Houses with Chimneys: A Curious Periodicity in $H_*(MO)$*

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In memory of Carlos B. de Lyra

The algebraic results described in this note arose from the investigation of a filtration on the unoriented cobordism ring associated to the geometric problem of immersing manifolds in Euclidean space up to cobordism ([1], [2], [4], [5]). The key algebraic result is as follows: let $A = 1 + a_1 t + a_2 t^2 + \ldots$ be a power series with coefficients in the polynomial ring $\mathbb{Z}[a_1, a_2, \ldots, a_n, \ldots]$, A_s^k the coefficient of t^s in A^k, and set $v(A_s^k) = r$ if $A_s^k \equiv 0 \ mod\ 2^r$, $A_s^k \not\equiv 0 \ mod\ 2^{r+1}$. We have: $v(A_s^k) = max\ (0, v(k) - v(s))$. It is unreasonable to suppose that this result in 2-adic arithmetic had not been noticed before — indeed a reason for circulating this note is to find out who had noticed this phenomenon first.

The paper is organized as follows: in the first section we give the background of the problem and exhibit a periodicity result in $H_*(MO;\ \mathbb{Z}_2)$. This periodicity is then related to the problem in 2-adic valuation mentioned above. The second section is purely algebraic and devoted to the examination of several questions in 2-adic arithmetic.

I am grateful to T. tom Dieck for pointing out that a generalization of the Schoolboy Multinomial Theorem is the key to the solution (indeed, the current proof of Theorem 5 is essentially a rearrangement of a proof sketched by tom Dieck). I am also grateful to J. F. Adams for an exchange of letters which clarified the experimental fact that m_{2^r-1} is the element in the basis of $H_{2^r-1}(MO;\ \mathbb{Z}_2)$ dual to the basis of monomials in Stiefel-Whitney classes corresponding to w_{2^r-1}.

1. Background and statement of results. Let $H_*(\ \ )$ henceforth denote homology with coefficients in the integers modulo 2. We let MO be the

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Thom spectrum for the orthogonal group and φ_*: $H_*(MO) \longrightarrow H_*(BO)$ the Thom isomorphism, h: $\pi_*(MO) \longrightarrow H_*(MO)$ the Hurewicz homomorphism. We let λ_s: $\tilde{H}_{*+s}(MO(s)) \longrightarrow H_*(MO)$ be the map into the direct limit and define the filtration on $\pi_*(MO)$ by the pullback diagram

$$\begin{array}{ccc} F_s & \longrightarrow & \pi_*(MO) \\ \downarrow & & \downarrow h \\ \tilde{H}_{*+s}(MO(s)) & \xrightarrow{\lambda_s} & H_*(MO). \end{array}$$

The filtration F_s and the structure of the associated graded group has been studied in [4] and [5]. The main algebraic difficulty in that study is this. Let b_n be the non-zero element in $H_n(MO)$ which is in the image of λ_1, then $H_*(MO)$ is a polynomial algebra on $b_1, \ldots, b_n, \ldots$ over $\mathbb{Z}_2$ and it is very easy to decide whether a given y in $H_*(MO)$ is in the image of λ_s: express y as a polynomial in the b_n – if its algebraic degree in the generators b_i is at most s then it lies in the image of λ_s. However there is an unfortunate aspect – it turns out that the b_n are far from the image of the mod 2 Hurewicz homomorphism h-polynomial generators for $\pi_*(MO)$ tend to have ever more complicated expressions in terms of the b_n under the monomorphism h. Clearly a more convenient set of generators for $H_*(MO)$ is needed, and it is furnished by the elements m_n which are defined as follows: consider a power series in t defined by

$$\omega = t + b_1 t^2 + b_2 t^3 + \ldots + b_n t^{n+1} + \ldots$$

and define the m_n as the coefficients of the inverse power series

$$t = \omega + m_1 \omega^2 + m_2 \omega^3 + \ldots + m_n \omega^{n+1} + \ldots$$

Notice that this process makes sense over $\mathbb{Z}$, not just over $\mathbb{Z}_2$ as in our application. There is a formula for the coefficients m_n (originally due to Burmann and Lagrange – see Hurwicz [3], p. 139, and rediscovered in the cobordism context by Miscenko – see Novikov [6], p. 936): let $\omega = tB$, then

$$m_n = \frac{1}{n+1} B_n^{-n-1}.$$

Notice that B^{-1} makes sense (the multiplicative inverse) since B has 1 as its constant term. Our m_n are obtained by taking these modulo 2. Notice that $m_{2n} = h(\text{class of } RP^{2n})$, so these elements are indeed close to the image of h. Since $m_n = b_n$ modulo decomposable elements of $H_*(MO)$ we have that the m_n will serve as polynomial generators of $H_*(MO)$. If they are to be of use for our purposes we must have a test for deciding whether a polynomial belongs to the image of λ_t – a test in terms of the m_n. This is furnished by a homomorphism of algebras

$$\Delta\colon\ H_*(MO) \longrightarrow H_*(MO)[s]$$

of degree 0 (if we let grade $s = 1$) which is defined by setting $\Delta(x) = \sum_i \Delta_i(x) s^i$

where Δ_i is the dual homomorphism to cupping with the $i-th$ Stiefel-Whitney class w_i (when we identify $H^*(MO)$ with $H^*(BO)$ under the Thom isomorphism). An element x in $H_*(MO)$ is in the image of λ_t if and only if the degree of $\Delta(x)$ as a polynomial in s is at most t. Here is a table of Δm_n for low values of n.

TABLE 1

Δm_n for $n \leq 15$

$$\Delta m_1 = m_1 + s$$
$$\Delta m_2 = m_2 + sm_1$$
$$\Delta m_3 = m_3 + s^3$$
$$\Delta m_4 = m_4 + s\mu_3 + s^2 v_2 + s^3 m_1$$
$$\Delta m_5 = m_5 + sm_2^2 + s^2 v_3$$
$$\Delta m_6 = m_6 + s\mu_5$$
$$\Delta m_7 = m_7 + s^7$$
$$\Delta m_8 = m_8 + s\mu_7 + s^2 v_6 + s^3 v_5^{\#} + s^4 \omega_4 + s^5 \mu_3 + s^6 v_2 + s^7 m_1$$
$$\Delta m_9 = m_9 + sm_4^2 + s^2 v_7 + s^4 \omega_5 + s^5 v_2^2 + s^6 v_3$$
$$\Delta m_{10} = m_{10} + s\mu_9 + s^4 \omega_6 + s^5 \omega_5^{\#}$$
$$\Delta m_{11} = m_{11} + s^3 m_2^4 + s^4 \omega_7$$
$$\Delta m_{12} = m_{12} + s\mu_{11} + s^2 v_{10} + s^3 v_9^{\#}$$
$$\Delta m_{13} = m_{13} + sm_6^2 + s^2 v_{11}$$
$$\Delta m_{14} = m_{14} + s\mu_{13}$$
$$\Delta m_{15} = m_{15} + s^{15}$$

Notation:
$$\mu_{2i+1} = m_{2i+1} + m_1 m_{2i} + m_2 m_{2i-1} + \ldots + m_i m_{i+1},$$
$$v_k = m_k + m_1^2 m_{k-2} + m_2^2 m_{k-4} + \ldots,$$
$$\omega_k = m_k + m_1^4 m_{k-4} + m_2^4 m_{k-8} + \ldots,$$
$$v_k^{\#} = \mu_k + m_1^2 \mu_{k-2} + m_2^2 \mu_{k-4} + \ldots,$$
$$\omega_k^{\#} = \mu_k + m_1^4 \mu_{k-4} + m_2^4 \mu_{k-8} + \ldots.$$

The reader has noticed the presence of a lot of zeroes in the table and the beginnings of a periodicity along the diagonal. The next table exhibits in a schematic way the occurences of non-zero entries in Δm_n for $n \leq 32$.

TABLE 2

Pattern of Nonzero Entries In Δm_n

```
 0  *
 1  * *
 2  * *
 3  *     *
 4  * * * *
 5  * * *
 6  * *
 7  *             *
 8  * * * * * * * *
 9  * * *   * * *
10  * *     * *
11  *     * *
12  * * * *
13  * * *
14  * *
15  *                             *
16  * * * * * * * * * * * * * * * *
17  * * *   * * *   * * *   * * *
18  * *     * *     * *     * *
19  *     * *       *     * *
20  * * * *         * * * *
21  * * *           * * *
22  * *             * *
23  *             * *
24  * * * * * * * *
25  * * *   * * *
26  * *     * *
27  *     * *
28  * * * *
29  * * *
30  * *
31  *                                                             *
32  * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
```

The pattern of nonzero entries can be described in terms of houses with a chimney. The r-th house will have 2^r stories; its top floor will be at row 2^r and its chimney will be in row $2^r - 1$ and column 2^r. If A_r denotes the r-th house, the $(r+1)-st$ house is obtained by the scheme

$$A_{r+1} = \begin{bmatrix} A_r^- & A_r \\ A_r & 0 \end{bmatrix}$$

where A_r^- indicates the house A_r with its chimney knocked off and 0 indicates a $2^r x 2^r$ block of zeroes. Thus

```
     *            *            *
A0   *      A1  * *      A2  * * * *
                *            * * *
                             * *
                             *
```

```
                    *
A3  * * * * * * * *
    * * *   * * *
    * *     * *
    *     * *
    * * * *
    * * *
    * *
    *
```

This pattern was noticed experimentally when Δm_n was being computed using a recursive formula in [5] (Theorem 5 of that note): let $S = 1 + + st + sm_1 t^2 + \ldots + sm_k t^{k+1} + \ldots$, S_j^i the coefficient of t^j in S^i, then

$$m_k = \sum_{i=0}^{k} (\Delta m_i) S_{k-i}^{i+1}.$$

It turned out that this recursive formula tended to give an ever more complicated presentation of zero for many $\Delta_i m_n$. To understand this pattern of zeroes it was found to be helpful to give an explicit formula for Δm_n, namely

$$\Delta m_n = \sum_{k=o}^{n} (-1)^k \frac{\binom{n+k}{k}}{n+1} B^{-n-1}_{n-k} s^k.$$

The reader may notice with surprise the presence of $(-1)^k$ in the formula – we put it in to signal that the formula is valid over the integers (that is, in $H_*(MU; \mathbb{Z})$) and our formula for $H_*(MO)$ is the reduction of this one modulo 2. Our next table gives the maximum power of 2 dividing the coefficients in the integral formula.

TABLE 3

2 − Adic Valuation of $\dfrac{\binom{n+k}{n}}{n+1} B^{-n-1}_{n-k}$

$\downarrow n$	$\xrightarrow{k}$
0	0
1	00
2	001
3	0110
4	00001
5	000211
6	0022112
7	02211220
8	000000001
9	0001000311
10	00110033112
11	011003321221
12	0000222211112
13	00032223111322
14	003322331133223
15	0332233113322330
16	00000000000000001
17	000100020001000411
18	0011002200110044112
19	01100221011004431221
20	000011110000333311112
21	0002111200043334111322
22	00221122004433441133223
23	022112200443344213322331
24	0000000022222222111111112
25	00010004222322241112111422
26	001100442233224411221144223
27	0110044323322443122114432332
28	00003333222233331111333322223
29	000433342224333411143334222433
30	0044334422443344114433442244334
31	04433442244334411443344224433440
32	000000000000000000000000000000001

If the reader treats Table 3 as instructions to color by number he will find that there is a beautiful periodicity (we see only its bottom layer when we reduce modulo 2 to obtain Table 2), moreover the table extends via this periodicity to all values of k (not just for $k \le n$ as in our definition). The author is currently unable to explain this.

2. Some results in 2-adic arithmetic. Let G be a free abelian group. We define a function v: $G \longrightarrow \mathbb{N} \cup \{\infty\}$ (called "the greatest power of 2 in") to the natural numbers together with ∞ by setting $v(g) = r$ if g determines the zero coset in $G/2^rG$ and determines a nonzero coset in $G/2^{r+1}G$. We set $v(0) = \infty$.

Here are some useful trivialities: if $f: G \longrightarrow G'$ is a homomorphism of abelian groups, then $v(f(g)) \ge v(g)$ for all g in G. If we let $\mathbb{Z}_{(2)}$ be the subring of the rational numbers consisting of all a/b where a and b are relatively prime and b is odd (that is $\mathbb{Z}_{(2)}$ is the localization of Z at the prime 2) and set $G_{(2)} = G \otimes \mathbb{Z}_{(2)}$, then G is embedded in $G_{(2)}$ and v extends uniquely to a function on $G_{(2)}$. We put the two remarks together for future reference:

LEMMA 1. *If h: $G \longrightarrow G'$ is a homomorphism of free abelian groups then $v(h(g)) \ge v(g)$ for all g in G. Moreover if h induces an isomorphism of $G_{(2)}$ with $G_{(2)}$ then $v(h(g)) = v(g)$.*

Our next task is to investigate the special case $G = \mathbb{Z}$. If k is a natural number, we let $\alpha(k)$ be the number of ones in the dyadic expansion of k. Our next lemma gives a relation between v and α.

LEMMA 2. *If n is a natural number then $v(n+1) = 1 + \alpha(n) - \alpha(n+1)$.*

PROOF. Let $n+1 = 2^r(2a+1)$, so $v(n+1) = r$, and $\alpha(n+1) = \alpha(a) + 1$. However $n = 2^{r+1}a + 2^r - 1$, so $\alpha(n) = \alpha(a) + r$, and the lemma follows by solving for $\alpha(a)$.

COROLLARY 3. $v(n!) = n - \alpha(n)$.

PROOF. Induction on n. For $n = 0$ the result is trivially true. Suppose it is true for n, then

$$
\begin{aligned}
v((n+1)!) &= v(n!) + v(n+1) \\
&= n - \alpha(n) + 1 + \alpha(n) - \alpha(n+1) \\
&= n + 1 - \alpha(n+1)
\end{aligned}
$$

and we are done.

COROLLARY 4. $v\left(\binom{n+k}{n}\right) = \alpha(n) + \alpha(k) - \alpha(n+k)$.

REMARK. It seems that this result is due to Lagrange. I am grateful to Donald Davis for this piece of historical intelligence.

Since Corollary 4 gives complete information about the greatest powers of 2 in binomial coefficients, we now plunge into the investigation of the 2-valuation of multinomial coefficients subject to certain constraints. Let $P = \mathbb{Z}[a_1, a_2, \ldots]$ be a polynomial ring in a countable number of indeterminates a_i, and consider the power series

$$A = 1 + a_1 t + a_2 t^2 + \ldots + a_r t^r + \ldots .$$

Denote by A_s^k the coefficient of t^s in A^k. We wish to examine $v(A_s^k)$ – in other words

$$v(A_s^k) = min\left\{ v\left(\binom{k}{j_0, j_1, \ldots, j_t}\right) \middle| \sum_{n=1}^{t} nj_n = s \right\}.$$

It is instructive to plot $v(A_s^k)$ as a function of k and s in order to obtain an idea of what to expect. We do this in Table 4.

TABLE 4

$v(A_s^k)$ for $k,\ s \leq 16$

s ↓ \ k →	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
2	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	3
3	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
4	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2
5	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
6	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	3
7	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
9	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
10	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	3
11	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
12	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2
13	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
14	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	3
15	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	4
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

One can already make some educated guesses about the behavior of $v(A_s^k)$ from this table – first: it is really a function of $v(k)$ and $v(s)$, so in particular it should be enough to know the columns for $k = 2^n$ which exhibit an interesting kind of periodicity. Second: the basic periodicity block becomes twice as long when we pass from 2^n to 2^{n+1} and is obtained by taking two periodicity blocks for 2^n, leaving the initial entry 0, and adding one to the remaining entries. We sum up these experimental results:

THEOREM 5. $v(A_s^k) = \max(0,\ v(k) - v(s))$.

Let us first prove that $v(A_s^k)$ depends only on $v(k)$. This it turns out will follow very easily from Lemma 1 – first we must provide the right setting:

LEMMA 6. *If* $B = 1 + b_1 t + b_2 t^2 + \ldots + b_n t^n + \ldots$ *where each* b_n *is a polynomial in the* a_i *then* $v(B_s^k) \geq v(A_s^k)$.

PROOF. Define a homomorphism of algebras h: $P \longrightarrow P$ where

$$P = \mathbb{Z}[a_1, \ldots, a_n, \ldots] \text{ by } h(a_n) = b_n,$$

then $hA_s^k = B_s^k$ for all k and s and the result follows from Lemma 1.

COROLLARY 7. $\nu(A_s^{ab}) \geq \nu(A_s^a)$ *for all a, b, s. Moreover if b is odd, equality holds.*

PROOF. Define a homomorphism of algebras h: $P \longrightarrow P$ by $h(a_n) = A_n^b$, then $h(A_s^a) = A_s^{ab}$ and the first sentence is proved. If b is odd, then h: $P_{(2)} \longrightarrow P_{(2)}$ is an isomorphism, and Lemma 1 gives the equality.

We have thus reduced the proof of Theorem 5 to the special case $k = 2^n$. Let us see how one could come to the basic idea of the proof by doing a very special case of the theorem: let's prove that if $\nu(s) \geq n$ then $\nu(A_s^{2^n}) = 0$, that is $A_s^{2^n} \not\equiv 0$ modulo 2. We have a fine tool for doing arithmetic modulo 2 – the Schoolboy Multinomial Theorem:

$$A^{2^n} = 1 + a_1^{2^n}t^{2^n} + \ldots + a_r^{2^n}t^{r2^n} + \ldots \text{ modulo } 2,$$

so in particular if $s = r2^n$ for some r, $A_s^{2^n} \equiv a_r^{2^n}$ *mod* 2, and therefore $\nu(A_s^{2^n}) = 0$. Let us jot this down:

LEMMA 8. *If* $\nu(s) \geq n$ *then* $\nu(A_s^{2^n}) = 0$.

We will now state and prove a mild generalization of the Schoolboy Multinomial Theorem. To save writing, let us introduce the operator S on power series in t which replaces t by t^2 and squares the coefficients: $SA = 1 + a_1^2t^2 + \ldots + a_r^2t^{2r} + \ldots$.

LEMMA 9 (Generalized Schoolboy Multinomial Theorem). *For each j with* $0 \leq j \leq n$ *we have* $A^{2^n} \equiv (S^{n-j}A)^{2^j}$ *mod* 2^{j+1}.

PROOF. The result is trivially true for $n = 0$, and for $j = 0$ it is the usual Schoolboy Multinomial Theorem. Assume the result for $n-1$ and for a $j < n$ – we are asked to prove the result for n and $j + 1 \leq n$. Using the inductive hypothesis we have $A^{2^{n-1}} = (S^{n-1-j}A)^{2^j} + 2^{j+1}B$ for some B, and squaring both sides we obtain $A^{2^n} \equiv (S^{n-1-j}A)^{2^{j+1}}$ modulo 2^{j+2}, so the inductive step works and the lemma is proved.

PROOF OF THEOREM 5. Notice that since Lemma 8 checks the theorem for s with $\nu(s) \geq n$ it will be sufficient to prove the theorem for s with $\nu(s) = n-j, 0 < j \leq n$. According to Lemma 9 we have $A^{2^n} \equiv (S^{n-j+1}A)^{2^{j-1}}$ *mod* 2^j, but since the only nonzero coefficients of $S^{n-j+1}A$ are associated with powers of t divisible by at least 2^{n-j+1}, so $A_s^{2^n} \equiv 0$ *mod* 2^j. Again Lemma 9 gives $A^{2^n} \equiv (S^{n-j}A)^{2^j}$ *mod* 2^{j+1}, so if $s = 2^{n-j}m$, m odd, then $A_s^{2^n} \equiv 2^j a_m^{2^{n-j}}$ modulo $(2^{j+1}, a_1, \ldots, a_{m-1})$, hence $\nu(A_s^{2^n}) = j = n - \nu(s)$, and the theorem is proved.

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