Foliations of Knot Complements in the Bicylinder Boundary*

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Abstract

In this expository paper, we shall analyze a particularly important class of examples of surfaces and hypersurfaces in Euclidean 4-space, namely those which arise by considering real 4-space as the space of two *complex* variables z and w, and by taking geometric loci of the form $f(z, w) = 0$ or hypersurfaces associated with such loci. Such surfaces and hypersurfaces are important in the study of the singularities of algebraic curves, as described for example in the book of Milnor [3], and they have been used recently in the construction of foliations of the 3-dimensional sphere by Lawson [2]. The examples of this paper were first presented at the International Symposium of Dynamical Systems and Foliations at Salvador in the summer of 1971, and the author expresses his gratitude for the opportunity to participate in that conference.

The examples constructed in this paper are closely related to another paper of the author [1] concerning minimal surfaces in the bicylinder boundary.

1. Introduction. Often when we are studying a surface in space in the neighborhood of a point, it is useful to see how the surface cuts the boundary of a small ball around the point. The nature of the intersection curve can often yield important topological and geometric information, and this process reduces the problem from the study of a surface in 3-space to the study of a curve on a 2-dimensional surface where it may be easier to picture the behavior of the function.

When we work with surfaces or hypersurfaces in a 4-dimensional space, such a method of reducing dimensions is almost essential if we are to get a good picture of what is happening in the neighborhood of a point. The device most often used is that of intersecting the hypersurface or surface with the boundary of a 4-dimensional ball about the point. This reduces the problem to an

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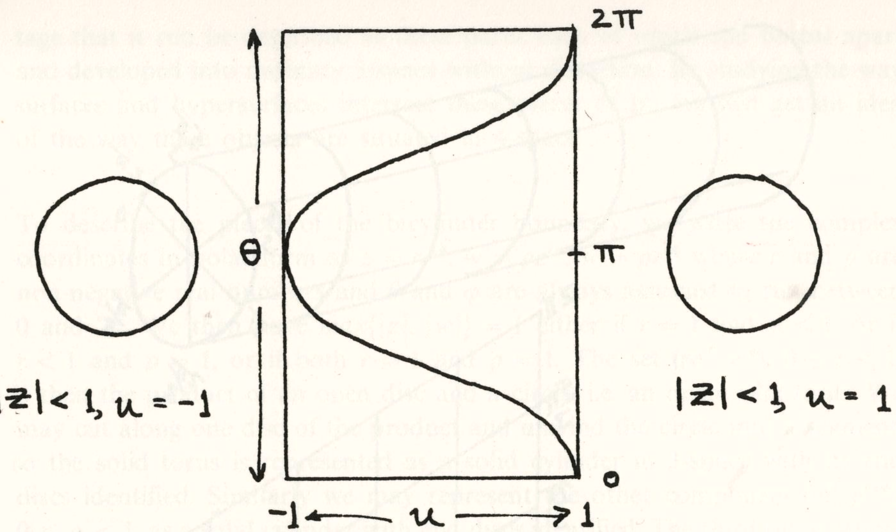
analysis of an ordinary curve or surface in something 3-dimensional, the boundary of the 4-ball.

There is still a difficulty however in visualizing exactly what is happening ina a 3-dimensional ball, even though the space we inhabit is itself 3-dimensional. In a way, the problem is analogous to that of getting a truly accurate picture of curve on a round 2-sphere if we wish to do this by means of representations on a flat page. There are various methods of making maps which serve different purposes, but each one must make some compromises.

In many problems, however, it may be possible to study a surface in the neighborhood of a point by intersecting the surface with a small ball that is not perfectly round. A differently shaped ball may in fact be better suited to the problem under consideration. If for example the problem concerns a surface which is the graph of a real-valued function of a complex variable, then there is no reason at all to prefer a metric ball which treats all of the dimensions alike. We can do better to work with a cylindrical metric, where we refer all of 3-space not to the three real coordinates x, y, u, but rather to the coordinates z, u where u is still a real number but $z = x + iy$ is a complex number. We then introduce the metric which measures the distance from a point z, u to the origin by setting $d(z, u) = max\{|z|, |u|\}$, where $|u|$ is the absolute value of the real number u and $|z|$ stands for the modulus of the complex number z, i.e. its distance to the origin. If we take the "sphere" in this metric with radius 1, then we want all points z, u whith $max\{|z|, |u|\} = 1$, and this sphere is a cylinder consisting of two flat discs $\{|z| < 1, u = 1\}$, $\{|z| < 1, u = -1\}$, a circular cylinder $\{|z| = 1, |u| < 1\}$, and a pair of circles $\{|z| = 1, u = 1\}$ and $\{|z| = 1, u = -1\}$. The advantage of working with such spheres is that the parts can be cut apart and developed into the plane with no distortion whatsoever within each piece.

For example, the graph of the function $f(u)$ = real part of z would meet the cylinder with $|z| = 1$ in the curve $(e^{i\theta}, \cos\theta)$, $0 \leq \theta \leq 2\pi$, where we have set $z = e^{i\theta}$ since the modulus of z is 1 on this cylinder. Since the absolute value of the real part of z is never greater than 1 in this ball, there will be no intersection of this graph with either the top or bottom discs.

In this last example, it turned out to be useful to introduce polar coordinates in the description of the complex variable, setting $z = re^{i\theta}$. Using this no-

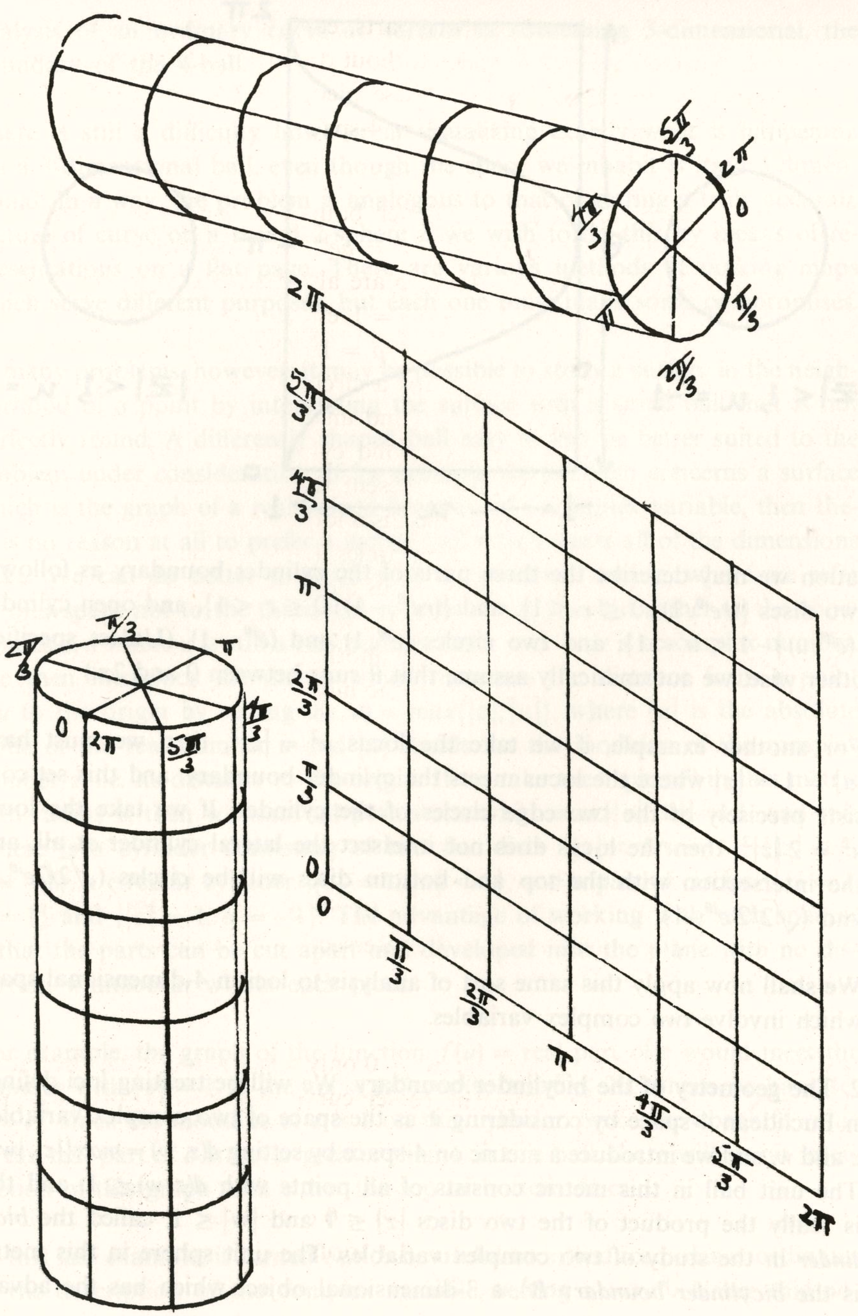


tation we may describe the three parts of the cylinder boundary as follows: two discs $\{(re^{i\theta}, 1) \mid 0 \leq r < 1\}$, and $\{(re^{i\theta}, -1) \mid 0 \leq r < 1\}$, and open cylinder $\{(e^{i\theta}, u) \mid -1 < u < 1\}$, and two circles $(e^{i\theta}, 1)$ and $(e^{i\theta}, -1)$. (Unless specified other wise, we automatically assume that θ runs between 0 and 2π.)

For another example, if we take the locus $u^2 = |z|^2$, then we must have $|u| = 1 = |z|$ where the locus meets the cylinder boundary, and this set consists precisely of the two edge circles of the cylinder. If we take the locus $u^2 = 2|z|^2$, then the locus does not intelsect the lateral cylinder at all, and the intersection with the top and bottom discs will be circles $(\sqrt{2}/2e^{i\theta}, 1)$ and $(\sqrt{2}/2\, e^{i\theta}, -1)$.

We shall now apply this same sort of analysis to loci in 4-dimensional space which involve *two* complex variables.

2. The geometry of the bicylinder boundary. We will be treating loci defined in Euclidean 4-space by considering it as the space of two complex variables, z and w, and we introduce a metric on 4-space by setting $d(z, w) = max\{|z|, |w|\}$. The unit ball in this metric consists of all points with $d(z, w) \leq 1$, and this is really the product of the two discs $|z| \leq 1$ and $|w| \leq 1$, called the *bicylinder* in the study of two complex variables. The unit sphere in this metric is the *bicylinder boundary* B^3, a 3-dimensional object which has the advan-



tage that it can be described as three parts, each of which can be cut apart and developed into ordinary 3-space without distortion. By studying the way surfaces and hypersurfaces intersect these pieces of B^3, we can get an idea of the way these objects are situated in 4-space.

To describe the pieces of the bicylinder boundary, we write the complex coordinates in polar form as $z = re^{i\theta}$, $w = \rho e^{i\varphi}$, $w = \rho e^{i\varphi}$ where r and ρ are non-negative real numbers and θ and φ are always assumed to run between 0 and 2π. We then have $max\{|z|, |w|\} = 1$ either if $r = 1$ and $\rho < 1$, or if $r < 1$ and $\rho = 1$, or if both $r = 1$ and $\rho = 1$. The set $(re^{i\theta}, e^{i\varphi})$, $0 \le r < 1$, is then the product of an open disc and a circle, i.e. an open *solid torus*. We may cut along one disc of the product and unwind the circle into a segment, so the solid torus is represented as a solid cylinder in 3-space with its end discs identified. Similarly we may represent the other component $(e^{i\theta}, \rho^{ei\varphi})$, $0 \le \rho < 1$, as a solid cylinder with end discs identified. The third piece, where $r = 1 = \rho$, is then the product of two circles, $(e^{i\theta}, e^{i\varphi})$, and it is the so-called *flat torus* in 4-space. We may cut this torus along two circles and fold it down into the plane so that it becomes a square with opposite sides identified. All three pieces may be placed next to one another in 3-space so that the identifications are clearly indicated. This is the model which we shall use for our study.

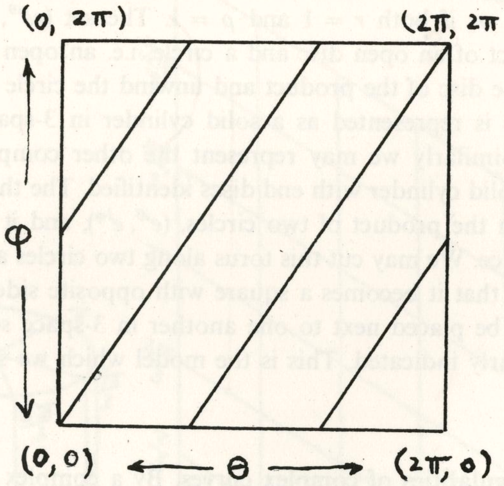
3. Knots and singularities of complex curves. By a complex curve we mean the locus of points in $\mathbb{C}^2$ which are the zeros of a polynomial function in z and w. Setting a complex number equal to zero imposes two real conditions, so the set $V_f = \{(z, w) \text{ in } \mathbb{C} \mid f(z, w) = 0\}$ is a surface. We assume that $f(0, 0) = 0$, and we now consider the intersection of such a surface V_f with the bicylinder boundary B^3.

EXAMPLE 1. If $f(z, w) = z$, then $f(z, w) = 0$ only when $z = 0$. Thus the surface V_f meets B^3 in just one curve, the center curve of the solid torus $(re^{i\theta}, e^{i\varphi})$, $0 \le r < 1$, i.e. the curve $(0, e^{i\varphi})$.

EXAMPLE 2. If $f(z, w) = z - w$, then $f(z, w) = 0$ only when $z = w$. In particular $|z| = |w|$ so V_f meets B^3 only in the flat torus, so $e^{i\theta} = e^{i\varphi}$ and $\theta = \varphi$ since both θ and φ are between 0 and 2π.

Example 3. If $f(z, w) = z^2 - w^2$, then $f(z, w) = 0$ only when $z = w$ or $z = -w$, so again $e^{i\varphi} = e^{i\theta}$ or $e^{i\varphi} = e^{i(\theta+\pi)}$. Thus we get two curves on the flat torus: $\theta = \varphi$ and $\varphi = \theta + \pi$.

Example 4. If $f(z, w) = z^3 - w^2$, then $f(z, w) = 0$ only if $|z|^3 = |w|^2$ so again $|z| = 1 = |w|$ and $(e^{i\theta})^3 = (e^{i\varphi})^2$. Thus $3\theta = 2\varphi$ or more generally $3\theta = 2\varphi + 2n\pi$. Thus either $\theta = \frac{2}{3}\varphi$ or $\theta = \frac{2}{3}\varphi \pm \frac{2n\pi}{3}$, and the intersection is a single curve on the flat torus known as (3,2) torus knot.



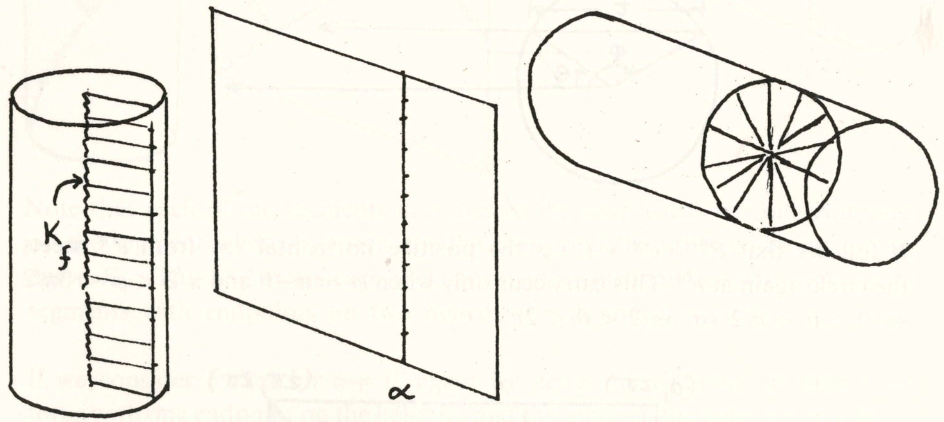
More generally, if $f(z, w) = z^p - w^q$, then the curve $V_f \cap B^3$ is contained in the flat torus, and the number of components of the curve is the greatest common denominator of p and q. Such a curve is known as a (p, q) torus knot.

4. Fibering the complement of a knot. In the previous section, we considered the locus $K_f = V_f \cap B^3 = \{(z, w) \in B^3 \mid f(z, w) = 0\}$. The complement of this curve K_f in B^3 can be decomposed into a family of surfaces F_α, one for each angle $0 \leq \alpha \leq 2\pi$, each with the same boundary curve K_f. We shall examine these surfaces and show how B^3 can be rotated around the curve K_f so that any one of these surfaces can be transformed into any other.

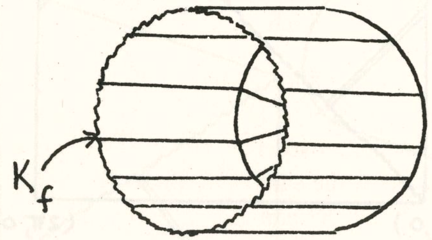
In the complement $B^3 - K_f$, $f(z, w) \neq 0$ so we may divide by the modulus of this value. We define the map $h : B^3 - K_f \longrightarrow S^1$ by $h(z, w) = \dfrac{f(z, w)}{|f(z, w)|}$,

where $S^1 = \{e^{i\alpha} \mid 0 \leq \alpha \leq 2\pi\}$ is the circle of complex numbers of unit modulus. The surface F_α is defined to be $h^{-1}(e^{i\alpha})$. We describe this family explicitly for the examples of the previous section.

Example 1. If $f(z, w) = z$, then $h(z, w) = \dfrac{z}{|z|} = e^{i\theta}$. Thus $h^{-1}(\alpha) = \{(z, w) \in B^3 \mid \dfrac{z}{|z|} = e^{i\alpha}\} = \{(re^{i\alpha},\ e^{i\varphi}),\ 0 \leq r < 1\} \cup \{(e^{i\alpha},\ e^{i\varphi})\} \cup \{(e^{i\alpha},\ \rho e^{i\varphi}) \mid 0 \leq \rho < 1\}$.



The fibre F_α is then a cylinder attached to a disc, giving a figure that can be represented precisely in 3-space as half of a closed circular cylinder.

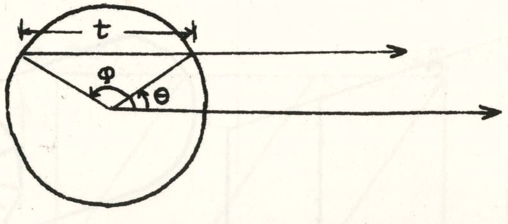


As α changes, the cylindrical parts rotate in the first solid torus and the discs move along the second solid torus. The union of the surfaces F_0 and F_π will be a circular cylinder, and so will $F_\alpha \cup F_{\pi+\alpha}$ for any α.

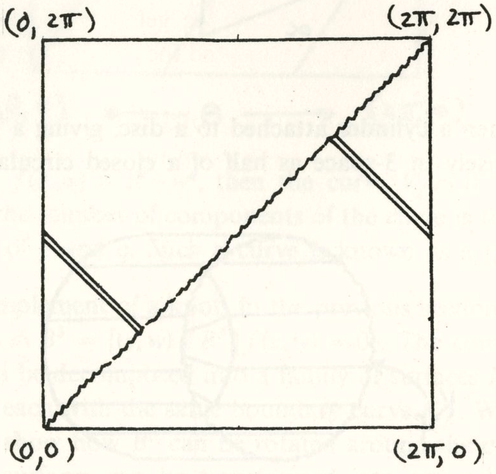
EXAMPLE 2. If $f(z,w) = z-w$, then $h(z,w) = \dfrac{z-w}{|z-w|}$. Consider a specific value $\alpha = 0$, so $h^{-1}(e^{i\theta}) = \{(z,w) \mid h(z,w)$ is a positive real number$\}$. Since $|z-w|$ is a positive real number, we want

$$F_0 = \{(z,w) \mid z-w \text{ is a positive real number}\}.$$

Note first of all that F_0 meets the flat torus only when $e^{i\theta} - e^{i\varphi}$ is a positive real number t.

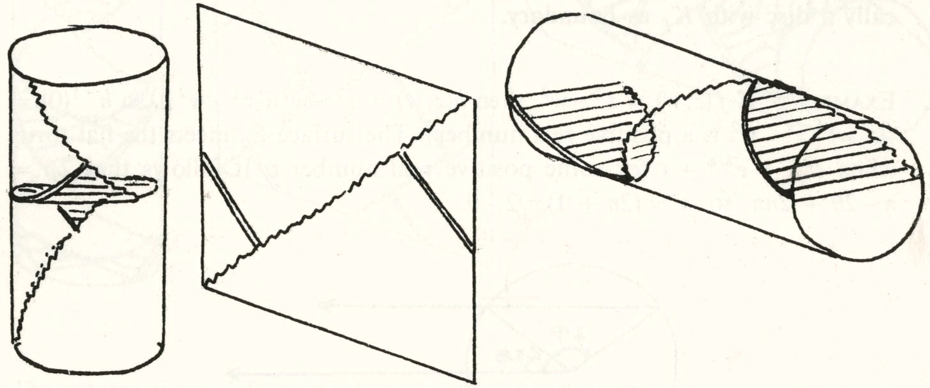


It follows that $e^{i\theta} = e^{i\varphi} + t$, i.e. the positive horizontal ray from $e^{i\varphi}$ meets the circle again at $e^{i\varphi}$. This can occur only when $\varphi = \pi - \theta$ and $\pi/2 < \varphi < 3\pi/2$ so $0 \leq \theta < \pi/2$ or $3\pi/2 < \theta \leq 2\pi$.



Now we may ask as well for the intersection of F_0 with a disc $\{(re^{i\theta}, e^{i\varphi_0})\}$ for some constant φ_0. As above, there will be a *segment* of values $re^{i\theta}$ with $re^{i\theta} = e^{i\varphi} + t$ for φ_0 in the interval $\pi/2 < \varphi_0 < 3\pi/2$, and no locus at all

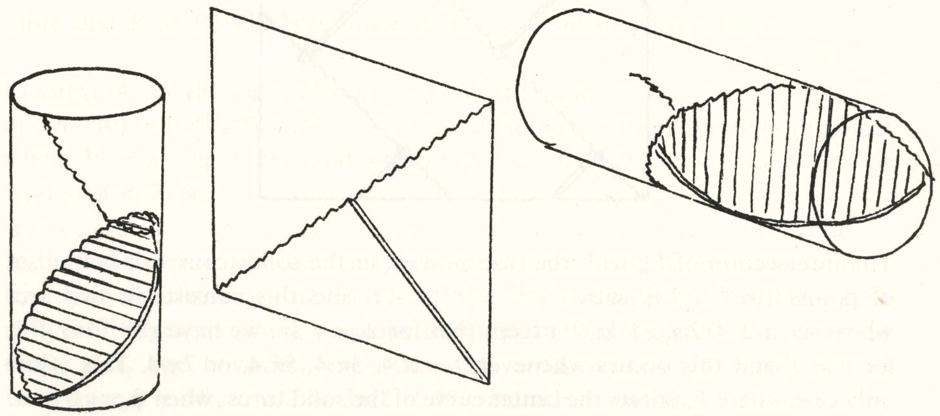
outside of this interval. Similarly for a fixed θ_0, we have F_0 meeting $\{(e^{i\theta_0}, \rho e^{i\varphi})\}$ in a segment if and only if θ_0 is in one of the intervals $0 \leq \theta_0 < \pi/2$ or $3\pi/2 < \theta_0 \leq 2\pi$.



Note that each of the segments in a disc of the solid torus has one endpoint on the segment where F_0 meets the flat torus and the other endpoint on the curve K_f. Thus F_0 meets each solid torus in a ruled surface with horizontal segments with endpoints on two helices, $\theta = \varphi$ and $\varphi = \pi - \theta$.

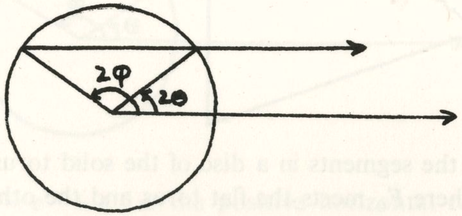
If we consider $h^{-1}(\alpha)$ for $\alpha \neq 0$, again we get a ruled surface in each solid torus with one endpoint on the helix K_f and the other on the helix $\varphi = \pi + 2\alpha - \theta$.

As α changes, the leaves move up one solid torus and along the other. For example, if $\alpha = -\pi/2$, we have $F_{\pi/2}$ meeting the flat torus when $e^{\theta} - e^{\varphi}$ is imaginary of the form ti, $t < 0$.

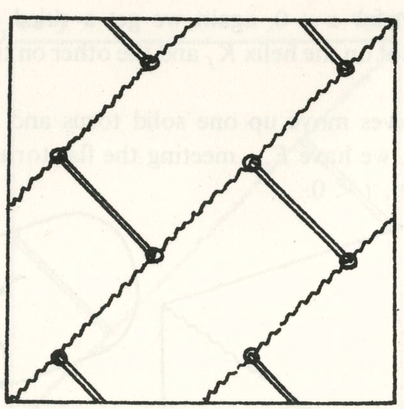


This locus consists of all points on the helix $\varphi = 2\pi - \theta$, with $0 < \varphi < \pi$, and the ruled surfaces in each solid torus have segments with one endpoint on this helix and the other on K_f. In each case, the surface F_α is topologically a disc with K_f as boundary.

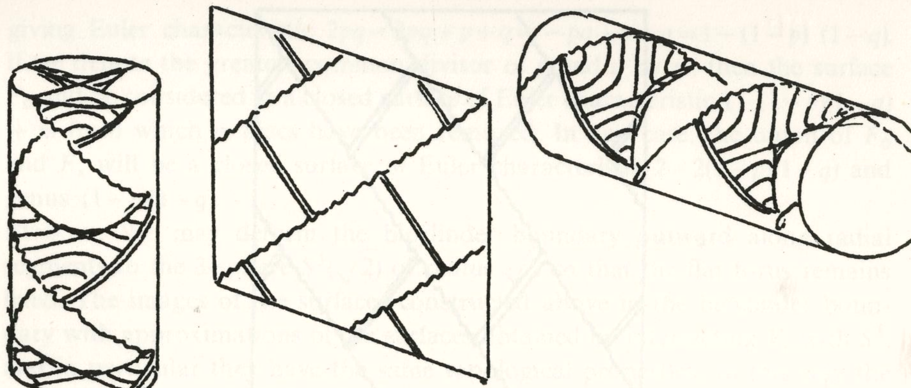
EXAMPLE 3. If $f(z, w) = z^2 - w^2$, then $h(z, w) = z^2 - w^2/(|z^2 - w^2|)$, so $h^{-1}(0) = \{(z, w) \mid z^2 - w^2$ is a positive real number$\}$. The surface F_0 meets the flat torus when $e^{2i\theta} = e^{2i\varphi} + t$ for some positive real number t. It follows that $2\varphi = \pi - 2\theta + 2n\pi$, so $\varphi = (2n + 1)\pi/2 - \theta$.



Also we must have $\pi/2 < 2\varphi < 3\pi/2$, so $\pi/4 < \varphi < 3\pi/4$ or $5\pi/4 < \varphi < 7\pi/4$.



The intersection of F_0 with the disc $\varphi = \varphi_0$ in the solid torus will be the set of points $\{(re^{i\theta}, e^{i\varphi_0})\}$ with $r^2 e^{i2\theta} = e^{i2\varphi_0} + t$, and this consists in two arcs whenever $\pi/2 < 2\varphi_0 < 3\pi/2$, except that for $2\varphi_0 = 2\pi$, we have $r^2 e^{i2\theta} = t + 1$, for $t > 0$, and this occurs whenever $\theta = \pi/4, 3\pi/4, 5\pi/4$, or $7\pi/4$. This is the only case where F_0 meets the center curve of the solid torus, when $\varphi_0 = 0$ or π.



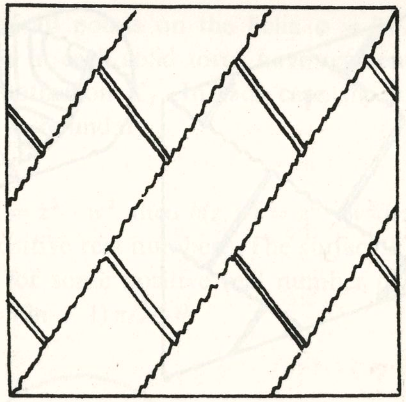
The intersection of F_0 with the solid torus $(re^{i\theta}, e^{i\varphi})$ then consists of a pair of saddle-shaped discs with two boundary edges on K_f and two on $F_0 \cap (S_1 \times S_2)$. The same configuration occurs in the other solid torus, so that the entire boundary of the set F_0 is K_f.

We can determine the topological nature of the surface F_0 by computing its Euler characteristic. There is a natural cell decomposition of the surface with four open 2-cells given by the discs in the solid tori, eight vertices, and twelve edges on the flat torus. The Euler characteristic is then $8 - 12 + 4 = 0$. Since K_f is a curve with two components, it follows that the surface F_0 is topologically a cylinder.

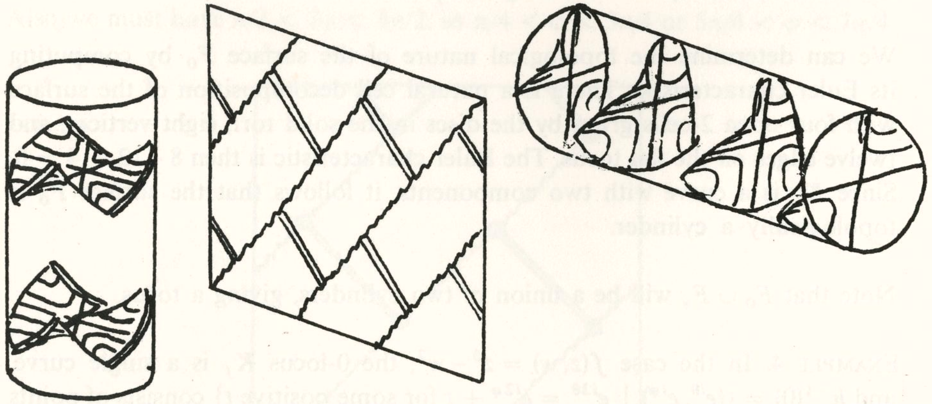
Note that $F_0 \cup F_\pi$ will be a union of two cylinders, giving a torus.

EXAMPLE 4. In the case $f(z, w) = z^3 - w^2$, the 0-locus K_f is a single curve, and $h^{-1}(0) = \{(e^{i\theta}, e^{i\varphi}) \mid e^{i3\theta} = e^{i2\varphi} + t$ for some positive $t\}$ consists of points where $3\theta = \pi - 2\varphi + 2\pi n$, and where $\pi/2 < 2\varphi < 3\pi/2$ so $\pi/4 < \varphi < 3\pi/4$ or $5\pi/4 < \varphi < 7\pi/4$.

In the solid tori we get different phenomena since z and w appear unsymmetrically in the defining equation f. In $S_1 \times D_2$, we have three surfaces each with four edges, as in the previous example, but for a disc $\theta = \theta_0$ constant in $D_1 \times S_2$, we have $3\theta_0 = \pi + 2\pi n - 2\varphi$ so $2\varphi = (2n + 1)\pi - 3\theta_0$ and we must have $0 < 3\theta_0 < \pi/2$ or $3\pi/2 < 3\theta_0 < 2\pi$. For such θ_0, the intersection



of the disc with F_0 consists of three arcs, except when $3\theta_0 = 2\pi m$, when the intersection consists of three straight lines through the central curve of $D_1 \times S_2$.



In this example, we have a cell decomposition with five 2-cells, eighteen 1-cells, and twelve 0-cells, giving an Euler characteristic of $12 - 18 + 5 = -1$. The surface is then a torus with one disc removed, and the closed figure formed by taking $F_0 \cup F_\pi$ is a surface of characteristic -2 and genus 2.

Similarly we may analyze the fibres F_α for the complement of a knot $f(z, w) = z^p - w^q$. We get a surface F_0 with $p + q$ 2-cells, $2pq$ 0-cells, and $3\,pq$ 1-cells,

giving Euler characteristic $2pq - 3pq + p + q = -pq + p + q = 1 - (1-p)\ (1-q)$. If we denote the greatest common divisor of p and q by m, then the surface F_0 and be considered as a closed surface of Euler characteristic $1 - (1-p)(1-q) + m$ from which m discs have been removed. In any case, the union of F_0 and F_π will be a closed surface of Euler characteristic $2 - 2(1-p)(1-q)$ and genus $(1-p)(1-q)$.

REMARK. We may deform the bicylinder boundary outward along radial segments to the 3-sphere $S^3(\sqrt{2})$ of radius $\sqrt{2}$ so that the flat torus remains fixed. The images of the surfaces constructed above in the bicylinder boundary with approximations of the surfaces obtained by intersecting V_f with S^3, and in particular they have the same topological properties. In this way the simpler geometry of the bicylinder boundary leads to a better understanding of the intersection with S^3 of loci related to functions $f(z, w)$.

BIBLIOGRAPHY

[1] BANCHOFF, T. F., *Minimal surfaces in the bicylinder boundary* (to appear).

[2] LAWSON, H. B., *Codimension-one Foliations of Spheres*, Annals of Mathematics **94** (1971), 494-503.

[3] MILNOR, J., *Singular Points of Complex Hypersurfaces*, Princeton University Press.

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