The Mapping Torus Construction and Concordance of Diffeomorphisms*

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Dedicated to the memory of Carlos B. de Lyra

1. If M is a smooth closed simply connected manifold and $\varphi\colon M \longrightarrow M$ is a diffeomorphism, then the mapping torus $S^1 \times_\varphi M$ is a smooth manifold which carries some information about the diffeomorphism φ. Thus we may expect to obtain a homomorphism from $\pi_0 \mathbf{Diff}(M)$ into some cobordism theory via the mapping torus construction. A technical problem arises in that the mapping torus construction does not produce a map $S^1 \times_\varphi M \longrightarrow X$ into some space X which act as the base space of the cobordism theory. One way to circumvent this problem is to study pairs (φ, h) where $\varphi\colon M \longrightarrow M$ and h is a homotopy from φ to the identity. Then the mapping torus construction produces a map $S^1 \times_\varphi M \longrightarrow M$, and we obtain a homomorphism-with-indeterminacy(one to many) from $\pi_0 \mathbf{Diff}^+(M)$ to, for instance, the oriented bordism group $\Omega_{n+1}(M)$; here $n = dim\, M$ and $\pi_0 \mathbf{Diff}^+(M)$ is the group of isotopy classes of diffeomorphisms of M homotopic to the identity.

Instead of $\pi_0 \mathbf{Diff}^+(M)$, we will study $\pi_0 \mathbf{Diff}^\lambda(M)$, the group of isotopy classes of diffeomorphisms $M \xrightarrow{\varphi} M$ which are *regularly* homotopic to the inclusion when restricted to a neighborhood of a λ-skeleton of M. In this case, the mapping torus construction produces a homomorphism-with-indeterminacy $\pi_0 \mathbf{Diff}^\lambda(M) \xrightarrow{t} \Omega_{n+1}(v_\lambda)$, where $v_\lambda : X_\lambda \longrightarrow BSO$ is the fibration such that $M \longrightarrow X_\lambda \xrightarrow{v_\lambda} BSO$ is the λ-th Moore-Postnikov factorization of the normal classifying map $M \longrightarrow BSO$, and $\Omega_*(v_\lambda)$ is the Lashof cobordism theory associated with that fibration. More precisely, we construct a certain group $D^\lambda(M)$ toghether with an epimorphism $D^\lambda(M) \longrightarrow \pi_0 \mathbf{Diff}^\lambda(M)$ and a homomorphism $D^\lambda(M) \xrightarrow{\beta} \Omega_{n+1}(v_\lambda)$. Then our main theorem is the following.

THEOREM. *For* $n \geq 6$ *and* $\lambda \geq \left[\dfrac{n+3}{2}\right]$ *there exists a short exact sequence of groups*

$$L_{n+2}(1) \xrightarrow{\partial} D^\lambda(M) \xrightarrow{\beta} \Omega_n(v_\lambda) \xrightarrow{\sigma} L_{n+1}(1).$$

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Here $L_i(1)$ is the i-th Wall group of the trivial group; if "concordance" is substituted for isotopy, and $\pi_1(M)$ is assumed only to satisfy

$$WH(\pi_1(M)) = K_0(Z[\pi_1(M)]) = 0,$$

one expects the same theorem. The sequence of the theorem is a surgery exact sequence, but the surgery involved differs from ordinary surgery in that the target space is not a Poincaré duality space – it merely has a "top class" which induces a Poincaré isomorphism only in the middle dimension. It turns out that this weak Poincaré duality property suffices for the theorem.

To contrast the groups $\pi_0\mathbf{Diff}^\lambda(M)$ and $\pi_0\mathbf{Diff}^+(M)$, here is an example: Let r be even and ≥ 4. Then K. Wang's trick [4] with the usual surgery exact sequence shows that $\pi_0\mathbf{Diff}^+: SU(r)$ is isomorphic modulo finite groups to $\sum_i H^{4i}(SU(r))$. On the other hand, for any $\lambda \geq \frac{r^2}{2}$ the theorem above shows that $\pi_0\mathbf{Diff}^\lambda(SU(r))$ is finite. That is, almost every diffeomorphism of $SU(r)$ homotopic to the identity is non-trivial near any $\frac{r^2}{2}$-skeleton.

For another application of the theorem consider the homotopy behavior of $\pi_0\mathbf{Diff}^{n-1}(M)$ for M simply connected, s-parallelizable and $n \geq 6$. Since each representative is homotopic to the identity on M-pt., the only non-trivial homotopy behaviour occurs in the top cell. This behavior is completely described by a homomorphism-with-indeterminacy $\Theta: \pi_0\mathbf{Diff}^{n-1}(M) \longrightarrow \pi_n(M)$. In the case that, in addition, $n = 2l$ with $l \neq 0$ mod 4, an argument based on the theorem shows that the sequence

$$\pi_0\mathbf{Diff}^{n-1}(M) \xrightarrow{\Theta} \pi_n(M) \longrightarrow \pi_n^s(M)$$

is exact in the obvious sense, when $\pi_n(M) \longrightarrow \pi_n^s(M)$ is stablization.

From now on *dim* $M = n \geq 6$ and $\pi_1(M) = 1$ always.

2. Weak Poincaré Duality Spaces. We will need to surger maps of manifolds into spaces which are not Poincaré duality spaces. However, these spaces will be sufficiently like Poincaré duality spaces to enable us to carry over the machinery of surgery.

DEFINITION. A weak *PD* space of dimension n is a pair (Y, y), where Y is a space, $y \in H_n(Y)$ and

$$y \cap : H^{\left[\frac{n+1}{2}\right]}(Y) \longrightarrow H_{\left[\frac{n}{2}\right]}(Y)$$

is an isomorphism. A map $f: M \longrightarrow Y$ of a closed oriented n-manifold M will have *degree* 1 into (Y,y) if $f_*[M] = y$, where M is the orientation class of M. The map f is *normal* if it is covered by a bundle map of a normal bundle of M to some vector bundle over Y. For a normal map f of degree 1, we may define the simply connected surgery obstruction $\sigma(f) \in L_n(1)$ algebraically as in [1]. If $\pi_1(Y) = 1$, then $\sigma(f) = 0$ iff, there is a surgery from f to $f': M' \longrightarrow Y$ such that a fibration homotopy equivalent to f' has its fiber $\left[\frac{n}{2}\right]$-connected. For pairs, we have the corresponding situation:

DEFINITION. A *weak P.D.* pair of dimension n is a triple (Y, Y', y) with $y \in H_n(Y, Y')$ such that $y \cap : H^{\left[\frac{n+1}{2}\right]}(Y, Y') \longrightarrow H_{\left[\frac{n}{2}\right]}(Y)$ is an isomorphism. If $f: M, \partial M \longrightarrow Y, Y'$ is a map of a oriented n-manifold $M, \partial M$ with orientation class $[M, \partial M]$, we say f has *degree* 1 if $f_*[M, \partial M] = y$. For f normal of degree 1 we may define the surgery obstruction $\sigma(f) \in L_n(1)$ algebraically again. Then if $\pi_1(Y) = 1$ and $f \mid \partial M: \partial M \longrightarrow Y'$ is a homotopy equivalence, $\sigma(f) = 0$ iff f may be surgered mod boundary to $f': M', \partial M' \longrightarrow Y, Y'$ such that the fiber of $f': M' \longrightarrow Y$ is $\left[\frac{n}{2}\right]$-connected.

Now suppose that M is a simply connected oriented smooth n-manifold. Let $M \longrightarrow BSO$ be a classifying map for its oriented normal bundle and

$$\begin{array}{ccc} M & \xrightarrow{\quad \iota \quad} & \overline{M} \\ & \searrow \quad \swarrow & \\ & BSO & \end{array}$$

a commutative diagram with $\overline{M} \longrightarrow BSO$ a fibration and $i: M \longrightarrow \overline{M}$ a homotopy equivalence. Let. $\overline{M} \xrightarrow{g} X \xrightarrow{v} BSO$ be the λ-th stage in the

Moore-Postnikov factorization of $\bar{M} \longrightarrow BSO$; that is, the fiber of $\bar{M} \xrightarrow{g} X$ is λ-connected and the last possibly non-zero homotopy group of the fiber of v is π_λ. Then $S^1 \times \bar{M} \xrightarrow{d\times g} S^1 \times X \xrightarrow{v\circ pr_2} BSO$ is the λ-th Moore-Postnikov stage of $S^1 \times \bar{M} \longrightarrow BSO$. Let $[S^1]$ be the orientation of S^1 and suppose $\lambda \geq \left[\frac{n+2}{2}\right]$. Then $(S^1 \times X, [S^1] \times g_*\iota_*[M])$ is a weak *PD* space. More generally, we have the following proposition.

PROPOSITION 1. $([S^1] \times g_*\iota_*[M]) \cap : H^i(S^1 \times X) \longrightarrow H_{n+1-i}(S^1 \times X)$ *is an isomorphism for* $n + 1 - \lambda \leq i \leq \lambda$. *It is a monomorphism for* $i = \lambda + 1$ *and an epimorphism for* $i = n - \lambda$.

However, we have many more weak *PD* spaces associated with $S^1 \times X$.

PROPOSITION 2. *Let* $x \in H_{n+1}(X)$. *Then for* $n + 1 - \lambda \leq i \leq \lambda$ *the map*

$$(([S^1] \times g_*\iota_*[M]) + 1 \times x) \cap : H^i(S^1 \times X) \longrightarrow H_{n+1-i}(S^1 \times X)$$

is an isomorphism.

PROOF. Write $\gamma = ([S^1] \times g_*\iota_*[M] \cap$, so we have to show that

$$\gamma + (1 \times x) \cap : H^i(S^1 \times X) \longrightarrow H_{n+1-i}(S^1 \times X)$$

is an isomorphism for $n + 1 - \lambda \leq i \leq \lambda$. Let $pr: S^1 \times X \longrightarrow X$ be projection on the second factor, and let $j: X \longrightarrow S^1 \times X$ be the standard inclusion. Let $S \in H^1(S^1)$ be such that $S \cap [S^1] = 1$. Set $A^i = pr^* H^i(X)$ and $B^i = S \times H^{i-1}(X)$. Then $H^i(S^1 \times X) = A^i \oplus B^i$. Set

$$A_{n+1-i} = j_* H_{n+1-i}(X)$$

and $B_{n+1-i} = [S^1] \times H_{n-i}(X)$. Then $H_{n+1-i}(S^1 \times X) = B_{n+1-i} \oplus A_{n+1-i}$ and $\gamma: A^i \longrightarrow B_{n+1-i}$ is an isomorphism-call it β – and $\gamma: B^i \longrightarrow A_{n+1-i}$ is an isomorphism – call it α. Then $\gamma(a, b) = (\beta(a), \alpha(b))$ for $(a, b) \in A^i \oplus B^i$. Define $S: A^i \longrightarrow A_{n+1-i}$ by $S(y) = 1 \times (x \cap y)$, so that

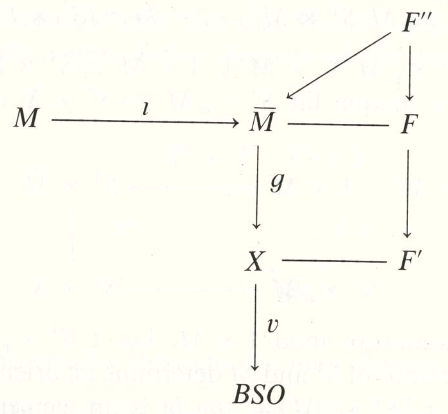
$$(\gamma + (1 \times x) \cap)(a, b) = (\beta(a), \alpha(b) + S(a)).$$

But then

$$(\gamma + (1 \times x) \cap)^{-1}(b', a') = (\beta^{-1}(b'), \alpha^{-1}(a' - S(\beta^{-1}(b'))),$$

and the proposition is proved.

3. The Mappng Torus Construction. As above we have the commutative diagram



with F the fiber of $v \circ g$, and F' the fiber of v, and F'' the fiber of both g and $F \longrightarrow F'$. We may assume that $M \longrightarrow BSO$ is the Gauss map of an embedding of M in Euclidean space, and we assume $\lambda \geq \left[\frac{n+2}{2}\right]$.

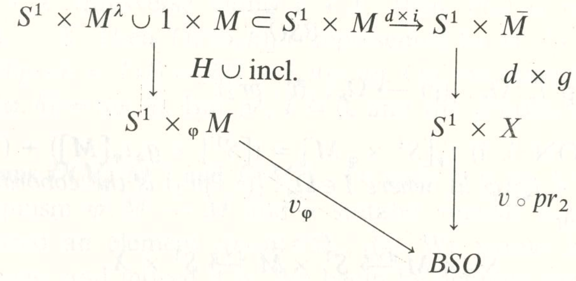
Let $\varphi: M \longrightarrow M$ be a diffeomorphism and

$$h: (I \times M^\lambda, \dot{I} \times M^\lambda) \longrightarrow (\dot{I} \times M, I \times M)$$

a regular homotopy from the inclusion of some regular neighborhood M^λ of a λ-skeleton of M to $\varphi \mid M^\lambda$. Then h defines an immersion

$$H: S^1 \times M^\lambda \longrightarrow S^1 \times_\varphi M$$

into the mapping torus of φ. Let $v_\varphi : S^1 \times_\varphi M \longrightarrow BSO$ be any Gauss map extending $1 \times M \longrightarrow M \longrightarrow BSO$. Thus we obtain the diagram



which commutes up to homotopy mod $1 \times M$. The fiber of $v \circ pr_2$ is $S^1 \times F'$ and $\pi_i(F') = 0$ for $i > \lambda$. Thus

$$H^{i+1}(S^1 \times_{\varphi} M, S^1 \times M^{\lambda} \cup 1 \times M : \pi_i(S^1 \times F')) = 0$$

for all i and $H^i(S^1 \times_{\varphi} M, S^1 \times M^{\lambda} \cup 1 \times M : \pi_i(S^1 \times F')) = 0$ for all i. But then there is a unique lift $S^1 \times_{\varphi} M \longrightarrow S^1 \times X$ of v_{φ} making the diagram

$$\begin{array}{ccc} S^1 \times M^{\lambda} \cup 1 \times M & \longrightarrow & S^1 \times \overline{M} \\ \downarrow & & \downarrow \\ S^1 \times_{\varphi} M & \longrightarrow & S^1 \times X \end{array}$$

commute up to homotopy mod $1 \times M$. Let $t: S^1 \times_{\varphi} M \longrightarrow X$ be any such lift; the orientations of S^1 and M determine an orientation $[S^1 \times_{\varphi} M]$ of $S^1 \times_{\varphi} M$. Then $t_*[S^1 \times_{\varphi} M] = x(\varphi, h)$ is an element of $H_{n+1}(X)$ independent of the choice of v_{φ} (subject to the condition that $v_{\varphi} | 1 \times M$ be the composition $1 \times M \xrightarrow{l} \overline{M} \xrightarrow{g} X \xrightarrow{v} BSO$). Similarily t represents an element $t(\varphi, h)$ of the $(n+1)$-st Lashof cobordism group $\Omega_{n+1}(v)$, independent of the choice of v_{φ}. In the same way again,

$$t'_*[S^1 \times_{\varphi} M] \in H_{n+1}(S^1 \times X) \quad \text{and} \quad t'(\varphi, h) \in \Omega_{n+1}(v \circ pr_2)$$

are independent of the choice of v_{φ}, where $t'(\varphi, h)$ is the element of $\Omega_{n+1}(v \circ pr_2)$ represented by any t'. However, we may say a little more. Let $j: X \longrightarrow S^1 \times X$ be the standard inclusion as above. Then the commutative diagram of fibrations

$$\begin{array}{ccc} X & \xrightarrow{j} & S^1 \times X \\ & {\scriptstyle v} \searrow & \downarrow {\scriptstyle v \circ pr_2} \\ & & BSO \end{array}$$

induces a map $j_*: \Omega_{n+1}(v) \longrightarrow \Omega_{n+1}(v \circ pr_2)$.

PROPOSITION 3. i) $t'_*[S^1 \times_{\varphi} M] = ([S^1] \times g_* l_*[M]) + (1 \times x(\varphi, h))$.
ii) $t'(\varphi, h) = I + j_* t(\varphi, h)$, *where* $I \in \Omega_{n+1}(v \circ pr_2)$ *is the cobordism class of the composition*

$$S^1 \times M \xrightarrow{d \times i} S^1 \times \overline{M} \xrightarrow{d \times g} S^1 \times X.$$

PROOF. Clearly ii) implies i). To prove ii), we replace X by a sufficiently high dimensional skeleton, but still call it X, and we replace v by a k-plane bundle ξ for k sufficiently large. Let $pr_*: \Omega_{n+1}(v \circ pr_2) \longrightarrow \Omega_{n+1}(v)$ be the map induced by the commutative diagram of fibrations

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{pr_2} & X \\ {\scriptstyle v \circ pr_2} \searrow & & \swarrow {\scriptstyle v} \\ & BSO & \end{array}$$

By the Thom transversality theorem, we may replace the sequence

$$\Omega_{n+1}(v) \xrightarrow{j_*} \Omega_{n+1}(v \circ pr_2) \xrightarrow{pr_*} \Omega_{n+1}(v)$$

with the sequence

$$\pi^s_{n+1+k}(T(\xi)) \longrightarrow \pi^s_{n+1+k}(T(S^1 \times \xi)) \longrightarrow \pi^s_{n+2+k}(T(\xi))$$

of stable homotopy groups, where T denotes Thom space, the map $T(\xi) \longrightarrow T(S^1 \times \xi)$ is induced by the vector bundle inclusion $1 \times \xi \subset S^1 \times \xi$, and the map $T(S^1 \times \xi) \longrightarrow T(\xi)$ is the map of Thom's spaces induced by the map of vector bundles $S^1 \times \xi \longrightarrow \xi$. However, the cofibration sequence

$$T(1 \times \xi) \longrightarrow T(S^1 \times \xi) \longrightarrow ST(\xi)$$

induces a long exact sequence in stable homotopy groups, which pr_* above splits. Thus we have a short split sequence,

$$0 \longrightarrow \Omega_{n+1}(v) \underset{pr_*}{\overset{j_*}{\rightleftarrows}} \Omega_{n+1}(v \circ pr_2) \xrightarrow{T} \Omega_n(v) \longrightarrow 0,$$

where $T(\alpha)$ is obtained geometrically by choosing a representative $f: \Gamma \longrightarrow S^1 \times X$ transverse along $1 \times X$; then $T(\alpha)$ is represented by $f^{-1}(1 \times X) \longrightarrow X$. Then $T(t'(\varphi, h))$ is represented by $M \xrightarrow{i} \overline{M} \xrightarrow{g} X$ as is $T(I)$, so $T(t'(\varphi, h)) = T(I) = T(I + j_* t(\varphi, h))$. On the other hand, by definition $pr_* t'(\varphi, h) = t(\varphi, h)$. But $pr_* I = 0$, and the proposition is proved.

4. The Groups $D(M, M^{\lambda})$ and $D(M^{\lambda})$. To each pair $(\varphi$, h) consisting of a diffeomorphism $\varphi: M \longrightarrow M$ and a suitable regular homotopy h, we have associated an element $t(\varphi, h) \in \Omega_{n+1}(v)$. We would like t to be a homomorphism, and indeed it is. We begin by defining the domain of t:

We let M be a smooth closed n-manifold and M^λ a smooth regular neighborhood of some λ-skeleton of M. Set

$$\begin{aligned}\bar{D}(M, M^\lambda) = \{(\varphi, h) \mid\ & \varphi: M \longrightarrow M \text{ is a diffeomorphism,}\\ & h: I \times M^\lambda, \dot{I} \times M^\lambda \longrightarrow I \times M, \dot{I} \times M \text{ is}\\ & \text{an immersion s.t. } (h(0,x) = (0,x) \text{ and}\\ & h(1,x) = (1, \varphi(x)) \text{ for all } x \in M^\lambda\}.\end{aligned}$$

Then define $D(M, M^\lambda)$ to be the quotient set of $\bar{D}(M, M^\lambda)$ by the equivalence relation $(\varphi, h) \sim (\psi, k)$ iff. there is a pair (Φ, H) such that

$$\Phi: M \times I,\ M \times I \longrightarrow M \times I,\ M \times I$$

is a diffeomorphism with $\Phi(x, 0) = (\varphi(x), 0)$ and $\Phi(x, 1) = (\psi(x), 1)$ for all $x \in M$, and

$$H: I \times M^\lambda \times I, \dot{I} \times M^\lambda \times I, I \times M^\lambda \times \dot{I} \longrightarrow I \times M \times I, \dot{I} \times M \times I, I \times M \times \dot{I}$$

is an immersion such that $H(t, x, 0) = (h(t, x), 0)$,

$$H(t, x, 1) = (k(t, x), 1), H(0, x, t) = (0, x, t),$$

and

$$H(1, x, t) = (1, \Phi(x, t))$$

for all $x \in M^\lambda$ and $t \in I$. We will denote the class of (φ, h) by $[\varphi, h]$. We define an operation

$$D(M, M^\lambda) \times D(M, M^\lambda) \overset{\circ}{\longrightarrow} D(M, M^\lambda)$$

as follows: For $\alpha, \beta \in D(M, M^\lambda)$ we may choose representatives $\alpha \in [\varphi, h]$ and $\beta = [\psi, k]$, such that $h(t, x) = (t, \varphi(x))$ for t near 1 and $k(t, x) = (t, x)$ for t near 0. Then $((\varphi \times 1) \circ k + h)(x, t) = h(x, 2t)$ for $t \leq \frac{1}{2}$ and $= (\varphi \times 1) \circ$ $\circ\, k(x, 2t - 1)$ for $\frac{1}{2} \leq t$ defines an immersion

$$(\varphi \times 1) \circ k + h: I \times M^\lambda, \dot{I} \times M^\lambda \longrightarrow I \times M, \dot{I} \times M$$

such that $((\varphi \times 1) \circ k + h)_0$ is the inclusion $M^\lambda \subset M$ and $((\varphi \times 1) \circ k + h)_1$ $= \varphi \circ \psi$. Set $\alpha \circ \beta = [\varphi \circ \psi, (\varphi \times 1) \circ k + h]$. Then $\circ$ is a well-defined, associative map. The map $\circ$ has a two-sided identity $e = [1, \text{const.}]$. Define $r(t) = 1 - t$ and $rev(h) = (r \times id) \circ h \circ (r \times id)$. For $\alpha = [\varphi, h]$ define $\zeta\alpha = [\varphi^{-1}, (\varphi^{-1} \times 1) \circ rev\, h]$. Then ζ is well defined and clearly $\alpha \circ \zeta\alpha =$ $= \zeta\alpha \circ \alpha = e$ so that $D(M, M^\lambda)$ is a group.

Now we have $t: \bar{D}(M, M^\lambda) \longrightarrow \Omega_{n+1}(v)$, which clearly factors through the quotiente map $\bar{D}(M, M^\lambda) \longrightarrow D(M, M^\lambda)$ to define $t: D(M, M^\lambda) \longrightarrow \Omega_{n+1}(v)$. We could prove directly that t is a homomorphism, but it is convenient to introduce another group $D^\lambda(M)$ at this point, and to show that t factors into two homomorphisms τ and β,

$$\begin{array}{ccc} & D^\lambda(M) & \\ {\scriptstyle\tau}\nearrow & & \searrow{\scriptstyle\beta} \\ D(M, M^\lambda) & \xrightarrow{\quad t \quad} & \Omega_{n+1}(v). \end{array}$$

As we shall see, $D^\lambda(M)$ is a more "canonical" version of $D(M, M^\lambda)$ independent of the choice of regular neighborhood M^λ.

As with $D(M, M^\lambda)$, we begin with a set

$$\begin{aligned}\bar{\bar{D}}^\lambda(M) = \{(\varphi, l) \mid\ & \varphi: M \longrightarrow M \text{ is a diffeomorphism,}\\ & l: S^1 \times_\varphi M \longrightarrow X \text{ is a lifting of a}\\ & \text{Gauss map } S^1 \times_\varphi M \text{ such that}\\ & l \mid 1 \times M = g \circ L \circ pr_2\}.\end{aligned}$$

We introduce an equivalence relation $\approx$ defined by $(\varphi_0, l_0) \approx (\varphi_1, l_1)$ if there is a concordance $\Phi: M \times I \longrightarrow M \times I$ from φ_0 to φ_1, and a lifting $L: S^1 \times_\Phi (M \times I) \longrightarrow X$ of a Gauss map, such that 1) $L \mid S^1 \times_\Phi (M \times 0) = \varphi_0$ and $L \mid S^1 \times_\Phi (M \times 1) = \varphi_1$, and 2) $L \mid 1 \times M \times I = g \circ \iota \circ pr_2$. Then the relation $\{((\varphi, h), (\varphi, t(\varphi, h)) \mid (\varphi, h) \in \bar{D}(M, M^\lambda)\}$ is compatible with the two equivalence relations $\sim$ and $\approx$, and on passing to the quotients $D(M, M^\lambda)$ and $\bar{D}^\lambda(M) = \bar{\bar{D}}^\lambda(M)/\approx$, it defines a map $\bar{\tau}: D(M, M^\lambda) \longrightarrow \bar{D}^\lambda(M)$. A map $\bar{\beta}: \bar{D}^\lambda(M) \longrightarrow \Omega_{n+1}(v)$ is obtained by factoring through the quotient map $\bar{\bar{D}}^\lambda(M) \longrightarrow \bar{D}^\lambda(M)$ the map $\bar{\bar{D}}^\lambda(M) \longrightarrow \Omega_{n+1}(v)$ which assigns to (φ, l) the Lashof bordism class of l. Clearly $t = \bar{\beta} \circ \bar{\tau}$. To define the product $x \circ y$ of two elements $x, y \in \bar{D}^\lambda(M)$, let (φ_0, l_0) represent x and (φ_1, l_1) represent y. The complex.

$$S^1 \times_{\varphi_0} (M \times 0) \cup (M \times I) \cup S^1 \times_{\varphi_1} (M \times 1),$$

where $1 \times M \times 0$ is glued to $M \times 0$ by the identity and $M \times 1$ to $1 \times M \times 1$ by the identity, is a strong deformation retract of the manifold

$$\Gamma = S^1 \times_{(\varphi_0 \times id)} M \times [-1, 0] \cup (S^1_+ \times M \times I) \cup S^1 \times_{(\varphi_1 \times id)} M \times [1, 2]$$

which is glued in the same way, with S^1_+ the right half circle. Then Gauss maps may be chosen, and a lift

$$L : S^1 \times_{\varphi_0}(M \times 0) \cup (M \times I) \cup S^1 \times_{\varphi_1}(M \times 1) \longrightarrow X$$

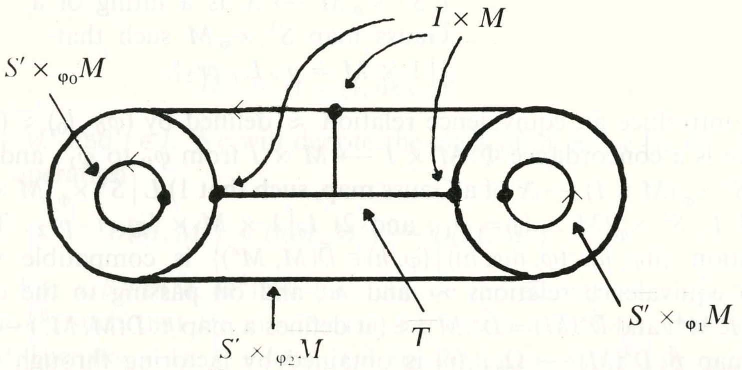
such that L restricted to $S^1 \times \varphi_0 M$ is l_0, restricted to $S^1 \times \varphi_1 M$ it is l_1, and restricted to $M \times I$ it is $M \times I \xrightarrow{pr} M \xrightarrow{i} \overline{M} \xrightarrow{g} X$. Let S^1_{++} be the part of S^1 in the first quadrant. Then L may be extended to

$$S^1 \times_{\varphi_0} M \times 0 \cup 1 \times M \times I \cup S^1_{++} \times M \times \frac{1}{2} \cup S^1 \times_{\varphi_1} M \times 1 = T$$

so that the restriction to $1 \times M \times I \cup S^1_{++} \times M \times \frac{1}{2}$ is

$$1 \times M \times I \cup S^1_{++} \times M \times \frac{1}{2} \xrightarrow{pr} M \xrightarrow{l} \overline{M} \xrightarrow{g} X.$$

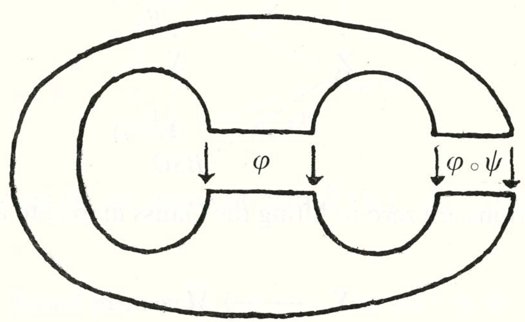
Then that lift may be extended to all of Γ:



Then the outer boundary of Γ is $S^1 \times_{\varphi_2} M$ for some diffeomorphism φ_2, the lift L restricts to $l_2 : S^1 \times_{\varphi_2} M \longrightarrow X$, and by choosing the identifications correctly, we have that $l_2 \mid 1 \times M = g \circ \iota \circ pr_2$. Then we set $x \circ y =$ class of (φ_2, l_2), and we leave to the reader the verification that $\circ$ is an associative map. Since $\bar{\tau}$ is epimorphic, we will know that $\bar{D}^\lambda(M)$ is a group as soon as we know that $\bar{\tau}$ is a homomorphism. And since (Γ, L) is a cobordims from $(S^1 \times_{\varphi_0} M, l_0) \amalg (S^1 \times_{\varphi_1} M, l_1)$ to $(S^1 \times_{\varphi_2} M, l_2)$, it is clear that $\bar{\beta}$ is already a homomorphism.

PROPOSITION 4. $\bar{\tau} : D(M, M^\lambda) \longrightarrow \overline{D}^\lambda(M)$ is a homomorphism.

PROOF: Let K be the displayed subset of the plane:

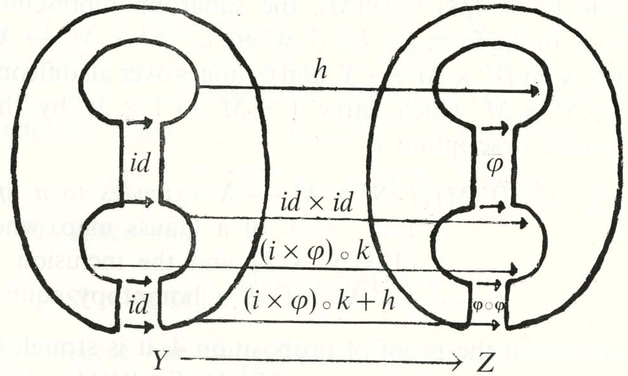
$K =$ 

We obtain a manifold with boundary by starting with $K \times M$ and identifying as the above picture suggests.

Let Z be the resulting manifold; we will have canonically

$$\partial X = S^1 \times_{\varphi} M \coprod S^1 \times_{\psi} M \coprod S^1 \times_{\varphi \circ \psi} M.$$

We obtain another manifold from $K \times M^\lambda$ by identifying with identity maps instead. Let Y be the resulting manifold. We may fill in to immerse Y in Z as the following picture suggests:



There is a canonical map $Y \cup * \times M \longrightarrow \overline{M}$, where $*$ is a suitable point of ∂K, so that we have a homotopy commutative diagram

$$\begin{array}{ccc} Y \cup * \times M & \longrightarrow & \overline{M} \\ \downarrow & & \downarrow g \\ Z & & X \\ & \searrow^{v_z} & \downarrow v \\ & & BSO \end{array}$$

and the obstructions are zero to lifting the Gauss map v_z to a map $Z \longrightarrow X$, making

$$\begin{array}{ccc} Y & \longrightarrow & \overline{M} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

homotopy comute. But the lift $Z \longrightarrow X$ then restricts to the lifts $S^1 \times_\varphi M \longrightarrow X$, $S^1 \times_\psi M \longrightarrow X$, and $S^1 \times_{\varphi\circ\psi} M \longrightarrow X$ defining $t(\varphi, h)$, $t(\psi, k)$, and (on the outer boundary) $t(\varphi \circ \psi, (1\times \varphi) \circ k + h)$. Thus $\bar{\tau}(\varphi \circ \psi, (1 \times \varphi) \circ k + h) = \bar{\tau}(\varphi, h) \circ \bar{\tau}(\psi, k)$, and the propostion is proved.

Finally, $\bar{D}^\lambda(M)$ is not quite the group we want. The group we want will correspond to the term $h_s(M \times I, \partial)$ in the surgery exact sequence, and we have not yet "divided out by the diffeomorphisms." One way to do so is to divide by $\mathscr{H}^\lambda(M) \subset \bar{D}^\lambda(M)$, the subgroup represented by lifts $l: S^1 \times M \longrightarrow X$ of the form $l = l_0 \circ \psi$, where $l_0 : S^1 \times M \longrightarrow X$ is a fixed lift extending to a lift $D^2 \times M \longrightarrow X$, and ψ ranges over all diffeomorphisms $\psi: S^1 \times M \longrightarrow S^1 \times M$ which carry $1 \times M \longrightarrow 1 \times M$ by the identity map. An alternate description is

$$\mathscr{H}^\lambda(M) = \{[1, l] \in \bar{D}^\lambda(M) \mid l : S^1 \times M \longrightarrow X \textit{ extendes to a lift } L : \Gamma \longrightarrow X \text{ of a Gauss map, where } \partial\Gamma = S^1 \times M \text{ and the inclusion } 1 \times M \subset \Gamma \text{ is a homotopy equivalence}\}.$$

Using the pictures in the proof of proposition 4, it is straightforward to check that $\mathscr{H}^\lambda(M)$ is a *normal* subgroup of $\bar{D}^\lambda(M)$. Set $D^\lambda(M) = D^\lambda(M)/\mathscr{H}^\lambda(M)$.

Clearly $\mathscr{H}^\lambda(M)$ is in the kernel of $\bar{\beta}: \bar{D}^\lambda(M) \longrightarrow \Omega_{n+1}(v)$ so that $\bar{\beta}$ factors thus to define β:

$$\begin{array}{ccc} \bar{D}^\lambda(M) & & \\ & \searrow^{\bar{\beta}} & \\ \downarrow & & \Omega_{n+1}(v) \\ & \nearrow_{\beta} & \\ D^\lambda(M) & & \end{array}$$

Let τ be $\bar{\tau}$ followed by the quotient map $\bar{D}^\lambda(M) \longrightarrow D^\lambda(M)$. Then $t = \beta \circ \tau$ as we sought.

5. The Surgery Exact Sequence. Let $L_j(\ \ )$ be the j-th Wall functor. Our next objective is to define a homomorphism $\Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(1)$ such that the sequences

$$D(M, M^\lambda) \xrightarrow{t} \Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(1)$$

and

$$D^\lambda(M) \xrightarrow{\beta} \Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(1)$$

are exact.

Let $\alpha \in \Omega_{n+1}(v)$ be represented by the lifting $\Gamma \xrightarrow{f} X$ of the Gauss map $v_\Gamma: \Gamma \longrightarrow BSO$. Then the inclusion $j: X \longrightarrow S^1 \times X$ gives us the element $j_*\alpha \in \Omega_{n+1}(v \circ pr_2)$ represented by $j \circ f$, and $(j \circ f)_*[\Gamma] = 1 \times z$ with $z \in H_{n+1}(X)$. Recall that we have a homotopy equivalence $\imath: M \longrightarrow \overline{M}$ and fibration $g: \overline{M} \longrightarrow X$. The map $id \times g \circ \imath: S^1 \times M \longrightarrow S^1 \times X$ represents the element $[id \times g \circ \imath]$, and $(id \times g \circ \imath)_*[S^1 \times M] = [S^1] \times g_*\imath_*[M]$. Thus $[id \times g \circ \imath] + j_*\alpha$ represents an element of $\Omega_{n+1}(v \circ pr_2)$ with homology characteristic class $[S^1] \times g_*\imath_*[M] + 1 \times z$. Now

$$(S^1 \times X,\ [S^1] \times g_*\imath_*[M] + 1 \times z)$$

is a weak PD space and

$$S^1 \times M \amalg \Gamma \xrightarrow{id \times (g \circ \imath) \amalg 1 \times f} S^1 \times X$$

is a normal, degree one map. There is a surgery $F_1 : \Lambda_1 \longrightarrow S^1 \times X$ from this map to $f_1 : \partial^+\Lambda_1 \longrightarrow S^1 \times X$ such that f_1 is homotopy equivalent to a fibration with $\left(\left[\frac{n+1}{2}\right]-1\right)$-connected fiber. We may assume that the surgeries all miss $1 \times \bar{M}$; that is, the first takes connected sum of $S^1 \times M$ with Γ, with the "hole" in $S^1 \times M$ off $1 \times M$, and all thereafter may be assumed to take place on the Γ-side of a sphere that originally separated "Γ" from "$S^1 \times M$". For that reason, and because

$$f_1 : \partial^+\Lambda_1 \longrightarrow S^1 \times X$$

has degree one, the kernel of

$$\pi_{\left[\frac{n+1}{2}\right]}(\partial^+\Lambda_1) \longrightarrow \pi_{\left[\frac{n+1}{2}\right]}(S^1 \times X)$$

is free abelian on a generating set represented by a finite bouquet of spheres. By attaching cells to these spheres, we obtain a Poincaré Duality space P_1^+ and a factorization

$$\partial^+\Lambda_1 \xrightarrow{f_1^+} P_1^+ \longrightarrow S^1 \times X$$

with f_1^+ a normal degree one map. Notice that we already have a homotopy equivalence $1 \times M \longrightarrow 1 \times M \subset P_1^+$ so that the only obstruction to surgering f_1^+ to a homotopy equivalence is the simply connected surgery obstruction $\sigma(f_1^+) \in L_{n+1}(1) \subset L_{n+1}(J)$, where J is the infinite cyclic group. We set $\sigma(f) = \sigma(f_1^+)$, and the following proposition tells us that $\sigma(f)$ is well defined.

PROPOSITION 5. *If $F_2 : \Lambda_2 \longrightarrow S^1 \times X$ is another surgery as above from*

$$id \times g \circ \iota \amalg 1 \times f \quad to \quad f_2 : \partial^+\Lambda_2 \longrightarrow S^1 \times X$$

with $\left(\left[\frac{n+1}{2}\right]-1\right)$-connected fiber, then $\sigma(f_1^+) = \sigma(f_2^+)$.

PROOF. We may glue the two surgeries along $id \times g \circ \iota \amalg 1 \times f$ to obtain a cobordism $F : \Lambda \longrightarrow S^1 \times X$ from $\partial^+\Lambda_1 \xrightarrow{f_1} S^1 \times X$ to $\partial^+\Lambda_2 \xrightarrow{f_2} S^1 \times X$. But F may be factored through the map

$$(\Lambda; \partial^+\Lambda_1, \partial^+\Lambda_2) \xrightarrow{G} (P; P_1, P_2),$$

where P_1^+ and P_2^+ are obtained by adding cells along spheres as above, and $P = \Lambda \cup P_1^+ \cup P_2{}^+$. Then G is a cobordism of degree one in the sense so that $\sigma(f_1^+) = \sigma(f_2^+)$, and the proposition is proved.

PROPOSITION 6. $\sigma : \Omega_{n+1}(v) \longrightarrow L_{n+1}(1)$ *is a homomorphism.*

PROOF. Suppose we have $f_1 : \Gamma_1 \longrightarrow S^1 \times X$ and $f_2 : \Gamma_2 \longrightarrow S^1 \times X$ representing elements of $\Omega_{n+1}(v.)$ Recall that the surgery Λ above started by connected-summing $S^1 \times M$ and Γ, and continued by surgering only on the Γ-side. Let Λ_1 and Λ_2 be two such surgeries starting with

$$S^1 \times M \amalg \Gamma_i \xrightarrow{\text{id} \times \text{g} \circ \iota \amalg 1 \times \text{f}_i} S^1 \times X,$$

for $i = 1$ and 2 respectively. Then we may construct a new surgery $\Lambda_1 * \Lambda_2$ by starting with

$$S^1 \times M \amalg \Gamma_1 \amalg \Gamma_2 \xrightarrow{id \times g \circ \iota \amalg 1 \times f_1 \amalg 1 \times f_2} S^1 \times X,$$

connected-summing Γ_1 and Γ_2 to $S^1 \times M$, and then performing the surgeries of Λ_1 on the Γ_1-side, and the surgeries of Λ_2 on the Γ_2-side. Then $\Lambda_1 * \Lambda_2$ leads to

$$\partial^+\Lambda_1 * \Lambda_2 = \partial^+\Lambda_1 \underset{M}{\#} \partial^+\Lambda_2 \xrightarrow{f_1^+ \underset{M}{\#} f_2} S^1 \times X,$$

where the connected sum is taken along M in the obvious sense. On the other hand, there is a surgery Λ_{12} as above, starting from

$$S^1 \times M \amalg \Gamma_1 \# \Gamma_2 \xrightarrow{id \times g \circ \iota \amalg f_1 \# f_2} S^1 \times X$$

and leading to $\partial^+\Lambda_{12} \xrightarrow{f_{12}^+} S^1 \times X$ with $\left(\left[\frac{n+1}{2}\right]-1\right)$-connected fiber. By gluing these two surgeries along their common base, we obtain a bordism $\Lambda \xrightarrow{F} S \times X$ from $f_1^+ \underset{M}{\#} f_2^+$ to f_{12}^+. Now, the composition $\Lambda \xrightarrow{F} S^1 \times X \xrightarrow{pr} S^1$ is transverse to $1 \in S^1$, with inverse image $I \times M$. Cutting along this inverse image we obtain a manifold $\bar{\Lambda}$ with lateral boundary $I \times M \amalg I \times M$, bottom boundary $\overline{\partial^-\Lambda} = \overline{\partial^+\Lambda_{12}}$ = result of cutting $\partial^+\Lambda_{12}$ along $0 \times M \times 1$, and top boundary $\overline{\partial^+\Lambda} = \overline{\partial^+\Lambda_1 \underset{M}{\#} \partial\Lambda_2}$ = result of cutting $\partial^+\Lambda_1 \underset{M}{\#} \partial\Lambda_2$ along $1 \times M$. Let $\overline{\partial^+\Lambda_1}$ and $\overline{\partial^+\Lambda_2}$ be the results of cutting $\partial^+\Lambda_i$ along M for $i = 1$ and 2 respectively. Then we see that $\overline{\partial^+\Lambda} = \overline{\partial^+\Lambda_1} \cup \overline{\partial^+\Lambda_2}$ where the component $M \times 1$ of the lateral boundary of $\overline{\partial^+\Lambda_1}$ is canonically glued to the component $M \times 0$ of the lateral boundary of $\overline{\partial^+\Lambda_2}$. The sphere bouquets representing the kernels of

$$\pi_{\left[\frac{n+1}{2}\right]}(\partial^+\Lambda_i) \longrightarrow \pi_{\left[\frac{n+1}{2}\right]}(S^1 \times X)$$

for $i = 1, 2$ and 12 give rise to bouquets B_1, B_2 and B_{12} with $B_i \subset int\, \overline{\partial^+\Lambda_i}$. Then $\bar{P}_i = \overline{\partial^+\Lambda_i} \cup CB_i^-$ is the Poincaré Duality pair obtained by cutting along M the Poincaré Duality space P_i of the definition of $\sigma(f_i)$. And then, according to Shaneson [2], the simply connected surgery obstruction $\sigma(f_i)$ of $\partial^+\Lambda_i \longrightarrow P_i$ is the surgery obstruction of $\overline{\partial^+\Lambda_i}$, $M \times \{0, 1\} \longrightarrow \bar{P}_i$, $M \times \{0, 1\}$. Let $\bar{Q} = \overline{\partial\Lambda} \cup \bar{P}_1 \cup \bar{P}_2 \cup \bar{P}_{12}$. Then

$$(\bar{\Lambda}; \partial_-\bar{\Lambda}, \partial_+\bar{\Lambda}, I \times M \times \{0, 1\}) \longrightarrow (\bar{Q}; \bar{P}_{12}, \bar{P}_1 \cup \bar{P}_2, I \times M \times \{0, 1\})$$

is a degree one map of 4-ads, and by the usual addition theorem [3] we see that $\sigma(f_1) + \sigma(f_2) = \sigma(f_{12})$. But by $\sigma(f_{12}) = \sigma(f_1 \amalg f_2)$, so

$$\sigma(f_1) + \sigma(f_2) = \sigma(f_1 \amalg f_2),$$

and the proposition is proved.

PROPOSITION 7. *For $n \geq 6$ and $n > \lambda \geq \left[\frac{n+1}{2}\right]$, the sequences*

$$D(M, M^\lambda) \xrightarrow{t} \Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(v)$$

and

$$D^\lambda(M) \xrightarrow{\beta} \Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(v)$$

are exact.

PROOF. Since $D(M, M^\lambda) \longrightarrow D^\lambda(M)$ is an epimorphism, it suffices to prove the second sequence is exact. According to proposition 3, $t'[\varphi, h] = I + j_* t[\varphi, h] = [id \times g \circ \imath] + j_* t[\varphi, h]$. But $t'[\varphi, h]$ is represented by a map $t' : S^1 \times_\varphi M \longrightarrow S^1 \times X$ with $\left[\frac{n+1}{2}\right]$-connected fiber. Thus $\sigma(t[\varphi, h]) = 0$ and Image $(t) \subset ker(\sigma)$.

Now we have to show that $ker(\sigma) \subset Image(t)$. Suppose $f : \Gamma \longrightarrow X$ represents an element α of $\Omega_{n+1}(v)$ such that $\sigma(\alpha) = 0$. Then there is a bordism $F : \Lambda \longrightarrow S^1 \times X$ from

$$\partial^-\Lambda = S^1 \times M \amalg \Gamma \xrightarrow{id \times g \circ \imath \amalg 1 \times f} S^1 \times X$$

to $\partial^+\Lambda \xrightarrow{f^+} S^1 \times X$ with $\left[\frac{n+1}{2}\right]$-connected fiber. We may suppose that

$\partial^+\Lambda \xrightarrow{f^+} S^1 \times X$ is transverse along $1 \times X$, with $(f^+)^{-1}(1 \times X) = M$ and $f^+ \mid M = g \circ \imath$. Thus, there is a simply connected manifold A with $\partial A = M \times 0 \amalg M \times 1$ and $\partial^+\Lambda$ obtained by gluing $(x, 0)$ to $(\varphi(x), 1)$ for $x \in M$. Moreover, $H_i(A, M \times 0) = H_i(A, M \times 1) = 0$ for $i \leq \left[\frac{n+1}{2}\right]$ so $H^i(A, M \times 0) = H^i(A, M \times 1) = 0$ for $i \geq \left[\frac{n}{2}\right] + 1$ and $A = M \times I$. Thus $\partial^+\Lambda = S^1 \times_\varphi M$ for some diffeomorphism $\varphi : M \longrightarrow M$.

Since the composition $1 \times M \subset S^1 \times_\varphi M \longrightarrow S^1 \times X \longrightarrow X$ is homotopy equivalent to $g \circ \imath$, it follows that the fiber of f^+ is λ-connected. Then in the following diagram

$$\begin{array}{ccc} 1 \times M^\lambda & \subset & S^1 \times_\varphi M \\ \cap & & \downarrow f^+ \\ S^1 \times M^\lambda & \longrightarrow & S^1 \times X, \end{array}$$

the obstructions are zero to finding $S^1 \times M^\lambda \longrightarrow S^1 \times_\varphi M$ making the top triangle commute and the bottom one homotopy commute. But then $S^1 \times M^\lambda \longrightarrow S^1 \times_\varphi M$ is a normal map, and by Hirsch's Immersion Theorem we may, up to homotopy, assume it is given by an immersion $S^1 \times M^\lambda \longrightarrow S^1 \times_\varphi M$ such that $1 \times M^\lambda \longrightarrow 1 \times M$ by the inclusion. But then this immersion determines an immersion

$$h : I \times M^\lambda, \dot{I} \times M^\lambda \longrightarrow I \times M, \dot{I} \times M$$

such that $h_0 =$ inclusion and $h_1 = \varphi \mid M^\lambda$. And now it follows that $f^+ : \partial^+\Lambda \longrightarrow S^1 \times X$ represents $t'(\varphi, h) = I + j_* t[\varphi, h)$. But f^+ already represents $I + j_*\alpha$ so $j_*\alpha = j_* t(\varphi, h)$, and since j_* is 1-1, $\alpha = t(\varphi, h)$. The proposition is proved.

Finally, we would like to add one term to the second surgery exact sesequence above. Let $\alpha \in L_{n+2}(1)$. Then α is represented by a normal degree one map$(P, \partial P) \longrightarrow (D^{n+2}, S^{n+1})$ which is a homotopy equivalence on the boundary. There is a diffeormorphism $\varphi_\alpha : D^n \longrightarrow D^n$, identity near S^{n-1}, and unique modulo concordance fixed near S^{n-1}, such that the diffeomorphism $\varphi_\alpha : S^n \longrightarrow S^n$ it determines produces ∂P via the pasting construction. The diffeomorphism φ_α determines a diffeomorphism $\varphi_\alpha : M \longrightarrow M$ in the same way – extension by the identity outside some smooth $D^n \subset M$. We may assume $D^n \cap M^\lambda = \phi$, and we set $\partial\alpha = \tau[\varphi_\alpha, \text{const.}]$. Then $\partial : L_{n+2}(1) \longrightarrow D^\lambda(M)$ is a well-defined homomorphism.

THEOREM. *For* $n \geq 6$ *and* $n > \lambda \geq \left[\dfrac{n+3}{2}\right]$, *the following sequence is exact:*

$$L_{n+2}(1) \xrightarrow{\partial} D^\lambda(M) \xrightarrow{\beta} \Omega_{n+1}(v) \xrightarrow{\sigma} L_{n+1}(1).$$

PROOF. We have only to check exactness at $D^\lambda(M)$. To see that $t \circ \partial = 0$ notice that $(S^1 \times M) \# \partial P = S^1 \times_{\varphi_\alpha} M$, and that $D^n \cap M^\lambda = \phi$ gives us

$$\begin{array}{ccc} & S^1 \times M^\lambda & \\ & \downarrow \text{incl.} \quad \searrow \text{incl.} & \\ (S^1 \times M) \# \partial P = S^1 \times_{\varphi_\alpha} M & \longrightarrow & S^1 \times M = (S^1 \times M) \# S^{n+1}, \end{array}$$

a commutative diagram, so that $\beta(\partial\alpha)$ is represented by

$$(S^1 \times M) \# \partial P \longrightarrow S^1 \times M \xrightarrow{pr_2} M \xrightarrow{g \circ \iota} X,$$

which bounds $(D^2 \times M) \# P \longrightarrow D^2 \times M \longrightarrow X$.

Now suppose $\beta[\varphi, l] = 0$. Then the lift $l : S^1 \times_\varphi M \longrightarrow X$ extends to a lift $L : \Gamma \longrightarrow X$ for some manifold Γ with $\partial\Gamma = S^1 \times_\varphi M$, and we obtain the commutative diagram

$$\begin{array}{ccc} \partial\Gamma = S^1 \times_\varphi M & \xrightarrow{l'} & S^1 \times X \\ \cap & & \cap i \\ \Gamma & \xrightarrow{L'} & D^2 \times X \end{array}$$

where $l' = (pr_1, l)$ and $L' = (\zeta, L)$ for some extension $\zeta : \Gamma \longrightarrow D^2$ of $pr_1 : \partial\Gamma \longrightarrow S^1$. The short exact sequence

$$0 \longrightarrow H_{n+2}(D^2 \times X, S^1 \times X) \xrightarrow{\partial} H_{n+1}(S^1 \times X) \xrightarrow{i_*} H_{n+1}(D^2 \times X) \longrightarrow 0$$

gives us $\partial L'_*[\Gamma, \partial\Gamma] = [S^1] \times g_*\iota_*[M] + 1 \times x$ for some

$$x \in H_{n+1}(X) = H_{n+1}(D^2 \times X).$$

But $i_*\partial L_*[\Gamma, \partial\Gamma] = 0$ and $i_*([S^1] \times g_*\iota_*[M] + 1 \times x) = x$, so $x = 0$. On the other hand

$$(D^2 \times M, S^1 \times M) \xrightarrow{id\, g \circ \iota} (D^2 \times X, S^1 \times X)$$

has $\partial(id \times g \circ \iota)_*[D^2 \times M, S^1 \times M] = [S^1] \times g_*\iota_*[M]$ so

$$(id \times g \circ \iota)_*[D^2 \times M, S^1 \times M] = L'_*[\Gamma, \partial\Gamma].$$

Now, using $\lambda \geq \left[\dfrac{n+3}{2}\right]$, it is easy to check that

$$(id \times g \circ \iota)_*[D^2 \times M, S^1 \times M] \cap : H^{\left[\frac{n+3}{2}\right]}(D^2 \times X, S^1 \times X) \longrightarrow H_{\left[\frac{n+2}{2}\right]}(D^2 \times X)$$

is an isomorphism. By assuming, as we may, that $\Gamma \xrightarrow{L} X$ has $\left(\left[\dfrac{n+2}{2}\right] - 1\right)$-connected fiber, we see that the surgery obstruction $\sigma(L) \in L_{n+2}(1)$ to making the fiber $\left[\dfrac{n+2}{2}\right]$-connected is defined. Let $\alpha = \sigma(T)$ and consider $[\varphi, l] \circ (\partial\alpha)^{-1}$. First, $\beta([\varphi, l] \circ (\partial\alpha)^{-1}) = 0\text{-}0 = 0$. Second, $[\varphi, l] = \tau[\varphi, h]$ for some regular homotopy h, so

$$[\varphi, l] \,.\, (\partial\alpha)^{-1} = \tau([\varphi, h] \,.\, [\varphi_\alpha^{-1}, \text{rev (const.)}]) = \tau[\varphi \circ \varphi_\alpha^{-1}, h],$$

and we obtain the commutative diagram

$$\begin{array}{ccc} S^1 \times_{\varphi \circ \varphi_\alpha^{-1}} M & \xrightarrow{t'[\varphi \circ \varphi_\alpha^{-1}, h]} & S^1 \times X \\ \cap & & \cap \\ \Gamma_1 & \xrightarrow{L'_1} & D^2 \times X, \end{array}$$

where $\Gamma_1 = \Gamma \# P$ for a suitable parallelizable manifold, such that $\sigma(L'_1) = 0$. Thus we may assume that the fiber of L'_1 is $\left[\dfrac{n+2}{2}\right]$-connected.

We may embed $I \times M$ in Γ_1 so that $1 \times M \xrightarrow{id} 1 \times M$ and $0 \times M \longrightarrow int\, \Gamma_1$. Using this embedding, the fact that L'_1 is $[\frac{n+2}{2}]$-connected and the relative s-cobordism theorem in the simply connected case, we obtain a diffeomorphism $\Gamma_1, \partial\Gamma_1, 1 \times M \longrightarrow D^2 \times M, S^1 \times M, 1 \times M$ which is the identity on $1 \times M$. It follows immediately that $\varphi \circ \varphi_\alpha^{-1}$ is concordant to the identity, so that $[\varphi \circ \varphi_\alpha^{-1}, h] = [1, h]$.

This is the point at which we need $D^\lambda(M)$ instead of the more attractive $D(M, M^\lambda)$: We have a commutative diagram

$$\begin{array}{ccc}
1 \times M & \xrightarrow{\quad L \quad} & 1 \times \overline{M} \\
\cap & & \downarrow g \\
S^1 \times M & \xrightarrow{\quad t'(1,\, h) \quad} & S^1 \times X \\
\cap & & \cup \\
\Gamma & \xrightarrow{\quad L_1' \quad} & D^2 \times X
\end{array}$$

with $1 \times M \longrightarrow \Gamma$ a homotopy equivalence. It follows that $(1, t'(1, h))$ represents the identity element of $D^\lambda(M)$; that is, $\tau[1, h] = 1$, which implies $[\varphi, l] = \tau[\varphi, h] = \tau[\varphi_\alpha, \text{const.}] = \partial\alpha$, and the theorem is proved. On the other hand, in $D(M, M^\lambda)$ we have no guarantee that $[1, h] = 1$.

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