow x_f$ is a continuous map $M^* \longrightarrow X$.

In applications, it is useful that the topology of M is finer than the topology of uniform convergence (cf. § 3). We shall see in the sequel that Condition (b) holds when M_0 is any class of contractions or of codensive operators with unique fixed point or of integral operators arising from ordinary differential equations with a locally Lipschitz second member. We recall that a *G_δ-set* is a countable intersection of open sets. The relation between G_δ-sets and the concept of generic property is that a dense G_δ-set in a second category space is of second category (since the complement of a dense G_δ-set is a first category set, while in a second category space the complement of a first category set is of second category). Then it follows at once the following

COROLLARY 1. *Under the hypotheses of Theorem 1, if M_0 is dense in M and if M is of second category, then* "existence, uniqueness and approximation of fixed points by fixed points of members of M" *is a generic property in M.*

In particular, if M is closed under uniform limits, then M is a complete metric space, hence of second category, for the topology of uniform convergence.

Proof of Theorem 1: Let $T: M_0 \longrightarrow X$ be the map defined by $f \longrightarrow x_f$. By (a) T is well defined, while by (b) T is continuous. Therefore from the statement and the proof of Theorem xiv.8.1 of Dugundji [7] we derive that T has a continuous extension $\tilde{T}$ over a G_δ-set M_1^* contained in the closure $\overline{M_0}$ of M_0 in M. We claim that every $f \in M_1^*$ has a fixed point. In fact, given $f \in M_1^*$ we can find a sequence $(f_n)_n$ in M_0 converging to f in M. The continuity of $\tilde{T}$ implies $\lim_n \tilde{T}(f_n) = \tilde{T}(f)$. This means that $(x_{f_n})_n$ converges to $\tilde{T}(f)$. Moreover, $(f_n)_n$ converges to f even uniformly. Then [12. Lemma] shows that $\tilde{T}(f)$ is a fixed point of f. For each $f \in M_1^*$ let $F(f)$ *be the set of all fixed points* of f. Define $\Phi : M_1^* \longrightarrow \mathbb{R}$ by

$$\Phi(f) = \sup_{x,y \in F(f)} d(x,y) \quad (= \operatorname{diam} F(f))$$

where d is a fixed bounded, complete metric for X (it is well known that every metric is equivalent to a bounded metric). The proof rests on the following statement:

(*) For every $f \in M_0$ and every $n \in \mathbb{Z}^+$ there exists a neighborhood U_n^f of f in M_1^* such that $\Phi(g) \leq \frac{1}{n}$ for all $g \in U_n^f$.

To prove (*), assume the contrary and argue for a contradiction. Then there are $f \in M_0$, $n \in \mathbb{Z}^+$ and (M being metric) a sequence $(f_\kappa)_\kappa$ in M such that $lim_k f_k = f$ in M and $\Phi(f_\kappa) > \frac{1}{n}$ for all κ. By $\Phi(f_\kappa) > \frac{1}{n}$, for every κ there are x_κ, $y_\kappa \in F(f_\kappa)$ such that $d(x_\kappa, y_\kappa) > \frac{1}{n}$. Since $lim_\kappa f_\kappa = f$ in M and $f \in M_0$, (b) implies

$$lim_\kappa \; x_\kappa = x_f = lim \; y_\kappa,$$

a contradiction. Therefore (*) holds. Then

$$U_n = \bigcup_{f \in M_0} U_n^f$$

is an open subset of M_1^*. Therefore

$$M^* = \bigcap_{n=1}^{\infty} U_n$$

is a G_δ-set in M_1^*, hence in M: for, there are two sequences $(V_n)_n$, $(W_n)_n$ of open sets of M such that

$$M_1^* = \bigcap_{n=1}^{\infty} V_n, \; U_n = M_1^* \cap W_n \qquad \text{(all } n).$$

Consequently M^* is the intersection of the sum of two sequences $(V_n)_n$, $(W_n)_n$. Each $f \in M^*$ has a unique fixed point since $M^* \subseteq \Phi^{-1}(0)$ by (*). The mapping $f \longrightarrow x_f$ coincides with $\tilde{T}$ on M^* because we shaw at the beginning of the proof that $\tilde{T}(f)$ was a fixed point of f, and each member of M^* has a unique fixed point. Therefore $f \longrightarrow x_f$ is continuous. q.e.d.

An interesting corollary of Theorem 1 is the following, which allows us to speak about generic properties as in Corollary 1.

COROLLARY 2. *Let X be a bounded complete metric space and M a set of condensive maps $X \longrightarrow X$ endowed with the topology of uniform convergence on X.*

If there exists a dense subsets M_0 of M such that every $f \in M_0$ has a unique fixed point, then (every $f \in M$ has a fixed point and) *there exists a G_δ-set M^* containing M_0 whose members have a unique fixed point. Moreover, if a sequence $(f_n)_n$ in M converges uniformly to $f \in M^*$ and if x_n is a fixed point of f_n for every n, then $(x_n)_n$ converges in X to the unique fixed point of f.*

Recall that a *condensive map* is a continuous map $f: X \longrightarrow X$ such that

$$\gamma(f(A)) < \gamma(A) \qquad (A \subseteq X)$$

where γ(A) denote the measure of non-compactness of A:

$$\gamma(A) = \inf\{\varepsilon > 0 | A \text{ can be covered by a finite number of sets of diameter} \leq \varepsilon\}.$$

Very important examples of codensive maps are furnished by the *k-set-contractions.* These are continuous maps $f: X \longrightarrow X$ with the property

$$\gamma(f(A)) \leq k\gamma(A) \qquad (A \subseteq X).$$

If $k < 1$, a k-set-contraction is condensive. This type of mappings was introduced by Darbo [6] under a different name. For recent contributions to their theory cf. Petryshyn [10] and Sadovskii [11]. We only note that the sum of a compact map and a contraction is a condensive map.

Corollary 2 does not require explicitly Condition (b) of Theorem 1 about the "continuous dependence" of fixed points. However this conditions holds in the class of condensive mappings with unique fixed point:

LEMMA. *Let X be a bounded complete metric space, $f: X \longrightarrow X$ a condensive map, $(f_n)_n$ a sequence of arbitrary maps $X \longrightarrow X$ converging uniformly to f. If x_n is a fixed point of f_n for every n, then $\{x_n | n \in \mathbb{Z}^+\}$ is a relatively compact set, and the limit of every convergent subsequence of (x_n) is a fixed point of f. In particular, if f has a unique fixed point x_f, then $lim_n x_n = x_f$.*

PROOF. By a well known theorem of Kuratowski, X is isometric to a bounded subset Y of a suitable Banach space B. Since X is complete, Y must be too. Thus Y is a closed subset of B. We identify the mappings f, f_n under consideration to the corresponding mappings $Y \longrightarrow Y$. Let I be the identity map of Y. By a well known property of condensive maps, I-f is a proper map $Y \longrightarrow B$. It is clear from the uniform convergence of $(f_n)_n$ to f that

$$\text{(1)} \qquad lim_n \; x_n - f(x_n) = 0$$

Therefore $K = \{0\} \cup \{x_n - f(x_n) \mid n \in \mathbb{Z}^+\}$ is a compact set, and hence $\widehat{I - f}^{-1}(K)$, too, is a compact set. Since $\{x_n \mid n \in \mathbb{Z}^+\} \subseteq \widehat{I - f}^{-1}(K)$, $\{x_n \mid n \in \mathbb{Z}^+\}$ is relatively compact. That the limit of every convergent subsequence of $(x_n)_n$ is a fixed point of f can be derived easily from (1) by the continuity of f. If f has a unique fixed point x_f, then every subsequence $(x_{nk})_k$ of $(x_n)_n$ has a subsequence which converges to x_f. Then $lim_n\ x_n = x$ by a well known theorem on limits. q.e.d.

Proof of Corollary 2. We have only to apply the above lemma and Theorem 1. q.e.d.

If Condition (b) of Theorem 1 is weakened to a continuous dependence of fixed points with respect to members of M_0 only, then we can guarantee only the genericity of the existence of fixed points, as shown by the following result.

THEOREM 2. *Assume the same hypotheses as in Theorem* 1, *except that* (b) *is substituted by the weaker condition*

(b)* *If* $f \in M_0$, $(f_n)_n$ *is a sequence in* M_0 *converging to* f, *and* x_n *is a fixed point of* f_n, *then* $lim_n\ x_n = x_f$.

Then there exists a G_δ*-set* M^* *of* M *containing* M_0 *such that every* $f \in M^*$ *has a fixed point and only one can be approximated by members of* M_0.

The statement

a fixed point x *of* $f \in M$ *can be approximated by fixed points of members of* M_0

means by definition that there is a sequence $(f_n)_n$ in M_0 converging to f in M such that $lim_n\ x_{f_n} = x$.

Sketch of the proof of Theorem 2. We proceed as in the proof of Theorem 1 to obtain M_1^*. For each $f \in M_1^*$, we define $F_0(f)$ to be the set of all fixed points x of f such that there is a sequence $(f_{x,n})_n$ in M_0 with the property

$$lim_n\ f_{x,n} = f \quad \text{and} \quad lim_n\ x_{f_{x,n}} = x.$$

(Note that $F_0(f)$ may not be the whole set of fixed points of f: for example, the identily map of I of $[0,1]$ has all points as fixed points, but only one of them can be approximated by the fixed points of the contractions in $M_0 = \left\{\left(1 - \frac{1}{n}\right) I \mid n \in \mathbb{Z}^+\right\}$.) Defining $\Phi : M_1^* \longrightarrow \mathbb{R}$ by

$$\Phi(f) = \sup_{x,y \in F_0(f)} d(x, y)$$

where d is a fixed bounded, complete metric for X, we prove statement (*) in the proof of Theorem 1 as follows. Assume (*) is false. Then there are $f \in M_0$ and $k \in \mathbb{Z}^+$ such that to every $i \in \mathbb{Z}^+$ there corresponds $f^i \in M_1^*$ for which

$$d'(f, f^i) < \frac{1}{i} \quad \text{and} \quad \Phi(f^i) > \frac{1}{k}$$

d' being any metric for M. Since $\Phi(f^i) > \frac{1}{k}$, for every i there are x_i, $y_i \in F_0(f^i)$ such that $d(x_i, y_i) > \frac{1}{k}$. By definition of $F_0(f^i)$, for every i there are two sequences $(f_{x_i,n})_n$, $(f_{y_i,n})_n$ in M_0 which converge to f^i such that

$$lim_n\ x_{f_{x_i,n}} = x_i,\ lim_n\ x_{f_{y_i,n}} = y_i.$$

There is n_i such that

$$d'(f^i, f_{x_i,n_i}) < \frac{1}{i},\ d'(f^i, f_{y_i,n_i}) < \frac{1}{i} \qquad (i \in \mathbb{Z}^+)$$

and

$$(**) \qquad d(x_{f_{x_i,n_i}}, x_{f_{y_i,n_i}}) > \frac{1}{k} \qquad (i \in \mathbb{Z}^+).$$

Since $lim_i\ f_{x_i,n_i} = lim_i\ f_{y_i,n_i} = f$ in M, (b)* implies

$$lim_i\ x_{f_{y_i,n_i}} = x = lim_i\ x_{f_{x_i,n_i}}$$

which contradicts (**). Thus (*) holds. Then we conclude like in the proof Theorem 1. q.e.d.

§ 2. Application to fixed points of nonexpansive mappings

In this section, the general results are applied to a fixed point problem which has received much attention during the last years. Recall that a map $f: A \subseteq X \longrightarrow X$, X a metric space with metric d, is said to be *nonexpansive if*

$$d(f(x), f(y)) \leq d(x, y) \qquad (x, y \in A).$$

There exist nonexpansive mappings on suitable bounded subsets of a Banach space which do not have a fixed point (cf. for example [13, § 3.6]). Moreover, it is not yet known whether a nonexpansive map on a convex and weakly compact subset of a Banach space has a fixed point, cf. Belluce-Kirk [2, p. 144].

The following theorem shows that most nonexpansive mappings must have a fixed point in convex or, more generally, star-shaped with respect to a point, bounded sets of an arbitrary Banach space. (Recall that a subset A of a Banach space is said to be *star-shaped with respect to a point* $x_0 \in A$ if for every $x \in A$, the segment joining x to x_0 is contained in A). The approximation of fixed points of nonexpansive mappings by fixed points of contractions has been studied by Browder [3]. His results are valid only for Hilbert spaces or for uniformly convex Banach spaces X with X^* strictly convex and with a weakly continuous daulity map. The following theorems show the genericity of this approximation for convex subsets of arbitrary Banach spaces.

Note that if a set A is bounded and star-shaped with respect to a point $x_0 \in A$, then A satisfies the approximation condition of the following theorem. For, the contractions $f_n : A \longrightarrow A$ defined by

$$f_n : x \longrightarrow \left(1 - \frac{1}{n}\right) x + \frac{1}{n} x_0 \qquad (n \in \mathbb{Z}^+)$$

satisfy to the inequality

$$\| x - f_n(x) \| \leq \frac{diam\,(A)}{n} \qquad (x \in A)$$

and hence the requirement. Therefore Theorem 3 applies to bounded, convex or, more generally, star-shaped subsets of arbitrary Banach spaces.

THEOREM 3. *Let X be any complete metric space whose identity mapping is approximated by contractions $X \longrightarrow X$. Then in the set M of nonexpansive mappings $X \longrightarrow X$ endowed with the topology of uniform convergence,* "existence, uniqueness and approximation of fixed points by fixed points of contractions" *is a generic property. More precisely, there is a dense G_δ-set M^* of M such each $f \in M^*$ has a unique fixed point x_f, and if $(f_n)_n$ is a sequence in M^* converging uniformly to $f \in M^*$, then $\lim_n x_{f_n} = x_f$.*

PROOF. Let M_o be the set of contractions $X \longrightarrow X$. By hypotheses, for every $n \in \mathbb{Z}^+$ there is $f_n \in M_0$ such that

$$d(x, f_n(x)) \leq \frac{1}{n} \qquad (x \in X)$$

where d is the metric of X. Therefore

$$d(f(x), f_n(f(x)) \leq \frac{1}{n} \qquad (x \in X\,;\; n \in \mathbb{Z}^+)$$

for every $f : X \longrightarrow X$. If $f \in M$, then $f_n f$ is a contraction $X \longrightarrow X$. We conclude that M_0 is dense in M. It is well known that every $f \in M_0$ has a unique fixed point x_f. Therefore, to apply Theorem 1, we have only to verify Condition (b). But this is a consequence of the lemma above since a contraction is a condensive map. q.e.d.

§3. Applications to the Cauchy problem for ordinary differential equations.

We show now how the main results of Orlicz [9] and of Lasota-Yorke [8] (whose analogue for "almost every-where" are proved in Cafiero [5]) follow from Theorem 1. Wo do this for bounded functions f while Lasota and Yorke avoid this restriction (however, their proof is based on a lemma, namely [8, Lemma 2], the proof of which is not clear to me unless we add the addtional hypothesis of the boundedness of f. In fact, it seems to me we need first to show that the sequence $(x_n)_n$ is equicontinuous. For this purpose I do not see any other way than the boundedness of f). The argument presented here is much simpler and shorter than the arguments of Orlicz and Lasota and Yorke mainly because of a topological viewpoint of the problem. We notice that the topology we need on M is finer than the topology of uniform convergence. Let X be a Banach space, $x_0 \in X$, I the interval $[a, b]$ or $[a, b[$ of $\mathbb{R}$, and B the Banach space of all bounded continuous functions $f : I \times X \to X$ with the sup norm. We derive from Theorem 1 that *existence on I, uniqueness and continuous dependence from data* of the solutions of the Cauchy problems

$$x' = f(t, x), \; x(a) = x_0$$

with $f \in B$, is a generic property on B. For, let Y be the topological space obtained by endowing $C(I, X)$, the set of all continuous functions $x : I \longrightarrow X$, with the topology of uniform convergence on compact subsets of I. Since I is locally compact with a countable base and X metric, Y is a metrizable space. Since X is a complete metric space, Y admits a complete metric. For every $f \in B$, define $T_f : Y \longrightarrow Y$ by

$$T_f(x)\,(t) = x_0 + \int_a^t f(s, x(s))ds.$$

It is easily seen that T_f is a continuous map $Y \longrightarrow Y$. If $f \neq g$, then $T_f \neq T_g$: for, the continuity of $f - g$ implies the existence of an open interval $J \subseteq I$ and a point $x \in X$ such that

$$f(t, x) \neq g(t, x) \qquad (t \in J).$$

Thus for $\bar{x}$ the constant map $t \longrightarrow x$ we have $T_f(\bar{x}) \neq T_g(\bar{x})$ since their derivatives differ on J. Then we define on $M = \{T_f \mid f \in B\}$ the metric topology τ which makes $T : f \longrightarrow T_f$ a homeomorphism. Since $\lim_n T_{fn} = T_f$ in (M, τ) means $\lim_n f_n = f$ uniformly, τ is finer than the topology of uniform convergence on Y. Let L be the set of bounded, locally Lipschitz maps $I \times X \longrightarrow X$. By Lemma 1 of Lasota-Yorke [8], L is dense in B. Therefore $M_0 = \{T_f \mid f \in L\}$ is dense in (M, τ). By a well known result, the Cauchy problem $x' = f(t, x)$, $x(a) = x_0$ has a unique solution on I for every $f \in L$. By Lemma 8 of [14], Condition (b) of Theorem 1 holds. Therefore, by Theorem 1, existence, uniqueness and "continuous dependence" of fixed points is a generic property in (M, τ). Since T_f has a fixed point if and only if the Cauchy problem

$$x' = f(t, x), \; x(a) = x_0$$

has a solution on I and since $T : B \longrightarrow (M, \tau)$ is a homeomorphism, we have proved the result of Lasota-Yorke [8] claimed at the beginning.

In [4] the following problem has been considered for $X = \mathbb{R}$:

CAFIERO'S PROBLEM: *Given $f : I \times X \longrightarrow X$, I an interval of $\mathbb{R}$ and X a Banach space, suppose that the Cauchy problem*

$$x' = f(t, x), \; x(t_0) = x_0$$

has a unique solution for (t_0, x_0) running over a dense subset A of $I \times X$. Then, what about the uniqueness outside A?

Cafiero [4] answers this question for $X = \mathbb{R}$ in the following manner: Uniqueness holds almost everywhere in $I \times \mathbb{R}$ if f satisfies Caratheodory hypotheses. His methods seems to be peculiar to $X = \mathbb{R}$, since it is based on the order relation of $\mathbb{R}$ via the use of maximal and minimal solutions. Noticing that

$$x \longrightarrow x_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad (t \in I)$$

is a compact operator on $C(I\mathbb{R}^n)$ if f is bounded and continuous or, more generally, if f satisfies the Caratheodory hypotheses, the following theorem solves Cafiero's problem in full generality for finite dimensional spaces.

THEOREM 4. *Let I be an interval of R, X a Banach space and $f : I \times X \longrightarrow X$ such that $\|f(t, x)\| \leq h(t)$ with $h \in L^1_{loc}(I, \mathbb{R})$ and, for every $(t_0, x_0) \in I \times X$,*

$$x \longrightarrow x_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad (t \in I)$$

is a compact operator in the space $C(I, X)$ endowed with the topology of uniform convergence on compact subsets of I. If there is a dense subset A of $I \times X$ such that the Cauchy problem

$$(1) \qquad x' = f(t, x), \; x(t_0) = x_0$$

has a unique solution on I for every $(t_0, x_0) \in A$, then "existence on I, uniqueness and continuous dependence of solutions" *of the Cauchy problem* (1) *is a generic property on $I \times X$*

PROOF. Let Y be the space $C(I, X)$ of all continuous functions $I \longrightarrow X$ with the topology of uniform convergence on compacta. As seen in the beginning of § 3, Y is a complete metric space. For every $(t_0, x_0) \in I \times X$, let $F(t_0, x_0)$ be the operator $Y \longrightarrow Y$ defined by

$$F(t_0, x_0)\,(x)\,(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

By hypothesis, each $F(t_0, x_0)$ is a compact operator $Y \longrightarrow Y$. Define

$$M = \{F(t_0, x_0) \mid (t_0, x_0) \in I \times X\}, \; M_0 = \{F(t_0, x_0) \mid (t_0, x_0) \in A\}.$$

Let M be topoligized by the topology of uniform convergence. It is easily seen that M_0 is dense in M. Then, by Corollary 2 to Theorem 1, there is a G_δ-set M^* of M such that $M_0 \subseteq M^*$ and every member of M^* has a unique fixed point. Since the map $F : I \times X \longrightarrow X$ defined by $(t_0, x_0) \longrightarrow F(t_0, x_0)$ is continuous, $A^* = F^{-1}(M^*)$ is a G_δ-set and is dense since $A^* \supseteq A$. Moreover, (1) has a unique solution for every $(t_0, x_0) \in A^*$. By Lemma in § 1 we have the continuous dependence. q.e.d.

When X is infinite dimensional, Cafiero's problem splits into an existence and a uniqueness problem. In other words, in the infinite dimensional case we have the following existence problem analogous to Cafiero uniqueness problem:

Given $f : I \times X \longrightarrow X$, I an integral of R and X a Banach space, if

$$x' = f(t, x), \; x(t_0) = x_0$$

has a solution for (t_0, x_0) running in a dense subset A of $I \times X$, what about the existence outside A?

Of course, Peano's theorem solves this problem for $X = \mathbb{R}^n$. For the infinite dimensional case, the author has no idea at present how to solve it.

§ 4. Remarks on further applications

The arguments of § 3 can be easily carried over to other situations as, for example, to integral equations and to hyperbolic equations.

In the case of integral equations, we can prove the analogue of Theorem 4 on Cafiero's problem, which should be a new result. However, it is an open problem if the results of Orlicz and Lasota-Yorke on the genericity of uniqueness can be extended to integral equations.

For hyperbolic equations

$$\frac{\partial^2}{\partial x \partial y} z = f\left(x, y, z, \frac{\partial}{\partial x} z, \frac{\partial}{\partial y} z \right)$$

we can generalize Theorem 4 of Alexiewicz-Orlicz [1] to arbitrary continuous f by proving also the continuous dependence of solutions. Moreover, we can prove the analogue of Cafiero's problem for the equation above.

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