Stable Homotopy Invariant Non Embedding Theorems in Euclideau Space*

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To the memory of Carlos B. de Lyra

1. We write $X \subset \mathbb{R}^n$ if there exists a finite simplicial complex of the homotopy type of $\Sigma^r X$ which embeds in S^{n+r}, where Σ denotes suspension; otherwise $X \not\subset \mathbb{R}^n$. We seek information on the smallest integer n such that $X \subset \mathbb{R}^n$ in terms of cohomological data derived from X. It is known that a differentiable n-manifold M can be differentiably embedded in $\mathbb{R}^{2n}$ but not necessarily in $\mathbb{R}^{2n-1}$, that a simplicial complex K of dimension n can be simplicially embedded in $\mathbb{R}^{2n+1}$, but not necessarily in $\mathbb{R}^{2n}$; which of course imply that a compact $M \subset \mathbb{R}^{2n}$ and a compact $K \subset \mathbb{R}^{2n+1}$. Particular attenction has been paid to the question of determining the Euclidean space of least dimension in which a projective space can be embedded and in [7] James wrote a survey article outlining some 30 years progress on the problem. We refer to this article for background material. Most attention has been given to $\mathbb{RP}^n$ but the methods of this paper are not appropriate in this case. We consider the complex and quaternionic projective spaces. As usual, let $\alpha(n)$ be the number of 1's in the dyadic expansion of n. It is known that $\mathbb{CP}^n$ embeds differentiably in $\mathbb{R}^{4n-\alpha(n)}$ and that $\mathbb{HP}^n$ embeds in $\mathbb{R}^{8n-\alpha(n)+4}$. More is known for particular values of n. The general results for non differentiable embeddings appear to be that $\mathbb{CP}^n$ will not embed in $\mathbb{R}^{4n-2\alpha(n)}$, ($\mathbb{CP}^n$ will not embed in $\mathbb{R}^{4n-2\alpha(n)+2}$ if n is even and $\alpha(n) \equiv 0 \ mod\ 3$) $\mathbb{HP}^n$ will not embed in $\mathbb{R}^{8n-2\alpha(n)-2}$ nor in $\mathbb{R}^{8n-2\alpha(n)}$ if $\alpha(n) \equiv 0 \ mod\ 4$. These negative results, except for that in parenthesis, are known to be homotopy invariant in the sense that a differentiable manifold homotopically equivalent to the projective space will not embed in these dimensions either. In proving such negative results one can either use the explicit topology and geometry of the spaces or alternatively apply "Integrality Theorems" for embedding differen-

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tiable manifolds due to Atiyah and Hirzebruch [4] which arose out of the differentiable Riemann-Roch Theorem (see also later work of Mayer [10]). The Integrality Theorems can be applied most successfully to suitable manifolds whose homology groups are free of 2-torsion. The purpose of this paper is to show different, though related, general methods of homotopy theory lead to Integrality Theorems for embedding CW-complexes in Euclidean space which are of comparable strength. In particular, these techniques are adequate to show that

(1.1) $\mathbb{C}P^n \not\subset \mathbb{R}^{4n-2\alpha(n)}$; $\mathbb{H}P^n \not\subset \mathbb{R}^{8n-2\alpha(n)-2}$ and $\mathbb{H}P^n \not\subset \mathbb{R}^{8n-2\alpha(n)}$ if $\alpha(n) \equiv 0 \; mod \; 4$.

More generally, non embedding theorems proved in [4] for differentiable manifolds involving the Chern character are shown to hold for complexes whose homology groups are free of 2-torsion. Related work of Gitler and Milgram can be found in [6] and this paper arose in an attempt to understand this, in particular to determine certain primary cohomology operations which occur there. As we succeed in doing this, we necessarily strengthen the main theorem of [6].

At the heart of the mathematics in this paper lies the relationship between higher order cohomology operations and the λ-ring structure of complex K-theory. The author was stimulated to think again along such lines after giving a course of lectures at the Institute of Mathematics and Statistics in the University of São Paulo during July and August 1972 at the invitation of Professor Carlos de Lyra.

2. The complex of origin and Spanier-Whitehead duality. In an attempt to prevent the straightforward ideas behind this paper becoming obscured by the algebraic details, we start by giving an interpretation of the mathematics which follows. Consider the complex $X = S^{2n} \cup e^{2t}$, $t > n$, where the homotopy class of the attaching map of the third cell is $\alpha \in \pi_{2t-1}(S^{2n})$. We wish to determine the smallest integer k such that there exists a complex $S^{2k} \cup e^{2t-2n+2k}$ with an iterated suspension homotopically equivalent to X. Equivalently one can ask how far α can be doubly desuspended. There is a similar problem for any CW-complex, in particular, for one of the form $X = S^{2n_1} \cup e^{2n_2} \cup \ldots \cup e^{2n_s}$, where $n_1 < n_2 < \ldots < n_s$. One method of gaining information on such questions is to use the Adams-Toda $e_{\mathbb{C}}$ or $e_{\mathbb{R}}$ invariants [3]. Recall that there exists a homomorphism $e_{\mathbb{C}}\colon \pi_{2t-1}(S^{2n}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ which is stable up to sign. One knows that $(k^t - k^n)e_{\mathbb{C}}(\alpha)$ is an integer for all k [3], and so if $e_X = -e_{\mathbb{C}}(\alpha) = 2^{-3}$ where X has three cells, it follows by taking $k=2$, that $n > 2$. More generally if X has $s+2$ cells, one can associate with it an $s \times s$-matrix $e_X = (a_{i,j})$ with entries in $\mathbb{Q}$, where $a_{i,j} = 0$ for $i > j$ and $a_{i,i} = -e_{\mathbb{C}}(\alpha_i)$ where α_i is the homotopy class of the attaching map of the $(i+2)'nd$ cell of X, pinching out the $2n_i - 1$ skeleton. The definition of e_X is elementary and well know, but I think unpublished. In the complex K-theory, $\tilde{K}(X) = \oplus \mathbb{Z}$, s summands, and one can choose generators u_i of exact CW-filtrations $2n_i$. The $e_{i,j} = a_{i,j-1}$ are defined by requiring that $ch(u_i + \sum_{j>i} e_{i,j} u_j) \in H^{2n}(X, \mathbb{Q})$, that is, the components of the Chern character are zero in the other dimensions. (This particular version of the definition follows work of J. Murdock.) Assuming that we are given homology orientations for the cells of X, it is clear that $e_{i,j}$ is well defined in $\mathbb{Q}/\mathbb{Z}$ mod $(1, e_{i,k}, k < j)$. Techniques similar to those used in [3], in particular the restrictions put upon the $e_{i,j}$ which are implied by the existence of the Adams operators ψ^k, lead to integrality conditions on the $e_{i,j}$ from which one can deduce a lower bound for n_1.

Now assume that X is homotopically equivalent to a simplicial complex which can be embedded in S^{2N}. A Spanier-Whitehead $2N$-dual Y [12] can be given a cellular structure of the form $S^{2N-2n_1} \cup e^{2N-2n_2} \cup \ldots \cup e^{2N-2n_s}$. A modified version of the proof of Theorem 3.2, working over $\mathbb{Z}$, shows how one can calculate e_Y directly from e_X. If $e_Y = (b_{i,j})$ where $b_{i,j} = f_{i,j+1}$ and we set $e_{l,l} = 1$ and $f_{s+1-m,s+1-m} = 1$, then for each pair l, m in $1 \leq l < m \leq s$,

$$\sum f_{s+1-m,s+1-m+i} \cdot e_{l,m-i} = 0$$

summing for $0 \leq i \leq m-l$. (To be quite precise, this matrix is e_{SY} but this does not affect the integrality conditions. It explains why the $e_{\mathbb{C}}$ invariants of the attaching maps of the cells in the dual complex have opposite signs from those expected.) For example if

$$e_X = \begin{pmatrix} e_\alpha & e_{\alpha\beta} \\ 0 & e_\beta \end{pmatrix}, \quad \text{then} \quad e_Y = \begin{pmatrix} -e_\beta & -e_{\alpha\beta} + e_\alpha e_\beta \\ 0 & -e_\alpha \end{pmatrix}.$$

One can then use the techniques mentioned above to obtain lower on $N - n_s$ and thus on N. The results (1.1) can be proved by this method and a few tricks. We do not pursue this line of thought further here in the interests of obtaining greater generality and because the quantity of elementary calculations needed to apply it effectively is prohibitively large. However it can be used, for example, to give information about stably self dual complexes with few cells.

3. We summarize some facts which follow by standard arguments from properties of complex K-theory [5] and Spanier duality [13]. We work at a fixed prime which for simplicity we choose to be 2. However all results, except those of § 4, generalize quite easily for odd primes. Let $\mathbb{Z}_{(2)}$ be the ring of integers localised at the prime ideal (2) and let $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$. The space X will always have the homotopy type of a finite CW-complex with integral homology free of 2-torsion, that is $X \in F_2$. Let Y be a $2N$-dual of X in the sense of [13] not [12], see Lemma 5.1 of the former. Clearly $Y \in F_2$. Then for sufficiently large l and m, there exists a map

$$\varphi: S^{2N+l+m} \longrightarrow \Sigma^l X \wedge \Sigma^m Y$$

inducing a non singular bilinear transformation

$$\varphi^*: \tilde{H}^*(\Sigma^l X, \mathbb{Z}_2) \otimes \tilde{H}^*(\Sigma^m Y, \mathbb{Z}_2) \longrightarrow H^{2N+l+m}(S^{2N+l+m}, \mathbb{Z}_2)$$

and thus a non singular pairing

$$(\,,): \tilde{H}^{even}(X) \otimes \tilde{H}^{even}(Y) \longrightarrow H^{2N}(S^{2N}),$$

where now we have $\mathbb{Z}_{(2)}$-coefficients. It is this pairing which concerns us and we can say that Y will exist if $X \subset S^{2N+1}$. If $X \subset S^{2N}$, we may assume that Y is a suspension. The map φ will also induce a non singular bilinear transformation $(\,,)_K: \tilde{K}(X) \oplus \tilde{K}(Y) \longrightarrow \tilde{K}(S^{2N})$ in the unitary K-theory with $\mathbb{Z}_{(2)}$-coefficients.

Implicit in the above are the facts that $H^{even}(W)$ and $K(W)$, $W = X$ or Y, are finitely generated free $\mathbb{Z}_{(2)}$-modules. Let $J: \tilde{H}^{even}(X) \longrightarrow \tilde{K}(X)$ be a filtration preserving isomorphism where $\tilde{H}^{even}(X)$ has the decreasing filtration given by its grading and $\tilde{K}(X)$ the CW-filtration. Define $K^{-1}: \tilde{K}(Y) \longrightarrow \tilde{H}^{even}(Y)$ by $(Jx, w)_K = (x, K^{-1}w)$ where $\tilde{K}(S^{2N})$ and $H^{2N}(S^{2N})$ are identified using the Chern character and where we have anticipated in the notation the fact that K^{-1} is an isomorphism. In fact K^{-1} is a filtration preserving isomorphism of free $\mathbb{Z}_{(2)}$-modules with inverse a filtration preserving isomorphism $K: \tilde{H}^{even}(Y) \longrightarrow \tilde{K}(Y)$. We are now in position to use the techniques of [9]. If $X \in F_2$, the homomorphism

$Q^q_J: H^{2n}(X) \longrightarrow H^{2n+2q}(X)$ can be defined by $Q^q_J x = 2^q ch_{n+q} Jx$. The reduction of Q^q_J mod 2 equals $\chi(Sq^{2q})$, where χ is the usual anti-automorphism in the mod 2 Steenrod algebra. The homomorphism $S^q_J: H^{2n}(X) \longrightarrow H^{2n+2q}(X)$ is then defined by requiring that S^0_J is the identity and the relations $\Sigma S^i_J Q^{q-i}_J = 0$ for each $q > 0$, summing for $0 \leq i \leq q$. The following lemma will be useful. The proof is routine.

LEMMA 3.1. $\Sigma Q^i_J S^{q-i}_J = 0$.

If $T: \tilde{H}^{even}(Y) \longrightarrow \tilde{H}^{even}(Y)$ is a homomorphism, define the dual homomorphism $\chi(T): \tilde{H}^{even}(X) \longrightarrow \tilde{H}^{even}(X)$ by $(x, Ty) = (\chi(T)x, y)$. The use of χ is justified by the following.

THEOREM 3.2. *$S^q_J = \chi(Q^q_K)$ and $Q^q_J = \chi(S^q_K)$.*

PROOF. Let $x \in H^{2l}(X)$ and $y \in H^{2n}(Y)$, $J(x) = u$ and $K(y) = v$. Then $(\psi^k u, \psi^k v)_K = \psi^k(u, v)_K = k^N(u, v)_K$, using the instability of the Adams operators ψ^k [2] and the fact that the ψ^k preserve tensor products. If $\Phi^k_J: H^{even}(X) \longrightarrow H^{even}(X)$ is defined to be $J^{-1}\psi^k J$, then

$$\Phi^k_J(x) = k \sum_{q \geq 0} 2^{-q} \sum_{0 \leq r \leq q} k^r S^{q-r}_J Q^r_J(x)$$

and similarly for $\Phi^k_K(y)$, see Lemma 2.10 of [9]. Thus $(k^l \Sigma 2^{-q} \Sigma k^r S^{q-r}_J Q^r_J x, k^m \Sigma 2^{-q} \Sigma k^r S^{q-r}_K Q^r_K y) = k^N(x, y)$.

Let $l+m = N-1$ and consider coefficients of k^N. The terms which are not necessarily zero for dimensional reasons give

$$2^{-1}(x, S^1_K y) + 2^{-1}(S^1_J x, y) = 0,$$

and so $(S^1_J x, y) = (x, Q^1_K y)$ and the result for $q = 1$ follows. Assume therefore that the theorem is true for $q < n$ and let $l+m = N-n$, and again equate coefficients of k^N. Therefore $2^{-n}\Sigma(S^i_J x, S^{n-i}_K y) = 0$, where the summation is for $0 \leq i \leq n$, and so by induction $(S^n_J x, y) = (x, -\Sigma^i_K S^{n-i}_K y)$, where the summation is now for $1 \leq i \leq n$. The latter equals $(x, Q^n_K y)$ by Lemma 3.1. This completes the proof.

A less elementary but possibly more informative proof of Theorem 3.2 can be given along the following lines. In Corollary 2.28 of [9], formulae are given to compute the deviation from naturality of S_J^n. Using the pairing $(\,,)$, one can compute in a similar manner the deviation from naturality of $\chi(Q_K^n)$. An inductive argument shows that $S_J^n - \chi(Q_K^n)$ is natural and is therefore a stable primary cohomomology operation defined in $\mathbb{Z}_{(2)}$-cohomology for spaces in F_2, which is zero.

Theorem 3.2 enables us to determine completely the λ-module structure of $K(Y)$ from that $K(X)$. More generally, a similar argument can be used to determine the λ-module structure of $K(Y)/\text{Torsion}$ from that of $K(X)/\text{Torsion}$ with Z-coefficients for the $2N$-dual Y of any finite complex X, using the rational γ-filtration.

We write $< \,, >_n : H^n(X, \mathbb{Q}) \oplus H_n(X, \mathbb{Z}_{(2)}) \longrightarrow \mathbb{Q}$ for the Kronecker index, $X \in F_2$. Define $V_J^q : H^{2n}(X) \longrightarrow H^{2n+2q}(X)$ to be the homomorphism $\Sigma\, 2^{q-i} S_J^{q-i} Q_J^i$, where the summation is for $0 \le i \le q$.

THEOREM 3.3. The Integrality Theorem. *Let* $x \in H^{2n-2q}(X)$ *and* $z \in H_{2n}(X)$.
(i) If $X \subset S^{2M+1}$, *then* $< 2^{M-n-q} V_J^q x, z >_{2n} \in \mathbb{Z}_{(2)}$.
(ii) If $X \subset S^{2M}$, *then* $< 2^{M-n-q-1} V_J^q x, z >_{2n} \in \mathbb{Z}_{(2)}$.

If X is an oriented differentiable $2n$ manifold, we could choose z to be the fundamental class. Part (i) of the theorem is clearly implied by part (ii), but we consider part (i) first.

It is sufficient to show that if $y \in H^{2N-2n+2q}(Y)$, then $(2^{M-n-q} V_J^q x, y) \in \mathbb{Z}_{(2)} \subset Q$ (with the obvious extension of the definition of $(\,,)$.) But

$$(2^{M-n-q} V_J^q x, y) = (x, 2^{M-n-q} \Sigma 2^i S_K^i Q_K^{q-i} y)$$

again summing for $0 \le i \le q$, by Theorem 3.2. But the second term in the last pairing is just the component of $\Phi_K^2(y)$ in dimension $2(M-n+q)$ (Lemma 2.10 of [9]), and so lies in $H^{2(M-n+q)}(Y)$ and the bracket lies in $\mathbb{Z}_{(2)}$, which proves (i). If Y is a suspension, cup products vanish in $\tilde{K}(Y)$ and so $\lambda^2 = -2\psi^2$ and this same expression is equal to the component of $-2K^{-1}\lambda^2 K(y)$ in dimension $2(M-n+q)$. This gives the additional power of 2 needed in part (ii).

Before interpreting Theorem 3.3 in more standard language, we give a direct application which shows how the homomorphism V_J^q can be calculated quite easily given the λ-module structure of $K(X)$.

EXAMPLE. Let $\mathbb{HP}^r \subset \mathbb{R}^{2M}$. Then $H^*(\mathbb{HP}^r) \cong \mathbb{Z}_{(2)}[x]/(x^{r+1})$, dim $x = 4$, $K(\mathbb{HP}^r) = \mathbb{Z}_{(2)}[u]/(u^{r+1})$ where for suitable and standard choice of u,

$$\psi^k(u) = k^2 u + \{k^2(k^2-1)/2^2(2^2-1)\}\,.\,u^2 + \dots + \\ + \{k^2(k^2-1)\dots(k^2-(r-1)^2)/r^2(r^2-1)\dots(r^2-(r-1)^2)\}\,.\,u^r.$$

Thus by considering the component of $\Phi_J^k(x)$ in dimension $4r$, we obtain

$$\Sigma k^i S_J^{2r-i} Q_J^i x = 2^{2(r-1)}(k^2-1)\dots(k^2-(r-1)^2)/r^2(r^2-1)\dots(r^2-(r-1)^2)\,.\,x^r,$$

where we have chosen J to be the ring isomorphism such that $Jx = u$. Therefore

$$\Sigma k^{2r-i} S_J^{2r-i} Q_J^i x = 2^{2(r-1)}(1-k^2)\dots(1-k^2(r-1)^2)/r^2(r^2-1)\dots(r^2-(r-1)^2)\,.\,x^r,$$

and so $V_J^{2r-2} x = 2^{2r-1}/(2r)!\,.\,x^r$.

Applying Theorem 3.4 (ii) with $n = 2r$ and $q = 2r-2$, we obtain $M \ge \nu_2(2r!) + 2r$. Since $\alpha(r) = r - \nu_2(r!)$ and $\alpha(2r) = \alpha(r)$, this becomes $M \ge 4r - \alpha(r)$. Therefore $\mathbb{HP}^r \not\subset S^{8r-2\alpha(r)-2}$.

Given $u \in K(X)$, write $ch_n(u) = s_n(u)/n! \in H^{2n}(X, \mathbb{Q})$. Then $s_n(u) \in H^{2n}(X)$. Let $\alpha: H^{2n}(X) \longrightarrow H^{2n}(X, \mathbb{Z}_2)$ be induced by the usual coefficient map. The following corollaries to Theorem 3.3 are closely connected with the theorems of § 3.2 of [3].

COROLLARY 3.4. *Let* $u \in K(X)$ *and* $q = 2^r(2s+1)$, $r \ge 0$. *If* $s_q(u) = 2^t y$, $0 \le t \le r$, *where* $y \in H^{2q}(X)$ *and* $\alpha y \ne 0$, *then* $X \not\subset S^{4q-2\alpha(q)-2t}$.

PROOF. Let $Jx = u$, where without loss of generality we may assume that $x \in H^{2v}(X)$. We know that $Q_J^s x = 2^s ch_{t+s} u = 2^s s_{t+s}(u)/(t+s)!$. Therefore

$$V_J^{q-v} x = 2^{q-v}\{s_q(u)/q! + S_J^1(s_{q-1}(u)/(q-1)!) + \dots + S_J^{q-v}(s_v(u)/v!)\}.$$

We need to establish (for the case $r = t$) that $S_J^1(s_{q-1}(u)) = 0$ *mod* 2. This follows from the Splitting Principle, the definition of ch and that fact that S_J^1 *mod* $2 = Sq^2$. For $Sq^2\alpha\{s_{q-1}(u)\} = \alpha\{(q-1)s_q(u)\}$. Thus $V_J^{q-t} x = 2^{q-v+t}/q!\,.\,(y+2z)$. The corollary follows from Theorem 3.3 (ii).

COROLLARY 3.5. *If* $d \in H^2(X)$ *and* $\alpha(d^q) \neq 0$, *then* $X \not\subset S^{4q-2\alpha(q)}$.

This corollary follows from Corollary 3.4.

EXAMPLE. $\mathbb{CP}^n \not\subset S^{4n-2\alpha(n)}$.

4. It is to be expected that if the complex K-theory of this paper could be replaced by real or quaternionic K-theory, some results would be strengthened by a power of 2, as in [4]. There are difficulties in doing this for a general $X \in F_2$ and we do not pursue this here. However we give an example to show how real and quaternionic K-theory can be used in certain circumstances.

Let $c: K_R(Y) \longrightarrow K(Y)$ and $c': K_H(Y) \longrightarrow K(Y)$ be the complexification transformations. If Y can be given a cellular structure in which all cells have dimensions congruent to zero modulo 4, then an inductive argument on the skeletons shows that each element of $\tilde{K}(Y)$ is of the form $c(u)+c'(v)$. Suppose that $\mathbb{HP}^r \subset S^{8r-2\alpha(r)}$ where $\alpha(r) \equiv 0$ (4). We shall obtain a contradiction. The dual space Y can be given a cellular structure of the form

$$S^{4r-2\alpha(r)} \cup e^{4r-2\alpha(r)+4} \cup \dots \cup e^{8r-2\alpha(r)-4}.$$

Let $y \in H^{4r-2\alpha(r)}(Y)$ be defined by $(x^r, y) = 1$, where x is a generator of $H^4(\mathbb{HP}^r)$. Let $K(y) = c(u) + c'(v)$, Then $\lambda^2 K(y) = \lambda^2 c(u) + \lambda^2 c'(v) = c(w)$, since $c(u)\,.\,c'(v) = 0$ as Y is a suspension. Now it is a routine matter to check that $\lambda^2 = 0$ *mod* 2 in $\tilde{K}(Y)/\tilde{K}_{8r-2\alpha(r)-6}(Y)$, that is, $K^{-1}\lambda^2 K = -2^{-1}\Phi_K^2$ is zero mod 2 in dimensions less than $8r - 2\alpha(r) + 4$, using the techniques introduced above. But $c: K_R(S^{8r-2\alpha(r)-4}) \longrightarrow K(S^{8r-2\alpha(r)-4})$ maps a generator to twice a generator since $\alpha(r) \equiv 0$ *mod* 4, and as $\lambda^2 K(y)$ is real, it follows that $K^{-1}\lambda^2 k(y) = 0$ *mod* 2. The second last sentence in the proof of Theorem 3.3 implies that $2^{\alpha(r)-2r} V_J^q(x) \in H^{4r}(\mathbb{HP}^r)$, $q = 2r - 2$. The calculation in the example following Theorem 3.3 shows that this is false. Therefore $\mathbb{HP}^r \not\subset S^{8r-2\alpha(r)}$ if $\alpha(r) \equiv 0$ (4).

5. The Theorem of Gitler and Milgram. As suggested earlier, Theorem 3.2 when reduced modulo 2 states the well known fact that $(\chi(Sq^{2i})x, y) = (x, Sq^{2i}y)$ in $\mathbb{Z}_2$, where x and y are $\mathbb{Z}_2$ cohomology classes. In particular, if $x \in H^{2j}(X, \mathbb{Z}_2)$ and $y \in H^{2N-2j-2i}(Y, \mathbb{Z}_2)$ where $i > N - j - i$, then $(\chi(Sq^{2i})x, y) = (x, Sq^{2i}y)$

is zero, since $Sq^{2i}y = 0$. Therefore $\chi(Sq^{2i})x = 0$ for $i > (N-j)/2$, which is part of a theorem of Thom (see § 1 of chapter 3 in [14]). Since $\chi(Sq^{2i}) = 0$, $Q_J^i \mid H^{2j}(X) = 0$ (2) and so $2^{i-1} ch_{2j+2i} J(x) \in \alpha H^{2j+2i}(X)$, an unstable integrality theorem on the Chern character. The technique of [6] is to use the system of higher order operations which generalize $\chi(Sq^{2i})$ due to Maunder [11]. These operations on the cohomology groups of spaces in F_2 are closely connected with the Q_J^q. We indicate why. We know that Q_J^q *mod* $2 = \chi(Sq^{2q})$. If $x \in H^{2r}(X, Z_2)$ and $\chi(S_q^{2q})x = 0$, choose $\xi \in H^{2r}(X)$ with $\alpha(\xi) = x$. If $J': \tilde{H}^{even}(X) \longrightarrow \tilde{K}(X)$ is another filtration preserving isomorphism, Corollary 2.28 of [9] implies that

$$Q_{J'}^q\, \xi = Q_J^q \xi + 2Q_J^{q-1} f_1 \xi \ mod\ 4.$$

Thus for each ξ, there is a well defined class $(Q_J^q(\xi)/2)$ *mod* 2 in $H^{2q+2r}(X, \mathbb{Z}_2)/\text{Image } \chi(Sq^{2q-2})$. Considering the differen. choices of ξ, we obtain a well defined homomorphism

$$[Q^q]_2: Ker\,\chi(Sq^{2r}) \subset H^{2r}(X, \mathbb{Z}_2) \longrightarrow H^{2q+2r}(X, \mathbb{Z}_2)/(Image\ \chi(Sq^{2q-2}) + \\ + Image\ \chi(Sq^{2q})),$$

which is indeed the second order Maunder operation. The process may be extended to obtain operations of arbitrarily high order. One can attempt the same programme for S_J^q and one does indeed obtain a secondary operation, but beyond the second order there are difficulties (which to a considerable extent are explained in the inner workings of [6]). Our approach to [6] is to use the unstable normalising conditions satisfied by the S_J^q directly.

LEMMA 5.1. *Let* $y \in H^{2q}(Y)$. *Then*

$$S_K^{q+k} y = 0 \ mod \begin{cases} 2^{t/2} & if\ t\ is\ even, \\ 2^{(t+2)/2} & if\ t\ is\ even\ and\ Sq^{2q+t}Sq^t \alpha y = 0, \\ 2^{(t+1)/2} & if\ t\ is\ odd. \end{cases}$$

PROOF. The component of $\Phi_K^2(y)$ in dimension $4q + 2t$ is $2^{-t}\Sigma 2^i S_K^{q+t-i} Q_K^i\, y$, for $0 \leq i \leq q + t$, and so $\Sigma 2^i S_K^{q+t-i} Q_K^i\, y = 0$ *mod* 2^t. The result now follows by a routine inductive argument together with the fact that

$$Sq^{2u} = \sum_{i_1+i_2+\dots+i_s=u} \chi(Sq^{2i_1})\,\chi(Sq^{2i_2}) \dots \chi(Sq^{2i_s}).$$

We can now prove the main theorem of [6] in a strengthened form.

THEOREM 5.2. *Let* $X \in F_2$, $X \subset S^M$ *and* $\mu \in K_{2q}(X)$. *Assume also that if* $M = 2N+1$ *and* $q+N$ *is even, then*

$$\chi(Sq^{2r+q-N})\,\chi(Sq^{N-q}) : H^{2q}(X, Z_2) \to H^{2q+2r}(X, Z_2)$$

is zero. Then $2^{r-t}ch_{q+r}\ \mu \in \alpha H^{2q+2r}(X)$ *where* $t = [(4r+2q-M+5)/4]$.

PROOF. First let $M = 2N+1$ and, without loss of generality, let $Jx = \mu$ with $x \in H^{2q}(X)$. If $y \subset H^{2u}(Y)$, $u = N-q-r$, then $(Q_J^r x, y) = (x, S_K^r y)$ by Theorem 3.2. Lemma 5.1 implies that

$$S_K^r y = 0 \begin{cases} mod\ 2^{(r-u+2)/2} & \text{if } (r-u) \text{ is even and } Sq^{r+u}Sq^{r-u}\alpha y = 0, \\ mod\ 2^{(r-u+1)/2} & \text{if } (r-u) \text{ is odd.} \end{cases}$$

Therefore $Q_J^r x = 0$ *mod* 2^t with t as above and so $2^{r-t}ch_{q+r}\ \mu \in \alpha H^{2q+2r}(X)$. If $r-u$ is even, $t = (r-u+2)/2 = (4r+2q-M+5)/4$, provided that $\chi(Sq^{2r+q-N})\,\chi(Sq^{N-q})\alpha x = 0$. If r$-u$ is odd, $t = (r-u+1)/2 = (4r+2q-M+3)/4$. If $M = 2N$ then $Sq^{r+u}Sq^{r-u}\alpha y$ is a cup square and so vanishes as Y is a suspension. Thus $t = (4r+2q-M+4)/4$ or $(4r+2q-M+2)/4$ depending upon whether $(r-u)$ is even or odd. The theorem is proved.

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