A Generator Theorem for Flows*

ERNST EBERLEIN**

Let $(\Omega, \mathscr{L}, m, (T_t)_{t\in R})$ be a flow on a Lebesgue measure space. We call a partition $\pi = \{P_j \mid j \in J\}$ of Ω into measurable sets a *generator of finite type* (for $(T_t)_{t\in R}$) if (1) there is a $t_0 \in R$ such that π generates under T_{t_0}, (2) for every P_j there is an $\alpha_j > 0$ such that every orbit entering P_j will stay there during a time-interval of length at least α_j, (3) orbit-pieces of finite length intersect only a finite number of the P_j (for a more formal definition see [1]).

This concept of generator for flows has shown its usefulness in [1], [2]. We prove the following theorem.

THEOREM. *Let $(\Omega, \mathscr{L}, m, (T_t)_{t\in R})$ be an aperiodic flow on a Lebesgue measure space with finite entropy $h((T_t)_{t\in R})$. Then there exists a (countable) generator of finite type σ having finite entropy $H(\sigma)$.*

We remark that the real number $t_0 \neq 0$ such that σ generates under T_{t_0} can be prescribed in advance.

The theorem above was proved by Ornstein [2] under the additional assumption that the flow is mixing. The new result will come out from a combination of Ornstein's proof with ideas developed in [1], where we showed the existence of countable generators of finite type for aperiodic flows in the absence of considerations of entropy.

1. Preliminaries. We repeat only some of the notation of [1]. For two countable measurable partitions π, γ we denote by $d(\pi, \gamma)$ the distance given by the measure of the symmetric difference. If $B \in \mathscr{L}$ is a set with $m(B) > 0$ and π a partition one of whose elements is $\complement B$ (the complement) then we say: π is a partition on B.

*Recebido pela SBM em 12 de fevereiro de 1974.
**This research was done during a stay at IMPA, Rio de Janeiro, which took place under the Brazilian-German Cooperation Agreement CNPq-GMD.

Examples of such partitions are the partitions

$$\pi \cap B = \{P \cap B\ (P \in \pi),\ \complement B\}$$

induced by a π on sets B with $m(B) > 0$. If I is an interval of finite length $|I|$, $B \subset \Omega$ and $w \in B$, then we write $T_I w$ for the finite orbit-piece $\{T_t w \mid t \in I\}$, and we define

$$l(B, w) = \sup \{s \geq 0 \mid \exists I \subset R,\ 0 \in I,\ |I| = s \text{ and } T_I w \subset B\}$$

and

$$l(B) = \inf_{w \in B} l(B, w).$$

Then $l(P_j)$ is the sup of all possible α_j occurring in (2) in the definition of a generator of finite type.

Every aperiodic flow is proper (see [1]) and therefore can be represented as a flow under a function. An essential part of our proof is lemma (2.4) of [1]. It reads

LEMMA. *Let $(M, \mathscr{L}, m, (T_t)_{t \in R})$ be an aperiodic flow under a function, $t_0 > 0$, and $(\varepsilon_n)_{n \in N}$, $\varepsilon_{n+1} < \varepsilon_n$, $\varepsilon_n > 0$ a sequence converging to 0. Then there are sets $R_n \in \mathscr{L}$ and nonnegative integers N_n, L_n $(n \in \mathbb{N})$ such that*

(1) $m(R_n) < \varepsilon_n \quad (n \in \mathbb{N})$

(2) *the sets* $T_{it_0} R_n$ $(-N_n \leq i \leq L_n)$ *are disjoint and*

$$m\Big(\sum_{i=-N_n}^{L_n} T_{it_0} R_n\Big) > 1 - \varepsilon_n \quad (n \in \mathbb{N})$$

(3) $R_n \cap R_m = \emptyset \quad (n \neq m)$

(4) $l(R_n) = t_0,\ l(\complement R_n) \geq (N_n + L_n)t_0,\ l\Big(\complement\big(\sum_{m=1}^{n} R_m\big)\Big) \geq t_0 2^{-1} \quad (n \in \mathbb{N}).$

REMARK. The inductive construction shows immediately that R_n, L_n, N_n do not depend on ε_m $(m > n)$ and N_n can be chosen arbitrarily large.

2. Proof of the theorem. (1) First we note the following (see [2]), using the fact that an aperiodic flow has a representation as a flow under a function: If π is a countable partition such that $H(\pi) < \infty$, then for any $\varepsilon > 0$, there exists a finite partition $\pi' = \{P'_1, \ldots, P'_n\}$ such that $H(\pi') < 2H(\pi)$, $d(\pi, \pi') < \varepsilon$, $l(P'_i) > 0$ $(1 \leq i \leq n)$, and there exists a partition ρ such that $H(\rho) < \varepsilon$ and $\pi' \vee \rho \supset \pi$.

(2) Let $t_0 \neq 0$ be given. Then T_{t_0} is an aperiodic discrete-time transformation with finite entropy $h(T_{t_0})$. Thus — by Rohlin's theorem ([3]) — there exists a countable generator $\pi = \{P_1, P_2, \ldots\}$ for T_{t_0} having finite entropy $H(\pi)$. We fix π and some $\varepsilon > 0$.

We choose $\varepsilon_1 > 0$ such that the following holds: if τ is a partition with $d(\tau, \pi) < \varepsilon_1$, then there exists a partition ρ' such that $\tau \vee \rho' \supset \pi$ and $H(\rho') < \varepsilon$. Let n be sufficiently large so that

(a) $$n^{-1} H\Big(\bigvee_{k=0}^{n-1} T_{-kt_0} \pi\Big) - h(T_{t_0}, \pi) < \varepsilon 2^{-1},$$

and

(b) $$-t \log t - (1-t)\log(1-t) < \varepsilon 2^{-1} \qquad t \in [0, n^{-1}[.$$

We choose $R_1 \in \mathscr{L}$ and integers N_1, L_1 according to the lemma above.

We can assume that $N_1 \geq n$, and denote

$$\pi(N_1, L_1) = \bigvee_{i=-N_1}^{L_1} T_{-it_0} \pi,$$

$$D_1 = \Omega \setminus \sum_{i=-N_1}^{L_1} T_{it_0} R_1,$$

$$\gamma_1 = \{D_1,\ T_{it_0} R_1\ (-N_1 \leq i \leq L_1)\}$$

and $v = \{\Omega\}$, i.e. v is the trivial partition. Then the following equation holds:

$$\sum_{k=-N_1}^{L_1} H(\pi(N_1, L_1) \cap T_{kt_0} R_1 \mid v \cap T_{kt_0} R_1) =$$

$$= \sum_{k=-N_1}^{L_1} [H(\pi(N_1, L_1) \cap T_{kt_0} R_1) - H(\nu \cap T_{kt_0} R_1)] =$$

$$= H(\pi(N_1, L_1) \vee \gamma_1) - H(\pi(N_1, L_1) \cap D_1) + H(\nu \cap D_1) - H(\gamma_1).$$

At least one term of the sum on the left-hand side cannot exceed the right-hand side divided by $N_1 + L_1 + 1$. We can assume that this is the term given by $k = 0$. But

$$H(\pi(N_1, L_1) \vee \gamma_1) - H(\gamma_1) = \\ = H(\pi(N_1, L_1) \mid \gamma_1) \leq H(\bigvee_{i=0}^{N_1+L_1} T_{-it_0} \pi)$$

and

$$H(\nu \cap D_1) - H(\pi(N_1, L_1) \cap D_1) \leq 0$$

and therefore

$$H(\pi(N_1, L_1) \cap R_1 \mid \nu \cap R_1) \leq \\ \leq (N_1 + L_1 + 1)^{-1} H(\bigvee_{i=0}^{N_1+L_1} T_{-it_0} \pi) < h(T_{t_0}, \pi) + \varepsilon 2^{-1} \leq \\ \leq H(\pi) + \varepsilon 2^{-1}.$$

From $m(R_1) < n^{-1}$ follows $H(\nu \cap R_1) < \varepsilon 2^{-1}$, and we conclude

$$H(\pi(N_1, L_1) \cap R_1) \mid < H(\pi) + \varepsilon$$

Furthermore we have

$$\bigvee_{l=-N_1}^{L_1} T_{lt_0} (\pi(N_1, L_1) \cap R_1) \overset{\varepsilon_1}{\supset} \pi,$$

Which means that there is a τ such that

$$\bigvee_{l=-N_1}^{L_1} T_{lt_0} (\pi(N_1, L_1) \cap R_1) \supset \tau$$

and $d(\tau, \pi) < \varepsilon_1$. By the choice of ε_1, there exists π_1 such that $H(\pi_1) < \varepsilon$ and $\tau \vee \pi_1 \supset \pi$. Therefore

$$\bigvee_{l=-N_1}^{L_1} T_{lt_0} [(\pi(N_1, L_1) \cap R_1) \vee \pi_1] \supset \pi.$$

Using (1) we can replace $\pi(N_1, L_1) \cap R_1$ by a finite partition $\sigma_1 = \{S_1^1, \ldots, S_{r_1}^1, \complement R_1\}$ on R_1 such that $H(\sigma_1) < 2H(\pi) + 2\varepsilon$, $l(S_k^1) > 0$ $(1 \leq k \leq r_1)$ and

$$\bigvee_{l=-\infty}^{\infty} T_{lt_0} (\sigma_1 \vee \pi_1) \supset \pi.$$

(3) We now repeat the procedure in (2), using π_1 instead of π and ε_1 instead of ε, in order to get a finite partition $\sigma_2 = \{S_1^2, \ldots, S_{r_2}^2, \complement R_2\}$ on a set R_2 (disjoint from R_1) such that

(a) $$H(\sigma_2) < 2\varepsilon + 2\varepsilon_1,$$

(b) $$l(S_k^2) > 0 \qquad (1 \leq k \leq r_2)$$

and

(c) There exists a partition π_2 such that $H(\pi_2) < \varepsilon_1$ and

$$\bigvee_{l=-\infty}^{\infty} T_{lt_0} (\sigma_1 \vee \sigma_2 \vee \pi_2) \supset \pi.$$

(4) Continuing this process we get finite partitions σ_k on sets R_k $(k \geq 1)$. Define $\sigma = \bigvee_{k \geq 1} \sigma_k$. By the disjointness of the R_n $(n \geq 1)$, σ is a countable partition consisting of the sets S_i^k $(k \geq 1,\ 1 \leq i \leq r_k)$ and $\complement(\sum_{k=1}^{\infty} R_k)$. Since $l(\complement(\sum_{k=1}^{\infty} R_k) \geq t_0 2^{-1}$, we conclude $l(S) > 0$ for all $S \in \sigma$. $l(R_n) = t_0$ $(n \geq 1)$ implies that any orbit-piece of finite length intersects only a finite number of the R_n, each of them consisting of a finite number (namely r_n) of elements of σ. This verifies (3) in the definition of generator of finite type. Since ε_k $(k \geq 1)$ can be chosen such that $\sum_{k \geq 1} \varepsilon_k < \infty$ we conclude

$$H(\sigma) \leq \sum_{k=1}^{\infty} H(\sigma_k) < \infty.$$

Furthermore we have $\bigvee_{l=-\infty}^{\infty} T_{lt_0} \sigma \supset \pi$, and since π was chosen to be a generator for T_{t_0} we are through.

REFERENCES

[1] EBERLEIN, E., *Einbettung von Stromungen in Funktionenraume durch Erzeuger vom endlichen Typ* (to be published in Zeitschrift für Wahrscheinlichkeitstheorie).

[2] ORNSTEIN, D., *The isomorphism theorem for Bernoulli flows*, Advances in Mathematics Vol. **10** (1973), p. 124-142.

[3] ROHLIN, V. A., *Lectures on the entropy theory of measure-preserving transformations*, Russian Math. Surveys Vol. **22** (5), (1967), p. 1-52.

Instituto de Matemática Pura e Aplicada
Rio de Janeiro – BRASIL