Almost Surely Convergent Versions of Sequences which Converge Weakly*

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1. Introduction

A. V. Skorokhod established in [5] the following theorem, Let (S, d) be a complete separable metric space, $\{\mu_n\}$ $n = 1, 2, \ldots$ a sequence of Borel probability measures which converge weakly to a probability measure μ. There exists a probability space $(\Omega, \mathscr{A}, P)$, (which can be taken as the interval $[0, 1]$, with the Borel σ-algebra and the Lebesgue measure on it) and a sequence of random variables $\{X_n\}_{n=0,1,\ldots}$ on this space taking values in S such that, the distribution of X_n is μ_n, the distribution of X is μ, and such that $X_n \longrightarrow X$ [a.s.P]. This result was extended by Dudley in [2] to the case of separable metric spaces. In [4] Pyke posed the question of whether this result could be extended further to the case in which the mode of convergence of the probability measures is the one used in [2].

This extension was obtained by Wichura in [6]. The purpose of this paper is to give a different proof of that result by properly modifying the arguments used by Dudley in [2]. Briefly, the methods used in [2] work in the general case.

The proof is full of measurability difficulties which seem to be unavoidable. Theorem 3.2 is an extension of Theorem 3.1. The proof is very close to that of Theorem 3.1 and therefore only the steps of the proof are indicated. One could of course prove Theorem 3.2 directy but we have chosen to prove Theorem 3.1 and sketch Theorem 3.2 in order not to complicate the proof even further with difficulties of a different nature.

The results are stated and proved for the case of sequences of measures. With the obvious modifications they are valid for nets.

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2. Basic Notation and Terminology

Throughout this paper the pair $(\Omega, \mathscr{A})$ and the triple $(\Omega, \mathscr{A}, P)$ will denote a measurable space and a probability space respectively. For $A \subseteq \Omega$, I_A will denote the function which takes the value 1 on A and 0 in the complement of A. We write $P(A)$ for the probability of the event $A \in \mathscr{A}$, and $\int f dP$ for the integral of an $\mathscr{A}$-measurable, P-integrable, real valued function f defined on Ω. Let $\mathbb{R}$ denote the set of all real numbers.

For $f : \Omega \longrightarrow \mathbb{R}$ we define $\int_* f dP = sup\{\int g dP : g\ \mathscr{A}\text{-measurable}, g \leq f, \int g dP$ is defined$\}$ Similarly we define $\int^* f dP$. If $A \subseteq \Omega$ we will write instead of $\int^* I_A dP$ and $\int_* I_A dP$, $\mu^*(A)$ and $\mu_*(A)$ respectively.

If $X : \Omega \longrightarrow \Omega'$ and $\mathscr{A}$ and $\mathscr{A}'$ are two σ-algebras in Ω and Ω' respectively we say that X is $\mathscr{A}$-$\mathscr{A}'$ measurable when for all $A' \in \mathscr{A}'$, $X^{-1}(A') \in \mathscr{A}$. We will often use the notation $[X \in A']$ to indicate the set $X^{-1}(A')$.

If $(\Omega, \mathscr{A}, P)$ and $(\Omega', \mathscr{A}', P')$ are two probability spaces $P \times P'$ will denote the product probability on $(\Omega \times \Omega', \mathscr{A} \times \mathscr{A}')$. Given $(\Omega, \mathscr{A}, P)$, $(\Omega', \mathscr{A}')$ and $X : \Omega \longrightarrow \Omega'$ $\mathscr{A} - \mathscr{A}'$ measurable, we denote by PX^{-1} a probability on $(\Omega', \mathscr{A}')$ defined by $PX^{-1}(A') = P(X^{-1}(A'))$. We will also use the symbol. $\mathscr{L}(X)$ for PX^{-1}. Let now (S, d) be a metric space, with distance function d. Let $C(S)$ denote the set of all bounded real valued continuous functions on S. We write $B_{x,r}$ (resp. $\overline{B_{x,r}}$) for the open (resp. closed) ball centered at $x \in S$ of radius $r > 0$:

$$B_{x,r} = \{y \in S : d(x, y) < r\} \quad \overline{B_{x,r}} = \{y : d(x, y) \leq r\}$$

For $A \subseteq S$, we let A^c, $\bar{A}$, ∂A and A^δ denote respectively the complement of A, the closure of A, the boundary of A and open δ-ball $\{y : y \in S, d(y, A) < \delta\}$ about A. We let $\mathscr{B}$ denote the Borel σ-algebra of S; this is the σ-algebra generated by the topology induced by d. It coincides with the minimal σ-algebra that makes measurable the bounded continuous real valued functions on S. Let S_0 denote the σ-algebra generated by the balls of S. If the metric space is separable then clearly $\mathscr{B} = S_0$.

It is easily seen that if K is compact $d(\cdot, K)$ is S_0-measurable, from which it follows that any compact set $K \in S_0$ and for any $\delta \geq 0$ also $K^\delta \in S_0$. If $\mathscr{C}$ is a σ-algebra, $S_0 \subseteq \mathscr{C} \subseteq \mathscr{B}$ and μ is a probability measure on $\mathscr{C}$ we say that

μ is tight iff $\forall \varepsilon > 0$, there exists K a compact such that $\mu(K) > 1 - \varepsilon$. A subset A of S is said to be a P-continuity set (P defined on $\mathscr{B}$) iff $P(\partial(A)) = 0$. The class of all P-continuity sets is easily seen to be an algebra.

For basic results and definitions concernin the weak convergence of measures, the reader is referred to [1] and [3].

3. Almost Surely Convergent Versions of Sequences which Converge Weakly

We pass now to the study of a problem posed by Pyke in [4]. For applications of this kind of results to processes whose laws converge weakly see also [4].

Let (S, d) be a metric space, $\mathscr{B}$ the Borel σ-algebra, S_0 the σ-algebra generated by the balls, $\{\mathscr{D}_n\}_{n=1,2,\ldots}$ a sequence of σ-algebras each of them containning S_0 and contained in $\mathscr{B}$, μ_n a probability measure defined on $\mathscr{D}_n$, μ a probability measure on $\mathscr{B}$. We have the following theorem.

THEOREM 3.1. *If $\mu_n \xrightarrow{\omega} \mu$ and the supportof μ is separable, then there exist a probability space $(\Omega, \mathscr{A}, P)$ and a sequence of random elements $\{X_n\}_{n=0,1,\ldots}$ such that X_0 is $\mathscr{A} - \mathscr{B}$ measurable, X_n is $\mathscr{A} - \mathscr{D}_n$ measurable, $\mathscr{L}(X_0) = \mu$, $\mathscr{L}(X_n) = \mu_n$ and $X_n \longrightarrow X_0$ [a.s. P].*

PROOF. Let $\{\eta_k\}_{k=1,2,\ldots}$, $\{\delta_k\}_{k=1,2,\ldots}$, $\{\varepsilon_k\}_{k=1,2,\ldots}$ be three sequences such that $1 > \eta_k > 0$, $1 > \varepsilon_k > 0$, $\delta_k > 0$, $\delta_k \to 0$, $\varepsilon_k \to 0$ and $\sum_{k=1}^{\infty} \eta_k < \infty$. Let F be the support of μ, ($F \in \mathscr{B}$, $\mu(F) = 1$).

Let $B_{x,r_x} = \{y : d(x, y) < r_x\}$ be an open ball such that $\mu(\partial(B_{x,r_x})) = 0$ with $r_x < \delta_k$. Since $\bigcup_{x \in F} B_{x,r_x} \supseteq F$ and F is separable we can select a countable subcovering $\{B_{x_n,r_{x_n}}\}_{n=1,2,\ldots}$.

If $C'_{k,1} = B_{x_1,r_{x_1}}$ and $C'_{k,n} = B_{x_n,r_{x_n}} - \bigcup_{i=1}^{n-1} B_{x_i,r_{x_i}}$ we have $\sum_{i=1}^{\infty} C'_{k,i} = \bigcup_{n=1}^{\infty} B_{x_i,r_{x_n}} \supseteq F$. Since the class of all sets A such that $\mu(\partial(A)) = 0$ is an algebra, it follows that $\mu(\partial(C'_{k,i})) = 0$ for all i. Clearly also $C'_{k,i} \in S_0$ for all i. Select i'_k in such a way that $\sum_{i=1}^{i'_k} \mu(C'_{k,i}) > 1 - \eta_k$ and disregard those $C'_{k,i}$'s such that $\mu(C'_{k,i}) = 0$. In this form we constructed a family of sets $\{C_{k,i}\}_{i=1,2,\ldots,i_k}$, $i_k \leq i'_k$ with the following properties: $C_{k,i} \cap C_{k,j} = \emptyset$ if $i \neq j$, $\mu(C_{k,i}) > 0$, $\mu(\partial(C_{k,i})) = 0$, $C_{k,i} \in S_0$, $\sum_{i=1}^{i_k} \mu(C_{k,i}) > 1 - \eta_k$ and the diameter of $C_{k,i}$ is smaller than δ_k. Let $\Omega_n = S \times [0, 1]$, $n = 0, 1, \ldots$. Consider in Ω_0 the σ-field $\mathscr{B} \times \mathscr{B}'$ where $\mathscr{B}'$ is the Borel σ-field of $[0, 1]$ and in Ω_n, $n \geq 1$, the σ-algebra $\mathscr{D}_n \times \mathscr{B}'$.

Let $(\Omega, \mathscr{A}) = (\prod_{n=0}^{\infty} \Omega_n, (\mathscr{B} \times \mathscr{B}') \times \prod_{n=1}^{\infty} (\mathscr{D}_n \times \mathscr{B}'))$.

Take $\bar{x}_0 \in F$ and define h: $S \longrightarrow S$ by

$$h(x) = \begin{cases} x & \text{if } x \in F \\ \bar{x}_0 & \text{if } x \notin F \end{cases}$$

If $\omega = ((x_0, t_0), (x_1, t_1), \ldots) \in \Omega$ then let $X_0(\omega) = h(x_0)$ and for $n \geq 1$ $X_n(\omega) = x_n$. It is easy to check that X_0 is $\mathscr{A} - \mathscr{B}$ measurable and X_n is $\mathscr{A} - \mathscr{D}_n$ measurable. Let

$$A_{n,k,i} = C_{k,i} \times [0, \alpha_{n,k,i})$$

$$B_{n,k,i} = C_{k,i} \times [0, \beta_{n,k,i})$$

where $\alpha_{n,k,i}$ and $\beta_{n,k,i}$ are determined by the equations:

$$\mu(C_{k,i}) \wedge \mu_n(C_{k,i}) = \mu(C_{k,i})\alpha_{n,k,i}$$

(3.1)

$$\mu(C_{k,i}) \wedge \mu_n(C_{k,i}) = \mu_n(C_{k,i})\beta_{n,k,i}$$

Define $$A_{n,k,0} = \Omega_0 - \sum_{i=1}^{i_k} A_{n,k,i}$$

and $$B_{n,k,0} = \Omega_n - \sum_{i=1}^{i_k} B_{n,k,i}$$

Determine $n_1 < n_2 < \ldots$ such that for $n \geq n_k$,

$$\max_{1 \leq i \leq i_k} |\mu_n(C_{k,i}) - \mu(C_{k,i})| < \varepsilon_k \min_{1 \leq i \leq i_k} \mu(C_k$$

and

$$\min_{1 \leq i \leq i_k} \mu_n(C_{k,i}) > 0.$$

Define $k(n)$ as equal to 1 if $n < n_1$ and equal to l if $n_l \leq n > n_{l+1}$. Let $D_{n,i} = A_{n,k(n),i}$ and $E_{n,i} = B_{n,k(n),i}$. If λ is the Lebesgue measure on $\mathscr{B}'$ define $\mu_0 = \mu \times \lambda$ and $\mu'_n = \mu_n \times \lambda$ on $\mathscr{B} \times \mathscr{B}'$ an[illegible] ive.

Let $A = \bigcup_{\{n : \mu_0(D_{n,0}) = 0\}} D_{n,0}$

Notice that $D_{n,i}$, $E_{n,i}$ and A belong to $S_0 \times \mathscr{B}'$ a[illegible] to (3.1),

$$\mu_0(D_{n,i}) = \mu_0(A_{n,k(n),i}) = \mu'_n(B_{n,k(n),i}) = \mu$$

Also $\mu_0(D_{n,i}) > 0$ for $i \geq 1$. Define $\tau(\omega_0, n) = i$ if $\omega_0 \in D_{n,i}$, and for $\omega_0 \notin A$ $\mu(\omega_0, n)$, a probability measure on $\mathscr{D}_n \times \mathscr{B}'$ by

$$\mu(\omega_0, n)(G) = \frac{\mu'_n(G \cap E_{n,\tau(\omega_0,n)})}{\mu'_n(E_{n,\tau(\omega_0,n)})}$$

Notice that $\mu'_n(E_{n,\tau(\omega_0,n)}) = \mu_0(D_{n,\tau(\omega_0,n)}) > 0$ because $\omega_0 \notin A$.

Now define a probability on $(\prod_{n=1}^{\infty} \Omega_n, \prod_{n=1}^{\infty} \mathscr{D}_n \times \mathscr{B}'))$ by

$$P_{\omega_0} = \begin{cases} Q & \text{for } \omega_0 \in A \\ \prod_{n=1}^{\infty} \mu(\omega_0, n) & \text{for } \omega_0 \notin A \end{cases}$$

where Q is a fixed probability measure on $(\prod_{n=1}^{\infty} \Omega_n, \prod_{n=1}^{\infty} \mathscr{D}_n \times \mathscr{B}'))$.

We will show now that for all $G \in \prod_{n=1}^{\infty} (\mathscr{D}_n \times \mathscr{B}')$, $P \cdot (G)$ is $\mathscr{B} \times \mathscr{B}'$ measurable. It is enough to show it for G of the form $F_1 \times F_2 \times \ldots \times F_n \times \Omega_{n+1} \times \ldots$

where $F_i \in \mathscr{D}_i \times \mathscr{B}', i = 1, 2, \ldots, n$, because this is a semi-algebra of sets which generates the σ-field $\prod_{n=1}^{\infty} (\mathscr{D}_n \times \mathscr{B}')$.

Since
$$P_{\omega_0}(F_1 \times F_2 \times \ldots \times F_n \times \Omega_{n+1} \times \ldots) = $$
$$= I_{A^c}(\omega_0)\mu(\omega_0, 1)(F_1) \times \ldots \times \mu(\omega_0, n)(F_n) + I_A(\omega_0)Q(G)$$

it is enough to show that for all n $\mu(\cdot, n)(F_n)$ is measurable on A^c. But

$$\mu(\cdot, n)(F_n) = \frac{\mu_n'(F_n \cap E_{n,\tau(\cdot, n)})}{\mu_0(D_{n,\tau(\cdot,\omega)})} = \sum_{i=0}^{i_{k(n)}} \frac{\mu_n'(F_n \cap E_{n,i})}{\mu_0(D_{n,i})} I_{D_{n,i}}(\cdot)$$

and this equation clearly indicates that $\mu(\cdot, n)(F_n)$ is $S_0 \times \mathscr{B}'$ measurable on A^c. Let P be the unique probability on $(\Omega, \mathscr{A})$ such that for $F_0 \in \mathscr{B} \times \mathscr{B}'$ and $G \in \prod_{n=1}^{\infty} (\mathscr{D}_n \times \mathscr{B}')$, $P(F_0 \times G) = \int_{F_0} P_{\omega_0}(G)\mu_0(d\omega_0)$.

We will check now that $\mathscr{L}(X_0) = \mu$ and $\mathscr{L}(X_n) = \mu_n$ for all n. If $C \in \mathscr{B}$ and $[\ \cdot\ ]_{\omega_0}$ indicates the section determined by ω_0, we have

(3.2)
$$P(X_0 \in C) = \int_{\Omega_0} P_{\omega_0}[X_0 \in C]_{\omega_0}\mu_0(d\omega_0) =$$
$$= \int_{F \times [0,1]} P_{\omega_0}[X_0 \in F \cap C]_{\omega_0}\mu_0(d\omega_0).$$

If $\omega_0 = (x_0, t_0) \in F \times [0, 1]$, then it is easy to show that $[X_0 \in F \cap C]_{\omega_0}$ equals $\prod_{i=1}^{\infty} \Omega_i$ if $x_0 \in F \cap C$ and it is empty otherwise. Therefore (3.2) equals

$$\mu_0((F \cap C) \times [0, 1] = \mu(F \cap C) = \mu(C)$$

which proves that $\mathscr{L}(X_0) = \mu$.

Now to check that $\mathscr{L}(X_n) = \mu_n$ we proced as follows:

$$P(X_n \in C) = \int_{A^c} P_{\omega_0}([X_n \in C]_{\omega_0})\mu_0(d\omega_0)$$
$$= \int_{A^c} \mu(\omega_0, n)(C \times [0, 1])\mu_0(d\omega_0)$$
$$= \int_{A^c} \sum_{i=0}^{i_{k(n)}} \frac{\mu_n'((C \times [0, 1]) \cap E_{n,i})}{\mu_0(D_{n,i})} I_{A^c \cap D_{n,i}}\mu_0(d\omega_0)$$
$$\sum_{i=0}^{i_{k(n)}} \mu_n'((C \times [0, 1]) \cap E_{n,i}) = \mu_n'(C \times [0, 1])$$
$$= \mu_n(C).$$

We will show now that $X_n \longrightarrow X_0$ [a.s.P].

Since $\sum_{n=1}^{\infty} \mu(S - \sum_{i=1}^{i_{k(n)}} C_{k(n),i}) \leq \sum_{n=1}^{\infty} \eta_{k(n)} \leq \sum_{k=1}^{\infty} \eta_k < \infty$

if
$$B = \limsup_n \, (S - \sum_{i=1}^{i_{k(n)}} C_{k(n),i}) \times [0, 1]$$

then $\mu_0(B) = 0$.

It is easy to see that if $n \geq n_k$ then

$$\min_{1 \leq i \leq i_k} \alpha_{n,k,i} \wedge \min_{1 \leq i \leq i_k} \beta_{n,k,i} \geq 1 - \varepsilon_k.$$

If $\omega_0 \notin B$, $\omega_0 = (x, t)$, and $t < 1$ determine $n_0(\omega_0)$ such that $\forall\, n \geq n_0(\omega_0)$, $1 - \varepsilon_{k(n)} > t$ and $x \in \sum_{i=1}^{i_{k(n)}} C_{k(n),i}$. If $x \in C_{k(n),i}$ since

$$\min_{1 \leq 1 \leq i_{k(n)}} \alpha_{n,k(n),i} \geq 1 - \varepsilon_{k(n)},$$

$$\omega_0 = (x, t) \in C_{k(n),i} \times [0, \alpha_{n,k(n),i}) = A_{n,k(n),i} = D_{n,i}.$$

This is, for $n \geq n_0(\omega_0)$, $\omega_0 \in D_{n,i}$ for some $i \geq 1$.

The result will follow if we prove that for all n, and all $a > 0$ the set $[d(X_0, X_n) > a]$ is measurable and then that

$$P(\limsup_n [d(X_0, X_n) > \delta_{k(n)}]) = 0.$$

Let F' be a countable dense subset of F, and (X_0, X_n) be the mapping which sends ω into $(X_0(\omega), X_n(\omega))$. Now

$$\{\omega : d(X_0(\omega), X_n(\omega)) > a\} = \{\omega : d(X_0(\omega), X_n(\omega)) > a\} \cap [X_0 \in F]$$
$$= (X_0, X_n)^{-1}(\{(x, y) : (x, y) \in S \times S, d(x, y) > a\} \cap (F \times S))$$
$$= (X_0, X_n)^{-1}(\bigcup_{x \in F} (\{x\} \times \overline{B_{x,a}}^c))$$
$$= (X_0, X_n)^{-1}(\bigcup_{x \in F'} (\{x\} \times \overline{B_{x,a}}^c))$$

but $\bigcup_{x \in F'} (\{x\} \times \overline{B_{x,a}}^c) \in S_0 \times S_0$ (since it is a countable union of sets in $S_0 \times S_0$). Since (X_0, X_n) is $\mathscr{A} - \mathscr{B} \times \mathscr{D}_n$ measurable and $S_0 \times S_0 \subseteq \mathscr{B} \times \mathscr{D}_n$

it follows that the set $[d(X_0, X_n) > a]$ is measurable. Now

$$P(\lim_n \sup [d(X_0, X_n) > \delta_{k(n)}])$$

$$= \int_{A^c \cap B^c \cap (F \times [0,1))} P_{\omega_0} (\lim_n \sup [d(X_0, X_n) > \delta_{k(n)}])_{\omega_0} \mu_0(d\omega_0)$$

$$= \int_{A^c \cap B^c \cap (F \times [0,1))} P_{\omega_0} ((\lim_n \sup [d(\pi(\omega_0), X_n) > \delta_{k(n)}]_{\omega_0})$$

$$= (\prod_{n=1}^{\infty} E_{n, \tau(\omega_0, n)}))\mu_0(d\omega_0)$$

where $\pi(\omega_0) = x_0$ if $\omega_0 = (x_0, t_0)$.

If $n \geq n_0(\omega_0)$ then $\omega_0 \in D_{n, \tau(\omega_0, n)}$ with $\tau(\omega_0, n) \geq 1$. If $(\omega_1, \omega_2, \ldots)$ belongs to

$$(\lim_n \sup [d(\pi(\omega_0), X_n) > \delta_{k(n)}]) \cap (\prod_{n=1}^{\infty} E_{n, \tau(\omega_0, n)})$$

we have $\omega_0 \in D_{n, \tau(\omega_0, n)}$ and $\omega_n \in E_{n, \tau(\omega_0, n)}$. Therefore $\pi(\omega_0)$ and $\pi(\omega_n)$ belong to $C_{k(n), \tau(\omega_0, n)}$ which implies $d(\pi(\omega_0), \pi(\omega_n)) < \delta_{k(n)}$ for $n \geq n_0(\omega_0)$. This is

$$(\lim_n \sup [d(\pi(\omega_0), X_n) > \delta_{k(n)}]) \cap (\prod_{n=1}^{\infty} E_{n, \tau(\omega_0, n)}) = \varnothing$$

which proves the result.

Now let (S, d) and (S', d') be two metric spaces, S_0 and S'_0 the σ-algebras generated by the balls in S and S' respectively, $\{\mathscr{D}_n\}_{n=1,2,\ldots}$ and $\{\mathscr{D}'_n\}_{n=1,2,\ldots}$ sequences of σ-algebras in S and S' such that for all n $\mathscr{D}_n \supseteq S_0$ and $\mathscr{D}'_n \supseteq S'_0$, $\{\mu_n\}_{n=1,2,\ldots}$ a sequence of probability measures where μ_n is defined on $\mathscr{D}_n$. Let μ be another probability measure defined on $\mathscr{B}$, the Borel σ-algebra of S, and g: $S \longrightarrow S'$ be a continuous function such that for all n, $g^{-1} \mathscr{D}'_n \subseteq \mathscr{D}_n$. Then g induces a sequence of probability measures $\{\mu_n g^{-1}\}_{n=1,2,\ldots}$ and a probability μg^{-1} on the sequence of σ-algebras $\{\mathscr{D}'_n\}_{n=1,2,\ldots}$ and on the Borel σ-algebra of S' respectively. We have the following result.

THEOREM 3.2. *If $\mu_n g^{-1} \ \omega \longrightarrow \mu g^{-1}$ and the support of μg^{-1} is separable, then there exists a probability space $(\Omega, \mathscr{A}, P)$ and a sequence of random elements $\{X_n\}_{n=0,1,\ldots}$ defined on Ω and taking values in S such that X_0 is $\mathscr{A} - \mathscr{B}$ measurable, $\mathscr{L}(X_0) = \mu$ and for all $n = 1, 2, \ldots$ X_n is $\mathscr{A} - \mathscr{D}_n$ measurable, $\mathscr{L}(X_n) = \mu_n$ and $g(X_n) \longrightarrow g(X_0)$ [a.s.P].*

PROOF. Since the argument follows the one used in Theorem 3.1 we will only prove those claims which are new or require further argument. Take sequences $\{\eta_k\}_{k=1,2,\ldots}$, $\{\delta_k\}_{k=1,2,\ldots}$ and $\{\varepsilon_k\}_{k=1,2,\ldots}$ as before. Let F' be the support of μg^{-1}. Let $F = g^{-1}(F')$. We have $F \in \mathscr{B}$ and $\mu(F) = 1$. Select $\bar{x}_0 \in F$ and define h: $S \longrightarrow S$ by

$$h(x) = \begin{cases} x & \text{if } x \in F \\ \bar{x}_0 & \text{if } x \notin F \end{cases}$$

Clearly h is $\mathscr{B}$ measurable.

Now construct as before sets $\{C'_{k,i}\}_{\substack{k=1,2,\ldots \\ i=1,\ldots,i_k}}$ with the following properties:

$$\mu g^{-1}(C'_{k,i}) > 0, \ \mu g^{-1}(\partial(C'_{k,i})) = 0,$$

$C'_{k,i} \in S'_o$, diameter of $C'_{k,i}$ smaller than δ_k and $\sum_{i=1}^{i_k} \mu g^{-1}(C'_{k,i}) > 1 - \eta_k$.

Define then $C_{k,i} = g^{-1}(C'_{k,i})$. We have that the sets $\{C_{k,i}\}_{\substack{k=1,2,\ldots \\ i=1,\ldots,i_k}}$ have the same properties as the sets $\{C'_{k,i}\}_{\substack{k=1,2,\ldots \\ i=1,\ldots,i_k}}$ the only difference being that now $C_{k,i} \in \bigcap_{n=1}^{\infty} \mathscr{D}_n$. Define $\{\Omega_i\}$ $i = 0, 1, \ldots$, Ω and $\mathscr{A}$ as in Theorem 3.1. With the sets $\{C_{k,i}\}_{\substack{k=1,2,\ldots \\ i=1,\ldots,i_k}}$ construct $A_{n,k,i}$, $B_{n,k,i}$, and determine $\{\eta_k\}_{k=1,2,\ldots}$ as before. Then define sets $\{D_{n,i}\}_{\substack{n=1,2,\ldots \\ i=0,1,\ldots,i_k}}$ and $\{E_{n,i}\}_{\substack{n=1,2,\ldots \\ i=0,1,\ldots,i_k}}$ and probability measures μ_0 and $\{\mu_n\}_{n=1,2,\ldots}$. Set $A = \bigcup_{\{n: \mu_0(D_{n,0}) = 0\}} D_{n,0}$.

Notice that now $D_{n,i}$, $E_{n,i}$ and A belong to $(\bigcap_{n=1}^{\infty} \mathscr{D}_n) \times \mathscr{B}'$ and $\mu_0(D_{n,i}) = \mu'_n(E_{n,i})$. If $\omega = ((x_0, t_0), (x_1, t_1), \ldots) \in \Omega$ define $X_0(\omega) = h(x_0)$ and for all n

$X_n(\omega) = x_n$. Finally define τ, $\mu(\omega_0, n)$ for $\omega_0 \notin A$, P_{ω_0} and P as in the proof of Theorem 3.1. With these definitions it follows as in that theorem that $\mathscr{L}(X_0) = \mu$ and $\mathscr{L}(X_n) = \mu_n$. Define B and for $\omega_0 \notin B$ determine $n_0(\omega_0)$ as before. If $(g(X_0), g(X_n))$ is the mapping which sends $\omega \longrightarrow (g(X_0(\omega), g(X_n(\omega)))$ then the following formula is easy to check

$$[d'(g(X_0), g(X_n)) > a] = (g(X_0), g(X_n))^{-1} (\bigcup_{x \in F'} (\{x\} \times B_{x,a}{}^c))$$

where $a > 0$ and $\bar{F}'$ is a countable set dense in F'. This equality shows that $[d'(g(X_0), g(X_n)) > a] \in \mathscr{A}$. We have to show that

$$P(lim_n sup\, [d'(g(X_0), g(X_n)) > \delta_{k(n)}]) = 0.$$

The proof of this equality is the same as the corresponding one in Theorem 3.1 provided we replace X_0, X_n and d by $g(X_0)$, $g(X_n)$ and d' respectively.

REMARKS. Theorem 3.2 could prove useful for example in situations in which the function g determines the convergence in S, in the sense that $x_n \longrightarrow x(x_n$ and $x \in S)$ iff $g(x_n) \longrightarrow g(x)$. This is equivalent to the following: $\delta_{x_n} \xrightarrow{\omega} \delta_x$ iff $\delta_x g^{-1} \xrightarrow{\omega} \delta_x g^{-1}$ where δ_y, for $y \in S$, denotes the probability measure which concentrates all its mass at the point y $(\delta_y(\{y\}) = 1)$. Theorem 3.2 says, that what is true for sequences of points is true for sequences of measures if we assume that the limit measure has separable support. Because if $\mu_n \xrightarrow{\omega} \mu$ then $\mu_n g^{-1} \xrightarrow{\omega} \mu g^{-1}$. If $\mu_n g^{-1} \xrightarrow{\omega} \mu g^{-1}$ we construct by Theorem 3.2 a sequence $\{X_n\}_{n=0,1,\ldots}$ such that $\mathscr{L}(X_0) = \mu$, $\mathscr{L}(X_n) = \mu_n$ and such that $g(X_n) \longrightarrow g(X_0)$ [a.s.P] which in tu n implies $\mu_n = \mathscr{L}(X_n) \longrightarrow \mathscr{L}(X_0) = \mu$.

Let $(E, ||\cdot||)$ be a separable, reflexive Banach space such that there exists a countable family of continuous linear functionals $\{y_k\}_{k=1,2,\ldots}$ $||y_k|| \le 1$ such that $||x|| = \sup_{k>1} |\langle x, y_k\rangle|$, and if $||x_n|| \longrightarrow ||x||$, and for all k $\langle x_n, y_k\rangle \longrightarrow \langle x, y_k\rangle$ then $||x_n - x|| \longrightarrow 0$. By defining $g(x) = (||x||, \langle x, y_1\rangle, \ldots)$, $g\colon E \longrightarrow \mathbb{R}^\infty$, it is possible to use our result to prove that if a sequence of probability measures is such that $\mu_n ||\cdot||^{-1} \xrightarrow{\omega} \mu ||\cdot||^{-1}$ and for all $y \in E'$ $\mu_n y^{-1} \longrightarrow \mu y^{-1}$ then $\mu_n \xrightarrow{\omega} \mu$.

This application was suggested to me, in a personal communication, by Professor P. Bickel.

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