On Completion and Shape*

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Dedicated to the memory of Carlos B. de Lyra.

O. Introduction. The purpose of this paper is to bring together two concepts which appear in current work in algebraic topology, namely completion and shape. In [4] the completion of an object Y in a category $\mathscr{T}$ with respect to a functor $h\colon \mathscr{T} \longrightarrow \mathscr{A}$ was discussed and in [5] it was generalised to the completion with respect to a family S of morphisms in $\mathscr{T}$, and a close relationship between global S-completion and idempotent triples was established. In [8] Le Van introduced the notion of shape category for a full embedding $K\colon \mathscr{P} \longrightarrow \mathscr{T}$ between any two categories $\mathscr{P}$ and $\mathscr{T}$, generalising the notion of shape first introduced by Borsuk [1] and further developed by many other authors, in the context of topology.

For the sake of simplicity we shall restrict ourselves to global S-completion or h-completion, i.e. to the case where every object in $\mathscr{T}$ admits a completion. We also restrict ourselves to considering shape for an embedding $K\colon \mathscr{P} \longrightarrow \mathscr{T}$ where $\mathscr{P}$ is a full reflective subcategory of $\mathscr{T}$, as this situation is well suited to match with global completion.

In section 1 we condense from [4] and [5] what we need on global completion. Section 2 is devoted to the notion of shape of a full reflective embedding $K\colon \mathscr{P} \longrightarrow \mathscr{T}$. We emphasize its connection with the family of morphisms rendered invertible by the left adjoint F to K and with the (idempotent) triple $\mathbb{T}$ generated by the adjoint pair $F \dashv K$. Some of the properties established hold for more general functors K, but we defer the study of the general situation to a forthcoming paper.

*Recebido pela SBM em 26 de março de 1975.

In section 3 we establish the link between completion and shape. If S is a family of morphisms in $\mathscr{T}$ (S may be defined as the family of morphisms rendered invertible by some functor with domain $\mathscr{T}$) such that S–completion exists globally, and E: $\mathscr{T}_S \longrightarrow \mathscr{T}$ is the embedding of the full subcategory consisting of the S–complete objects of $\mathscr{T}$, it turns out that two objects X, Y of $\mathscr{T}$ have isomorphic S–completions if and only if they are isomorphic in the shape category of E, i.e. they have the same E–shape.

Section 4 is devoted to examples. Any situation where completion exists gobally serves as example; we describe a few situations of global completion.

In the course of the paper we make use of the category of fractions $\mathscr{T}[S^{-1}]$ of a category $\mathscr{T}$, with respect to a family S of morphisms in $\mathscr{T}$. If $\mathscr{T}$ belongs to a given universe, $\mathscr{T}[S^{-1}]$ belongs, in general, to a higher universe. In our case however, the assumption that S–completion exists globally insures that $\mathscr{T}[S^{-1}]$ belongs to the same universe as $\mathscr{T}$.

Everything described can, of course, be dualised: completion dualising to cocompletion (see e.g. [4] section 2) and shape to coshape [8].

1. Global S-completion. Let $\mathscr{T}$ be a category and S a family of morphisms in $\mathscr{T}$. We denote $\mathscr{T}[S^{-1}]$ the category of fractions with respect to S and F_S: $\mathscr{T} \longrightarrow \mathscr{T}[S^{-1}]$ the canonical functor. As $\mathscr{T}$ and $\mathscr{T}[S^{-1}]$ have the same objects we will use the same symbols for them in both categories. We assume S to be saturated, i.e. to contain all morphisms rendered invertible by F_S; this immediately entails that S is closed under composition and contains all isomorphisms in $\mathscr{T}$.

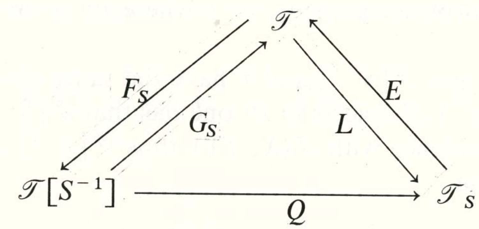
Given an object Y in $\mathscr{T}$, we say that Y is *S–completable* if the contravariant functor $\mathscr{T}[S^{-1}](-, Y)$: $\mathscr{T} \longrightarrow Ens$ is representable, i.e. if there is a natural equivalence τ: $\mathscr{T}[S^{-1}](-, Y) \longrightarrow \mathscr{T}(-, Z)$. We then call Z the *S–completion* of Y, and $e = \tau(1_Y)$: $Y \longrightarrow Z$ the canonical morphism. Thus, if it exists, S–completion is determined up to canonical isomorphism. If e is an isomorphism, we say that Y is *S–complete*.

If h: $\mathscr{T} \longrightarrow \mathscr{A}$ is any functor, let S be the family of morphism in $\mathscr{T}$ which are rendered invertible by h. Then, by Proposition 1.1 of [4], S is saturated.

*The author is grateful to Peter Hilton for valuable conversations on shape theory.

In this case the S–completion of Y is also called the h–completion of Y, and one also speaks about h–complete objects.

Throughout this section we assume that every object in $\mathscr{T}$ is S–completable, which is equivalent to assuming that the functor F_S has a right adjoint G_S. By Proposition 2.3 of [4] G_S is fully faithful whenever it exists. $G_S F_S Y$ is then the S–completion of Y and the unit η_Y: $Y \longrightarrow G_S F_S Y$ of the adjunction is the canonical morphism. The S–complete objects are precisely those Y for which η_Y is an isomorphism. The S–complete objects generate a full subcategory $\mathscr{T}_S$ of $\mathscr{T}$ and we denote E: $\mathscr{T}_S \longrightarrow \mathscr{T}$ the embedding. By Proposition 2.7 of [4] there is an equivalence Q: $\mathscr{T}[S^{-1}] \longrightarrow \mathscr{T}_S$ with $EQ = G_S$. The functor $L = QF_S$ is left adjoint to E, thus $\mathscr{T}_S$ is a full reflective subcategory of $\mathscr{T}$. The situation is illustrated by the diagram



For later use we collect some facts about our situation in

THEOREM 1.1. *If F_S has a right adjoint G_S with unit η and counit ε of the adjuction, then:*

(i) *S has a calculus of left fractions.*
(ii) *The triple $\mathbb{T} = (T, \eta, \mu) = (G_S F_S, \eta, G_S \varepsilon F_S)$ induced by the adjoint pair $G_S \dashv F_S$ is idempotent, i.e. μ is an equivalence.*
(iii) *η_X is in S, for all objects X in $\mathscr{T}$.*
(iv) *If $\mathscr{T} \xrightarrow{F_T} \mathscr{T}_T \xrightarrow{G_T} \mathscr{T}$ is the Kleisli situation of the triple $\mathbb{T}$, then there is a unique isomorphism of categories I: $\mathscr{T}[S^{-1}] \longrightarrow \mathscr{T}_T$ with $IF_S = F_T$ and $G_T I = G_S$.*
(v) *The adjoint pair $L \dashv E$ induces the Triple $\mathbb{T}$ of* (ii).
(vi) *The S–complete objects are precisely those which are isomorphic to $G_S F_S Y$, for some Y in $\mathscr{T}$.*

PROOF. (i) follows immediately from Theorem 2.6 of [4] and (ii) follows from Theorem 2.2 of [5] as G_S is full.

(iii) follows from (ii) and Proposition 2.1 of [5].

(iv) As F_T is onto objects, and $\mathbb{T}$ is idempotent, it follows, by Theorem 2.4 of [5], that G_T is fully faithful. But G_S is fully faithful as well, hence both F_T and F_S render invertible precisely the morphisms in S. The assertion then follows from the universal properties of F_S and F_T, G_T.

(v) follows from Proposition 1.1 of [5] as the functor Q, being an equivalence, is fully faithful.

(vi) If X is S-complete then $\eta_X\colon X \longrightarrow G_S F_S X = TX$ is an isomorphism. On the other hand if $X \xrightarrow{\phi} TY$ is an isomorphism, then $\eta_X = (T\phi)^{-1}.\ \eta_{TX}.\ \phi$. But this is an isomorphism as $\mu_X \eta_{TX} = 1$ where μ_X is an isomorphism.

REMARK. Notice that $\mathscr{T}[S^{-1}]$ and $\mathscr{T}$ have the same objects. Furthermore, for any pair X, Y of objects in $\mathscr{T}$ one has that $\mathscr{T}[S^{-1}](X, Y)$ is in one to one correspondence with $\mathscr{T}(X, TY)$, thus $\mathscr{T}[S^{-1}]$ and $\mathscr{T}$ belong to the same universe.

We have seen that whenever the S-completion exists globally it determines a full reflective subcategory $\mathscr{T}_S$ of $\mathscr{T}$. On the other hand given any full reflective subcategory $\mathscr{T}'$ of $\mathscr{T}$ with embedding E and left adjoint L to E, let S be the family of morphisms rendered invertible by EL. The triple $\mathbb{T} = (T, \eta, \mu)$ induced by the adjoint pair $L \dashv E$ is idempotent as E is full. By Theorem 2.4, (i), (iii) of [5] the S-completion exists globally and the category $\mathscr{T}_S$ is plainly isomorphic to $\mathscr{T}'$.

2. Shape. Let $K\colon \mathscr{P} \longrightarrow \mathscr{T}$ be any functor. The *shape category* $\mathscr{S}_K$ of K is defined by:

$$\mathscr{S}_K \text{ has the same objects as } \mathscr{T}$$

and

$$\mathscr{S}_K(X, Y) = Nat\,[\mathscr{T}(Y, K-), \mathscr{T}(X, K-)].$$

The composition in $\mathscr{S}_K$ is simply the composition of natural transformations. Notice that in general $\mathscr{S}_K$ belongs to a higher universe than $\mathscr{T}$.

We denote by D the canonical functor $\mathscr{T} \longrightarrow \mathscr{S}_K$ defined by $DX = X$ on objects and by $Df = \mathscr{T}(f, K-) = f^*$ on morphisms. We denote by K' the composite $K' = DK\colon \mathscr{T} \longrightarrow \mathscr{S}_K$.

Throughout this paragraph we assume that K is fully faithful and has a left adjoint F. We write η, ε for the unit and counit of the adjunction, and from Proposition 2.2 of [5] we immediately have:

PROPOSITION 2.1. *The triple $\mathbb{T} = (T, \eta, \mu) = (KF, \eta, K\varepsilon F)$ induced by the adjoint pair $F \dashv K$ is idempotent.*

For any X in $\mathscr{T}$, $\eta_X\colon X \longrightarrow KFX$ is rendered invertible by D, i.e.

$$D(\eta_X) = \eta_X^*\colon \mathscr{T}(KFX, K-) \longrightarrow \mathscr{T}(X, K-)$$

is an equivalence. Indeed, in the commnutative diagram

$$\begin{array}{ccc} \mathscr{T}(KFX, KZ) & \xleftarrow{(K)} & \mathscr{P}(FX, Z) \\ & {\scriptstyle \eta_X^*}\searrow \quad \swarrow{\scriptstyle \alpha} & \\ & \mathscr{T}(X, KZ) & \end{array}$$

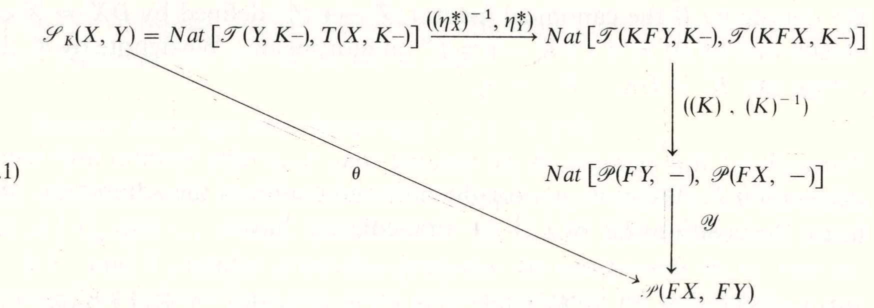
of natural transformations (K) is an equivalence as K is fully faithful and α is the adjunction-equivalence.

As η^* is an equivalence, the diagram

$$\begin{array}{ccc} \mathscr{T}(KFY, K-) & - - - - - - - - - \rightarrow & \mathscr{T}(KFX, K-) \\ \downarrow {\scriptstyle \eta_Y^*} & & \downarrow {\scriptstyle \eta_X^*} \\ \mathscr{T}(Y, K-) & - - - - - - - - - \rightarrow & \mathscr{T}(X, K-) \end{array}$$

defines a 1–1 correspondence θ

(2.1)



where $\mathscr{Y}$ is the Yoneda equivalence. θ clearly preserves the composition of morphisms and the identities. With this we define a functor $M: \mathscr{S}_K \longrightarrow \mathscr{P}$ by

$$M(X) = F(X) \qquad \text{on objects}$$

and

$$M(\omega) = \theta(\omega) \qquad \text{on morphisms,}$$

which is plainly fully faithful.

As K is fully faithful, the counit ε: $FK \longrightarrow 1$ of the adjunction $F \dashv K$ is an equivalence. Thus every object X in $\mathscr{P}$ is isomorphic to FKX hence to MKX, which shows that M is an equivalence of categories. A straightforward verification shows that $MD = F$. Thus $MK' = MDK = FK \xrightarrow{\varepsilon} 1$ is an equivalence. $K'MZ = K'FZ = DKFZ \xrightarrow{D\eta_Z} DZ = Z$ is an isomorphism for every object Z in $\mathscr{S}_K$ as shown above. Hence, M is a two sided adjoint of K'.

Summarising we have:

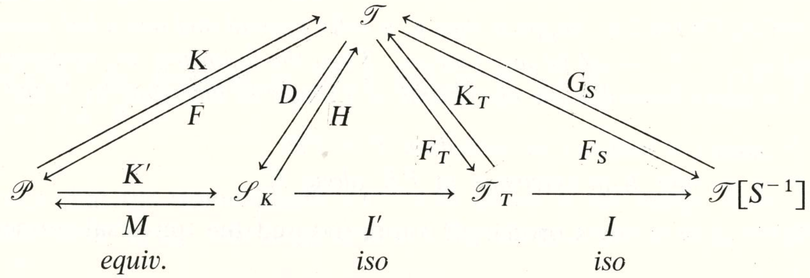
THEOREM 2.2. *If K: $\mathscr{P} \longrightarrow \mathscr{T}$ is fully faithful and has a left adjoint F, then there is an equivalence of categories M: $\mathscr{S}_K \longrightarrow \mathscr{P}$ which is a two-sided adjoint to K' and satisfies $MD = F$.*

We denote by H the composite $KM : \mathscr{S}_K \longrightarrow \mathscr{T}$ which is right adjoint to D as $K'F$ is equivalent to D.

COROLLARY 2.3. *If K: $\mathscr{P} \longrightarrow \mathscr{T}$ is fully faithful and has a left adjoint F, then:*

(i) *The adjoint pair $D \dashv H$ induces the same triple $\mathbb{T}$ on $\mathscr{T}$ as $K \dashv F$ does (and this triple is idempotent by Proposition 2.1).*
(ii) *The functors T, F and D render invertible the same family S of morphisms in $\mathscr{T}$, and S is saturated and has a calculus of left fractions.*
(iii) *If $\mathscr{T} \xrightarrow{F_T} \mathscr{T}_T \xrightarrow{K_T} \mathscr{T}$ is the Kleisli situation of $\mathbb{T}$, then there is a unique isomorphism of categories I': $\mathscr{S}_K \longrightarrow \mathscr{T}_T$ with $I'D = F_T$ and $KI' = H$.*
(iv) *The functor F_S: $\mathscr{T} \longrightarrow \mathscr{T}[S^{-1}]$ has a right adjoint G_S; the adjoint pair $F_S \dashv G_S$ induces the triple $\mathbb{T}$ of (i), and there is a unique isomorphism of categories I: $\mathscr{T}_T \longrightarrow \mathscr{T}[S^{-1}]$ with $IF_T = F_S$ and $G_SI = F_T$.*

The situation is illustrated by the diagram:



PROOF. (i) holds by the same argument as (v) of Theorem 1.1.

(ii) T, F and D render invertible the same family S as $T = KF$, $F = MD$ where K and M are fully faithful. S is saturated by Proposition 1.1 of [4] and has a calculus of left fractions by Proposition 2.5 of [4].

(iii) $\mathscr{S}_K$ and $\mathscr{T}_T$ both have the same objects as $\mathscr{T}$. Thus we define I' to be the identity on objects. Furthermore $\mathscr{S}_K(X, Y) \xrightarrow{\theta} \mathscr{P}(FX, FY) \xrightarrow{\alpha} \mathscr{T}(X, KFY) = \mathscr{T}_T(X, Y)$, where θ is as in (2.1) and α is the adjunction-equivalence, is a 1–1 correspondence which carries the composition of morphisms in $\mathscr{S}_K$ to the one in $\mathscr{T}_T$. Thus we define I' on morphisms by $I'(f) = \alpha\theta(f)$. I' is clearly an isomorphism satisfying $I'D = F_T$ and $KI' = H$. It is unique with this property as its inverse is uniquely determined by the universal property of the Kleisli situation.

(iv) The fact that F_S has a right adjoint G_S and that $F_S \dashv G_S$ generates $\mathbb{T}$ follows from Theorem 2.4 of [5] as $\mathbb{T}$ is idempotent. The rest follows from Theorem 1.1, (iv).

REMARK. As the functor F_S has a right adjoint, by the remark in Section 1 $T[S^{-1}]$ belongs to the same universe as $\mathscr{T}$. But by (iii) and (iv) of the corollary, $\mathscr{S}_K$ is isomorphic to $\mathscr{T}[S^{-1}]$, thus $\mathscr{S}_K$ belongs to the same universe as $\mathscr{T}$.

From Proposition 3 on page 245 of [9] we infer that, whenever $K\colon P \longrightarrow T$ has a left adjoint F with counit $\varepsilon\colon FK \longrightarrow 1$, then any functor $Q\colon P \longrightarrow A$ has a right Kan extension $\tilde{Q} = QF$ along K with universal transformation $Q\varepsilon$, and this Kan extension is preserved by any functor. Coming back to our situation we have the following statement about functors with domain $\mathscr{T}$:

PROPOSITION 2.4. *Suppose that K is fully faithful and has a left adjoint F, and let $P\colon \mathscr{T} \longrightarrow \mathscr{A}$ be any functor. Then the following are equivalent:*
(i) *P renders invertible the morphisms of S rendered invertible by F and D.*
(ii) *P admits a (unique) factorisation $P = \overline{P}D$.*
(iii) *P is a right Kan extension of PK along K.*

PROOF. (i)$\Leftrightarrow$(ii) by Corollary 2.3 (iii), (iv) and the universal property of F_S.

(i) $\Rightarrow$ (iii). η_X is in S for any X in $\mathscr{T}$ by Proposition 2.1 (iii) of [5], thus $P\eta\colon P \longrightarrow PKF$ is an equivalence. But PKF is a right Kan extension of PK along K, and so is P.

(iii) $\Rightarrow$ (ii). If P is a Kan extension of PK along K then there is a natural equivalence $P \longrightarrow PKF$ thus P renders invertible the elements of S as F does.

Given any functor $K\colon \mathscr{P} \longrightarrow \mathscr{T}$, K establishes an equivalence relation in the class of objects of $\mathscr{T}$. More precisely

DEFINITION 2.5. Given a functor $K\colon \mathscr{P} \longrightarrow \mathscr{T}$ we say that two objects X, Y in $\mathscr{T}$ have the same *K–shape* or that they are *K–shape equivalent* if DX and DY are isomorphic in $\mathscr{S}_K$.

Shape equivalence is clearly an equivalence relation. Notice that equivalence in $\mathscr{T}$ implies shape equivalence, and that the two equivalence

relations are the same if and only if K is codense. (Proposition 2 on page 242 of [9]).

3. Shape and Completion. Let $h\colon \mathscr{T} \longrightarrow \mathscr{A}$ be a functor, S the family of morphisms rendered invertible by h. As in section 1 we assume that S-completion exists globally and denote by $E : \mathscr{T}_S \longrightarrow \mathscr{T}$ the embedding of the S-complete and by L its left adjoint. By Corollary 2.3 (iii), (iv) the canonical functor $D\colon \mathscr{T} \longrightarrow \mathscr{S}_E$ renders invertible precisely the morphisms in S.

PROPOSITION 3.1. *If two objects X, Y in $\mathscr{T}$ have the same E-shape, then hX is isomorphic to hY.*

PROOF. By Proposition 2.4, h factors as $h = \overline{h}D$, hence $DX \cong DY$ entails $hX \cong hY$.

Notice that the isomorphism $hX \longrightarrow hY$ above is the image under $\overline{h}$ of a morphism in $\mathscr{S}_K$, but in general not the image under h of a morphism in $\mathscr{T}$.

DEFINITION 3.2. We say that two objects X, Y in $\mathscr{T}$ are *h-quisoconnected* if there are morphisms $X \xrightarrow{s} Z \xleftarrow{t} Y$ in S.

By Theorem 1.1 S has a calculus of left fractions, thus two objects X, Y in $\mathscr{T}$ are quisoconnected if and only if they satisfy the apparently weaker condition: there is a finite string

$$X \xrightarrow{s_1} \cdot \xleftarrow{s_2} \cdot \xrightarrow{s_3} \cdot \qquad \cdot \xleftarrow{s_n} Y$$

of morphisms in S.

PROPOSITION 3.3. *Let $h\colon \mathscr{T} \longrightarrow \mathscr{A}$ be a functor, S the family of morphisms rendered invertible by h. Suppose that the S–completion exists globally. Then two objects X, Y in $\mathscr{T}$ have isomorphic S–completions if and only if they are h–quisoconnected.*

PROOF. If X and Y have isomorphic S–completions, then there is an isomorphism $\phi\colon\ G_SF_SX \longrightarrow G_SF_SY$, and in $X \xrightarrow{\eta_X} G_SF_SX \xrightarrow{\phi} G_SF_SY$

$\xleftarrow{\eta_Y} Y$, $\phi\eta_X$ and η_Y are in S by Theorem 1.1 (iii). If X and Y are h–quisoconnected and $X \xrightarrow{s} Z \xleftarrow{t} Y$ are morphisms in S, then $(G_SF_St)^{-1} \circ G_SF_Ss$: $G_SF_SX \longrightarrow G_SF_SY$ is an isomorphism.

The classification of the objects of $\mathscr{T}$ as given in Proposition 3.3 can also be expressed in terms of shape, thus bringing together the notions of completion and of shape.

THEOREM 3.4. *Let* h: $\mathscr{T} \longrightarrow \mathscr{A}$ *be a functor, S the family of morphisms rendered invertible by h. Suppose that the S–completion exists globally and denote by* $E : \mathscr{T}_S \longrightarrow \mathscr{T}$ *the embedding of the full subcategory* $\mathscr{T}_S$ *of* $\mathscr{T}$ *generated by the S–complete objects. Then two objects X, Y in* $\mathscr{T}$ *have isomorphic S–completions if and only if they are E–shape equivalent.*

PROOF. Suppose that X and Y have isomorphic S–completions. Then they are h–quisoconnected by Proposition 3.3. But D renders invertible the morphisms in S, hence DX and DY are isomorphic. On the other hand let ϕ: $DX \longrightarrow DY$ be an isomorphism. Then $H\phi$: $HDX \longrightarrow HDY$ is clearly an isomorphism, but by Corollary 2.3, (i), (iv) $HD \cong G_SF_S$.

From Propositions 3.1 and 3.3 and Theorem 3.4 one infers immediately that hX is isomorphic to hY whenever X and Y are K–shape equivalent. In general it is not true that X and Y are shape equivalent whenever hX and hY are isomorphic. However we have

PROPOSITION 3.5. *Under the same hypotheses as in Theorem 3.4, if* f: $X \longrightarrow Y$ *is a morphism in T, then the following are equivalent:*

(i) $h(f)$ *is an isomorphism;*

(ii) $G_SF_S(f)$ *is an isomorphism, i.e. f induces an isomorphism on the completions of X and Y;*

(iii) $D(f)$ *is an isomorphism, i.e. f induces a shape-equivalence between X and Y.*

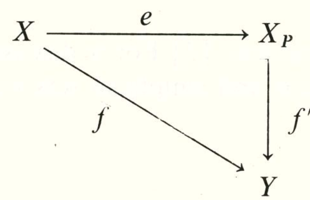
PROOF. Evident, as h, F_S and D render invertible the same family S of morphisms and G_S is fully faithful.

4. Examples. *Adams completion.* Let $\mathscr{T}$ be the category whose objects are the based CW–complexes and whose morphisms are based homotopy

classes of base-point preserving maps. Let h be a generalised additive homology (or cohomology) theory on $\mathscr{T}$ taking values in the category of graded abelian groups. Let S be the family of morphisms in $\mathscr{T}$ which are rendered invertible by h. Then in [3] A. Deleanu shows that the S–completion exists globally, provided that for every object Y in $\mathscr{T}$ the subcategory S_Y of $(Y \downarrow T)$ consisting of morphisms in S has a small cofinal subcategory. If K is the embedding of the full subcategory generated by the S–complete objects, then two objects have isomorphic S–completions iff they are K–shape equivalent iff they are h–quisoconnected.

Cochain homotopy. Let A be an abelian category with enough injectives, $K^+(A)$ the category of positive cochain complexes over A and homotopy classes of cochain maps. Denote h: $K^+(A) \longrightarrow A^{\mathbb{Z}_+}$ the cohomology functor and S the family of morphism of $K^+(A)$ rendered invertible by h. Then by Lemma 4.6 of [6] every object $X^{\cdot}$ in $K^+(A)$ admits a map s: $X^{\cdot} \longrightarrow X_S^{\cdot}$ in S, where $X_S^{\cdot}$ consists of injectives. It turns out that $X_S^{\cdot}$ is the S–completion of $X^{\cdot}$.

Localisation theories provide examples of completions. Often a localisation theory is defined in the following way: Given a category $\mathscr{T}$ and a family P of primes one defines a notion of P–local object in $\mathscr{T}$. A morphism e: $X \longrightarrow X_P$ is said to be P–localising if X_P is P–local and for every morphism f: $X \longrightarrow Y$ with Y P–local there is a unique morphism f' such that



commutes. This determines (e, X_P) up to canonical isomorphism. If every X in $\mathscr{T}$ has a P–localising morphism, the object-function $X \longrightarrow X_P$ extends to a functor L': $\mathscr{T} \longrightarrow \mathscr{T}_P$ where $\mathscr{T}_P$ is the full subcategory of $\mathscr{T}$ generated by the P–local objects. The couniversal property of e entails that L' is left adjoint to the embedding E: $\mathscr{T}_P \longrightarrow \mathscr{T}$ with counit e of the adjunction. The pair (L, e) where $L = EL'$ is then a P–localising theory.

By the comment at the end of section 1, a localisation theory gives rise to global S–completion with respect to the family S of morphisms rendered invertible by EL'. Furthermore the P–localising functor $L = EL'$ and the S–completing functor $G_S F_S$: $\mathscr{T} \longrightarrow \mathscr{T}$ are equal by Theorem 2.4, (i), (iii) of [5]; the two adjoint pairs even generate the same triple.

Two localisation theories have recently been studied in great depth [7].

Nilpotent groups. Let $\mathscr{N}$ be the category of nilpotent groups and P a family of primes. A nilpotent group G (or any group) is said to be P–local if $x \longrightarrow x^n$, $x \in G$, is bijective for all n prime to P. The fundamental theorem on page 7 of [7] then states that there is a P–localisation theory. The family S of the corresponding completion consists of the P–isomorphisms, i.e. of those homomorphisms ϕ: $G \longrightarrow K$ whose kernel consists of elements of finite order prime to P and have the property: for any $y \in K$ there is an n, prime to P, such that y^n is in the image of ϕ.

Nilpotent spaces. A connected CW–complex X is said to be nilpotent if $\pi_1(X)$ is nilpotent and operates nilpotently on $\pi_n(X)$ for every $n \geq 2$. A nilpotent CW–complex X is said to be P–local (where P is a family of primes) if $\pi_n(X)$ is P–local for all $n \geq 1$. Let $\mathscr{N}\mathscr{H}$ be the homotopy category of nilpotent CW–complexes. Theorem 3A of [7] then asserts that every X in $\mathscr{N}\mathscr{H}$ admits a P–localisation, i.e. that there is a P–localisation theory on $\mathscr{N}\mathscr{H}$. The family S of the corresponding completion are those morphisms ϕ in $\mathscr{N}\mathscr{H}$ for which $\pi_n(\phi)$ is a P–isomorphism for all $n \geq 1$.

Bousfield-Kan R-completion. [2] For a discussion of the Bousfield-Kan R–completion of groups and simplicial sets we refer to the examples in section 4 of [5].

References

[1] K. BORSUK, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), 223-264.

[2] A. K. BOUSFIELD, D. M. KAN, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304, Springer (1972).

[3] A. DELEANU, *Existence of the Adams completion for CW-complexes* (preprint).

[4] A. DELEANU, A. FREI and P. HILTON, *Generalised Adams Completion*, Cahiers de Topologie, Vol XV (1974) 61-82.

[5] A. DELEANU, A. FREI and P. HILTON, *Idempotent triples and completion*, Math. Zeitschr. (1975) (to appear).

[6] R. HARTSHORNE, *Resdues and duality*, Lecture Notes in Math., 20 Springer (1966).

[7] P. HILTON, G. MISLIN and J. ROITBERG, *Localization of nilpotent groups and spaces*, Mathematics Studies 15, North Holland (1975).

[8] J. LE VAN, *Shape theory*, Dissertation, University of Kentucky (1973).

[9] S. MACLANE, *Categories for the working mathematician*, Springer (1971).

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