Pre-Zariski Topology*

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Dedicated to Professor Carlos B. de Lyra.

1. Let $\mathbb{R}$ denote the field of all real numbers. Results on algebraic groups over $\mathbb{R}$ play important roles for Lie groups. However, when we use theorems of algebraic groups to study Lie groups, sometimes the unnaturalness of the Zariski topology in $\mathbb{R}^n$ causes some inconvenience. For example, the multiplicative group $\mathbb{R}^+$ of positive reals is not algebraic in $GL(1, \mathbb{R}) = \mathbb{R} - \{0\}$, although it is the connected component of an algebraic group. The concept of pre-algebraic groups was thus introduced for convenience, see e.g. Goto-Wang [1].

In this note, we shall show that the concept of pre-algebraic groups comes from a topology of $\mathbb{R}^n$, which we call pre-Zariski.

2. Unless specified otherwise, we consider $\mathbb{R}^n = \{(a_1, \ldots, a_n);\ a_i \in \mathbb{R}\}$ with respect to the usual euclidean topology. Let $\mathscr{E} = \mathscr{E}(n)$ and $\mathscr{Z} = \mathscr{Z}(n)$ denote the totality of closed sets, and the set of all algebraic (= Zariski closed) sets, in $\mathbb{R}^n$, respectively. Then $\mathscr{E} \supset \mathscr{Z}$ and $\mathscr{Z}$ satisfies the following chain condition: for $Z_1, Z_2, \ldots, \in \mathscr{Z}$,

(*) if $Z_1 \supsetneqq Z_2 \supsetneqq \ldots$, then the sequence is finite.

Let Z be in $\mathscr{Z}$. Then Z is known to be locally (arcwise) connected and every connected component of Z is open (and closed) in Z. Hence, every open and closed subset of Z is a union of connected components, and conversely, any union of connected components is open and closed in Z. Also, the number of connected components of Z is finite, by Whitney [2].

*Recebido pela SBM em 26 de março de 1935.
(1)Partially supported by grant ≃ NSF GP 439401.

Let us denote by $\mathscr{P} = \mathscr{P}(n)$ the totality of $P \subset \mathbb{R}^n$ which is open and closed in some $Z \in \mathscr{Z}$. Obviously, $\mathscr{E} \supset \mathscr{P} \supset \mathscr{Z}$.

3. First we given lemmas:

LEMA 1. *If* P_1 *and* P_2 *are in* $\mathscr{P}$, *then* $P_1 \cap P_2 \in \mathscr{P}$.

PROOF. Suppose that P_i is open and closed in $Z_i \in \mathscr{Z}$ for $i = 1,2$. Then $P_1 \cap P_2$ is a closed subset of $\mathbb{R}^n$ and is closed in $Z_1 \cap Z_2 \in \mathscr{Z}$. Next, since P_i is open in Z_i, $Z_i - P_i$ is closed in $\mathbb{R}^n$. Therefore, by

$$P_1 \cap P_2 = Z_1 \cap Z_2 - ((Z_1 - P_1) \cup (Z_2 - P_2)),$$

$P_1 \cap P_2$ is open in $Z_1 \cap Z_2$. Q.E.D.

LEMMA 2. $\mathscr{P}$ *has the chain condition* (*).

PROOF. Let $P_1 \supsetneq P_2 \supsetneq \ldots$ be a descending chain in $\mathscr{P}$. We choose $Z_i \in \mathscr{Z}$ such that P_i is open in Z_i $(i = 1,2,\ldots)$ and we set $Z_1 \cap Z_2 \cap \ldots \cap Z_i = Z_i'$. Then $Z_1' \supset Z_2' \supset \ldots$ and $\mathscr{Z} \ni Z_i' \supset P_i$. Since P_i is open in Z_i, we have that P_i is open in Z_i'. Since $\mathscr{Z}$ has the chain condition, we can find j with $Z_j' = Z_{j+1}' = \ldots$. Then $P_j, P_{j+1}, \ldots$ are all closed and open in Z_j', and by the Whiney's theorem mentioned above, there are only finitely many such sets. Q.E.D.

PROPOSITION 1. $\mathscr{P}$ *is the totality of closed sets with respect to a certain* T_1*-topology of* $\mathbb{R}^n$. (We call the topology *pre-Zariski.*)

PROOF. Let $\{P_\lambda;\ \lambda \in \Lambda\}$ be a subset of $\mathscr{P}$. By the chain condition $\bigcap P_\lambda = P_{\lambda_1} \cap \ldots \cap P_{\lambda_k}$ for some $\lambda_1, \ldots, \lambda_k \in \Lambda$, and by Lemma 1, $\bigcap P_\lambda \in \mathscr{P}$.

Next, let $P_1, \ldots, P_k$ be in $\mathscr{P}$. Then $P_1 \cup \ldots \cup P_k$ is a closed set of $\mathbb{R}^n$. Suppose that P_i is open and closed in Z_i $(i = 1,2,\ldots,k)$. Then $Z_i - P_i \in \mathscr{E}$, and we have $\bigcap_i (Z_i - P_i) \in \mathscr{E}$. Hence $P_1 \cup \ldots \cup P_k = Z_1 \cup \ldots \cup Z_k - \bigcap_i (Z_i - P_i)$ is open in $Z_1 \cup \ldots \cup Z_k$. Hence $P_1 \cup \ldots \cup P_k \in \mathscr{P}$.

Since $\mathscr{Z}$ gives a T_1-topology, so does $\mathscr{P}$. Q.E.D.

Now the following proposition is obvious:

PROPOSITION 2. *A member* P *of* $\mathscr{P}$ *is connected with respect to the pre-Zariski topology if and only if* P *is connected.*

4. In $\mathbb{R}^{n^2+1}$, we consider a hypersurface $\mathscr{G}$ defined by $det(x_{ij})x_0 = 1$, where x_{ij} $(i, j = 1, \ldots, n)$ and x_0 are independent indeterminates. For $(a_{11}, \ldots, a_{nn}, a_0) \in G$, we have $det(a_{ij}) \neq 0$, and so we have a one-to-one map $(a_{11}, \ldots, a_{nn}, a_0) \longrightarrow (a_{ij})$ from $\mathscr{G}$ onto the general linear group $\mathrm{GL}(n, \mathbb{R})$. After this, we identify $\mathrm{GL}(n, \mathbb{R})$ with $\mathscr{G}$ by the map.

By the pre-Zariski topology of $\mathrm{GL}(n, \mathbb{R})$, we mean the induced topology of $\mathscr{G} = \mathrm{GL}(n, \mathbb{R})$ in $\mathbb{R}^{n^2+1}$ with the pre-Zariski topology. Let us call a subgroup of $\mathrm{GL}(n, \mathbb{R})$ *pre-algebraic* if it is closed with respect to the pre-Zariski topology of $\mathrm{GL}(n, \mathbb{R})$. Obviously, a subgroup G of $\mathrm{GL}(n, \mathbb{R})$ is pre-algebraic if and only if G is of finite index in a suitable algebraic group A of $\mathrm{GL}(n, \mathbb{R})$, and this gives the equivalence of the definitions of pre-algebraic groups in this note and in Goto-Wang [1].

Bibliography

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